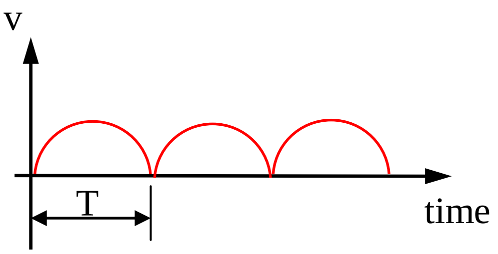
# 2. Fourier Series and Transform

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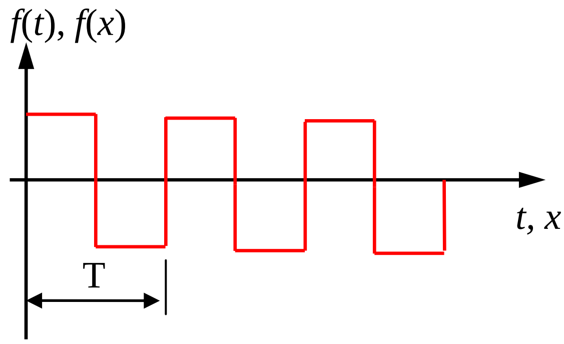



Periodic function: The function which repeats itself each "T" second, where "T" is called period.

Fourier Theorem: Any periodic function $f(t)$ can be rewritten as a sum of sine and cosine components as follows:-

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + b_n \sin \omega_n t$$

constant part mean, D.C. value — even part — odd part

$\frac{a_0}{2}$ **:** mean, average, DC-component**.**

$a_n$, $b_n$ : Coefficient of cosine and sine terms.

$\omega_n = \frac{2n\pi}{T}$ : radian frequency (rad/s)

$f_n = \frac{n}{T} = \frac{\omega_n}{2\pi}$ : frequency (Hz)

$f_1 = \frac{1}{T}$, first fundamental frequency.

$f_2 = \frac{2}{T} = 2f_1$, second fundamental frequency.

$f_3 = \frac{3}{T} = 3f_1$, Third fundamental frequency.

$p = \frac{T}{2}$ is half period.

$$\omega_n = 2\pi f_n = \frac{2n\pi}{T} = \frac{n\pi}{p}$$

$$a_0 = \frac{1}{p}\int_{d}^{d+2p} f(t)dt$$

$$a_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\cos\omega_n t dt$$

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\sin\omega_n t dt$$

**<u>Note:</u>** Fourier theorem can be proved by utilizing the Euler forms:

$$\int_{d}^{d+T} \sin\omega_n t dt = 0 \qquad , \qquad \int_{d}^{d+T} \cos\omega_n t dt = 0$$

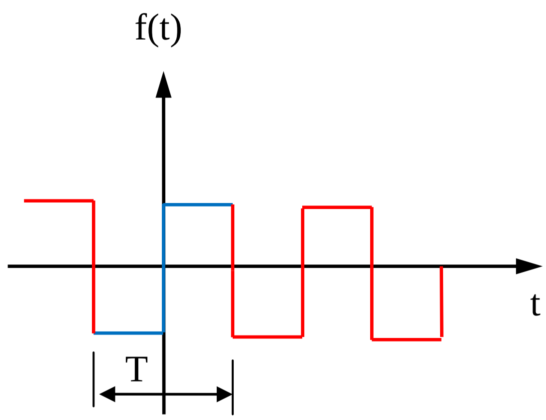
$$\int_{d}^{d+2p} \sin\omega_n t \cos\omega_m t dt = 0$$

$$\int_{d}^{d+2p} \sin\omega_n t \sin\omega_m t dt = \begin{cases} 0 & n \neq m \\ p & n = m \end{cases}$$

$$\int_{d}^{d+2p} \cos\omega_n t \cos\omega_m t dt = \begin{cases} 0 & n \neq m \\ p & n = m \end{cases}$$

**Example:** Find the Fourier expansion of the periodic function whose definition in one period as:

$$f(t) = \begin{cases} -1 & -1 < t < 0 \\ 1 & 0 < t < 1 \end{cases}$$




$$T = 2, \;\; p = 1, \;\; \omega_n = \frac{n\pi}{p} = n\pi$$

$$a_0 = \frac{1}{p}\int_{d}^{d+2p} f(t)dt = \frac{1}{1}\int_{-1}^{1} f(t)dt$$

$$= \int_{-1}^{0} (-1)dt + \int_{0}^{1} 1dt = -t\Big|_{-1}^{0} + t\Big|_{0}^{1} = 0$$

$$a_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\cos\omega_n t dt = \frac{1}{1}\int_{-1}^{1} f(t)\cos n\pi t dt$$

$$= \int_{-1}^{0} -\cos n\pi t dt + \int_{0}^{1} \cos n\pi t dt$$

$$= -\left[\frac{\sin n\pi t}{n\pi}\right]_{-1}^{0} + \left[\frac{\sin n\pi t}{n\pi}\right]_{0}^{1}$$

$$= -\left[0 + \cancelto{=0}{\frac{\sin n\pi}{n\pi}}\right] + \left[\cancelto{=0}{\frac{\sin n\pi}{n\pi}} - 0\right] = 0$$

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\sin\omega_n t dt = \frac{1}{1}\int_{-1}^{1} f(t)\sin n\pi t dt$$

$$= \int_{-1}^{0} -\sin n\pi t dt + \int_{0}^{1} \sin n\pi t dt$$

$$= \left[\frac{\cos n\pi t}{n\pi}\right]_{-1}^{0} - \left[\frac{\cos n\pi t}{n\pi}\right]_{0}^{1}$$

$$= \left[\frac{1}{n\pi} - \frac{\cos n\pi}{n\pi}\right] - \left[\frac{\cos n\pi}{n\pi} - \frac{1}{n\pi}\right]$$

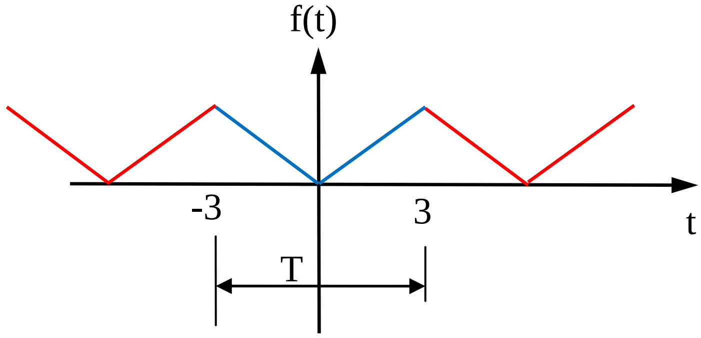
$$\therefore b_n = \frac{2}{n\pi}\left(1 - \cos n\pi\right) = \begin{cases} 4/n\pi & n: odd \\ 0 & n: even \end{cases}$$

$$\therefore a_0 = 0, \quad and \quad a_n = 0$$

$$\therefore f(t) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin n\pi t \qquad , \quad n: odd \;\; only$$

**Example:** Find the Fourier expansion of the periodic function whose definition in one period is;

$$f(t)=\begin{cases}-t & -3<t<0\\ t & 0<t<3\end{cases}$$




$$\therefore T=6,\ p=3,\ \omega_n=\frac{n\pi}{p}=\frac{n\pi}{3}$$

$$a_0=\frac{1}{p}\int_{d}^{d+2p} f(t)dt=\frac{1}{3}\left[\int_{-3}^{0}-tdt+\int_{0}^{3}tdt\right]=\frac{1}{3}\left[\frac{-t^2}{2}\bigg|_{-3}^{0}+\frac{t^2}{2}\bigg|_{0}^{3}\right]=3$$

$$a_n=\frac{1}{p}\int_{d}^{d+2p} f(t)\cos\omega_n t dt=\frac{1}{3}\int_{-3}^{0}-t\cos\frac{n\pi}{3}tdt+\frac{1}{3}\int_{0}^{-3}t\cos\frac{n\pi}{3}tdt$$

$$=\frac{-1}{3}\left[\frac{9}{n^2\pi^2}\cos\frac{n\pi}{3}t+\frac{3t}{n\pi}\sin\frac{n\pi}{3}t\right]_{-3}^{0}+\frac{1}{3}\left[\frac{9}{n^2\pi^2}\cos\frac{n\pi}{3}t+\frac{3t}{n\pi}\sin\frac{n\pi}{3}t\right]_{0}^{3}$$

$$=\frac{-3}{n^2\pi^2}(1-\cos n\pi)+\frac{3}{n^2\pi^2}(\cos n\pi-1)=\frac{6}{n^2\pi^2}(\cos n\pi-1)\qquad ,\ n\neq 0$$

$$a_n=\frac{-12}{n^2\pi^2}\qquad ,\ n:odd\ \ only\qquad n\neq 0$$

$$b_n=\frac{1}{p}\int_{d}^{d+2p} f(t)\sin\omega_n tdt=\frac{1}{3}\int_{-3}^{0}-t\sin\frac{n\pi}{3}tdt+\frac{1}{3}\int_{0}^{-3}t\sin\frac{n\pi}{3}tdt$$

$$=\frac{-1}{3}\left[\frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}t-\frac{3t}{n\pi}\cos\frac{n\pi}{3}t\right]_{-3}^{0}+\frac{1}{3}\left[\frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}t-\frac{3t}{n\pi}\cos\frac{n\pi}{3}t\right]_{0}^{3}$$

$$=\frac{3}{n\pi}\cos(-n\pi)-\frac{3}{n\pi}\cos(n\pi)=0$$

Substitute in Fourier series

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\omega_n t+b_n\sin\omega_n t)$$

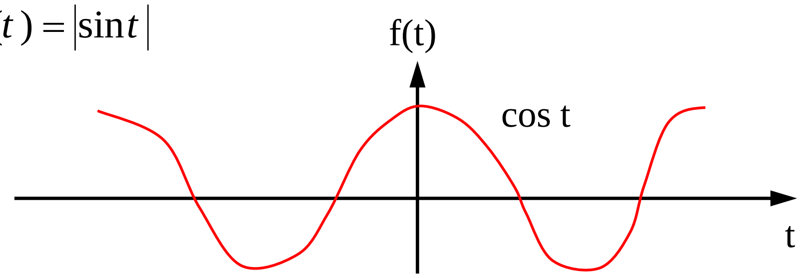
$$\therefore f(t)=\frac{3}{2}-\frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{3}t \qquad , \quad n: odd \quad only$$

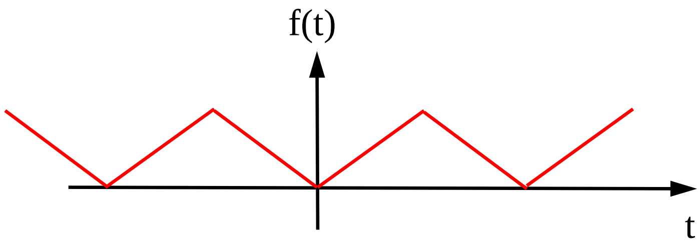
Or $$f(t)=\frac{3}{2}-\frac{12}{\pi^2}\left(\frac{1}{1}\cos\frac{\pi}{3}t+\frac{1}{9}\cos\frac{3\pi}{3}t+\frac{1}{25}\cos\frac{5\pi}{3}t+\cdots\cdots\right)$$

## Even and Odd functions

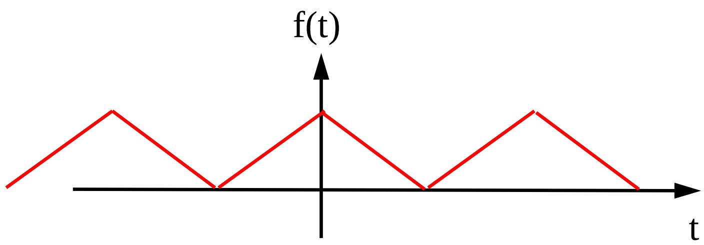
**Even function**: a function which has $f(t)=f(-t)$

Examples $\cos t,\ \sec t,\ \sin^2 t,\ f(t)=|\sin t|$



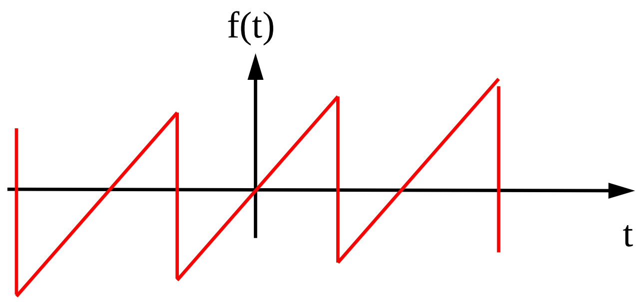

Example $f(t)=\begin{cases}-t & -3<t<0\\ t & 0<t<3\end{cases}$




Example $f(t)=\begin{cases}t+1 & -1<t<0\\ 1-t & 0<t<1\end{cases}$

**Odd function**: a function which has $f(t)=-f(-t)$

Example $\sin t,\ \tan t,\ \cot t$




**Example:** $f(t)=t \qquad -1<t<1$

**Notes:**

$$\begin{aligned} Even &\times Even = Even \\ Odd &\times Odd = Even \\ Even &\times Odd = Odd \end{aligned}$$

**For Even function:**

$$b_n = o$$

$$a_0 = \frac{2}{p}\int_{d}^{d+p} f(t)dt$$

$$a_n = \frac{2}{p}\int_{d}^{d+p} f(t)\cos\omega_n t\, dt$$

**For Odd function:**

$$a_n = o$$

$$b_n = \frac{2}{p}\int_{d}^{d+p} f(t)\sin\omega_n t\, dt$$

**Example:** Find the Fourier series for the following function defined in one period as;

$$f(t) = |t| \quad -1 < t < 1$$

$$T = 2,\ p = 1,\ \omega_n = \frac{n\pi}{p} = n\pi$$

$\therefore$ **Even function**

$$\therefore b_n = 0$$

$$a_n = \frac{2}{p}\int_d^{d+p} f(t)\cos\omega_n t\,dt = \frac{2}{1}\int_0^1 t\cos n\pi t\,dt$$

$$= 2\left[t\frac{\sin n\pi t}{n\pi} + \frac{\cos n\pi t}{n^2\pi^2}\right]_0^1 = 2\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$\therefore a_n = \begin{cases} -4/n^2\pi^2 & n: odd\ only \quad n \neq 0 \\ 0 & n: even \end{cases}$$

$$a_0 = \frac{2}{p}\int_d^{d+p} f(t)dt = \frac{2}{1}\int_0^1 t\,dt = 1$$

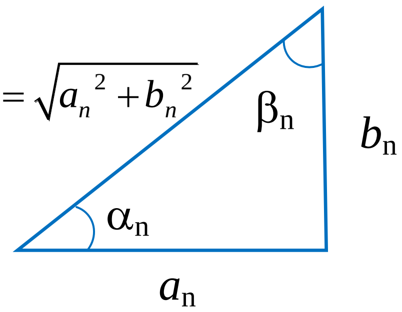
$$\therefore f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos n\pi t \qquad , \ n: odd \ \ only$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi t$$

**Alternative forms of Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n\left(\frac{a_n}{A_n}\cos\omega_n t + \frac{b_n}{A_n}\sin\omega_n t\right)$$

$$A_n = \sqrt{a_n^{\ 2} + b_n^{\ 2}}$$




$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n\left(\cos\alpha_n\cos\omega_n t + \sin\alpha_n\sin\omega_n t\right)$$

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\omega_n t - \alpha_n\right) \qquad \textbf{"Cosine series"}$$

Where $A_n = \sqrt{a_n^{\ 2} + b_n^{\ 2}}$ , $\alpha_n = \tan^{-1}\frac{b_n}{a_n}$ , $A_0 = \frac{a_0}{2}$

$A_n$ : Amplitude $\alpha_n$ : Phase angle for cosine series

**Or** equivalently:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n\left(\sin\beta_n\cos\omega_n t + \cos\beta_n\sin\omega_n t\right)$$

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(\omega_n t + \beta_n) \qquad \text{"Sine series"}$$

Where : $A_n = \sqrt{a_n^{\,2} + b_n^{\,2}}$ , $\beta_n = \tan^{-1}\dfrac{a_n}{b_n} \rightarrow Or \;\; \beta_n = \dfrac{\pi}{2} - \alpha_n$

$\beta_n$ : Phase angle for sine series

## Complex Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\omega_n t + b_n \sin\omega_n t$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\left(\frac{e^{i\omega_n t} + e^{-i\omega_n t}}{2}\right) + b_n\left(\frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2i}\right)$$

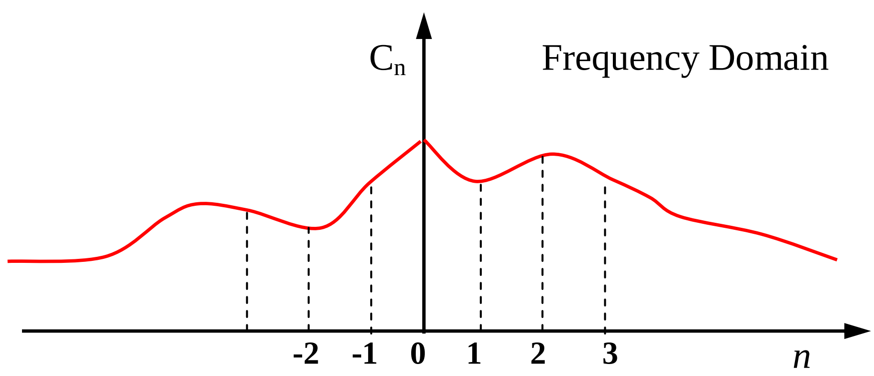
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{i\omega_n t} + \left(\frac{a_n + ib_n}{2}\right)e^{-i\omega_n t}$$

$$\therefore f(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{i\omega_n t} + C_{-n} e^{-i\omega_n t}$$

**where:** $C_0 = \dfrac{a_0}{2}$ , $C_n = \dfrac{a_n - ib_n}{2}$ , $C_{-n} = \dfrac{a_n + ib_n}{2}$ , $C_n = \overline{C_{-n}}$

Complex series can be rewritten as:

$$f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{i\omega_n t} \qquad \text{where:} \quad C_n = \frac{1}{2p}\int_{d}^{d+2p} f(t) e^{-i\omega_n t} dt$$



Proof:

- when n = 0

$$C_0 = \frac{a_0}{2} = \frac{1}{2p}\int_{d}^{d+2p} f(t)\,dt$$

- when n is positive

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{2}\left[\frac{1}{p}\int_{d}^{d+2p} f(t)\cos\omega_n t\,dt - i\,\frac{1}{p}\int_{d}^{d+2p} f(t)\sin\omega_n t\,dt\right]$$

$$= \frac{1}{2p}\int_{d}^{d+2p} f(t)\left(\cos\omega_n t - i\,\sin\omega_n t\right)dt$$

$$\therefore C_n = \frac{1}{2p}\int_{d}^{d+2p} f(t)e^{-i\omega_n t}\,dt$$

- when n is negative

$$C_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2}\left[\frac{1}{p}\int_{d}^{d+2p} f(t)\cos\omega_n t\,dt + i\,\frac{1}{p}\int_{d}^{d+2p} f(t)\sin\omega_n t\,dt\right]$$

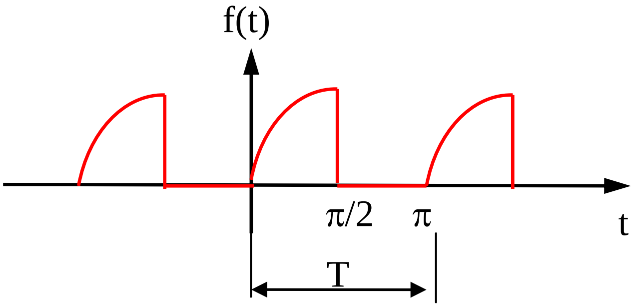
$$= \frac{1}{2p}\int_{d}^{d+2p} f(t)\left(\cos\omega_n t + i\sin\omega_n t\right)dt$$

$$\text{so far: } C_{-n} = \frac{1}{2p}\int_{d}^{d+2p} f(t)e^{i\omega_n t}\,dt$$

**Note:** $A_n = 2|C_n|$ , $\alpha_n = -Arg\,(C_n)$

**Example:** Find the complex Fourier series for the following function defined in one period.

$$f(t) = \begin{cases} \sin t & 0 < t < \pi/2 \\ 0 & \pi/2 < t < \pi \end{cases}$$




$$T = \pi,\ \ p = \frac{\pi}{2},\ \ \omega_n = \frac{n\pi}{p} = 2n$$

$$C_n = \frac{1}{2p}\int_{d}^{d+T} f(t)\, e^{-i\omega_n t}dt$$

$$= \frac{1}{\pi}\int_{0}^{\pi/2} \sin t\, e^{-i\omega_n t}dt + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0 * e^{-i\omega_n t}dt$$

$$= \frac{1}{\pi}\int_{0}^{\pi/2}\left(\frac{e^{it}-e^{-it}}{2i}\right)e^{-i\omega_n t}dt = \frac{1}{2\pi i}\int_{0}^{\pi/2}\left(e^{i(1-2n)t} - e^{-i(1+2n)t}\right)dt$$

$$C_n = \frac{1}{2\pi i}\left[\frac{e^{i(1-2n)t}}{i(1-2n)} + \frac{e^{-i(1+2n)t}}{i(1+2n)}\right]_0^{\pi/2}$$

$$= \frac{-1}{2\pi}\left\{\left[\frac{e^{i(1-2n)\pi/2}}{(1-2n)} + \frac{e^{-i(1+2n)\pi/2}}{(1+2n)}\right] - \left[\frac{1}{(1-2n)} + \frac{1}{(1+2n)}\right]\right\}$$

$$= \frac{-1}{2\pi}\left[\frac{i\cos n\pi}{(1-2n)} + \frac{-i\cos n\pi}{(1+2n)} - \frac{2}{(1-4n^2)}\right]$$

Note:

$$e^{i(1-2n)\pi/2} = e^{i\left(\frac{\pi}{2}-n\pi\right)} = e^{i\frac{\pi}{2}}.e^{-in\pi}$$

$$= \left(\cos(\pi/2)^{=0} + i\,\sin(\pi/2)^{=1}\right)\left(\cos n\pi - i\,\sin n\pi^{=0}\right)$$

$$= i\,\cos n\pi$$

$$= \frac{-1}{2\pi}\left[\frac{i4n\cos n\pi}{1-4n^2} - \frac{2}{1-4n^2}\right]$$

$$\therefore C_n = \frac{-1}{2\pi}\left[\frac{i4n(-1)^n - 2}{1-4n^2}\right]$$

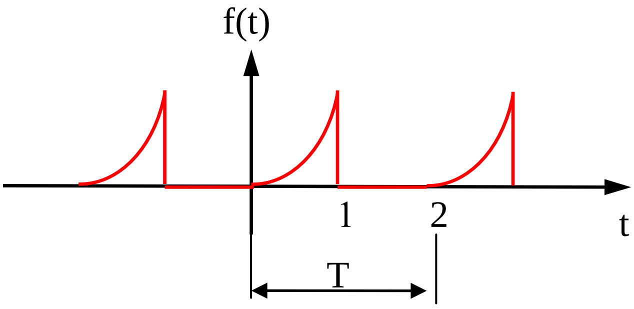
$$\therefore f(t) = \sum_{n=-\infty}^{n=\infty} C_n\, e^{-i\omega_n t} = \sum_{n=-\infty}^{n=\infty} \frac{-1}{2\pi}\left[\frac{i4n(-1)^n - 2}{1-4n^2}\right]e^{-i\omega_n t}$$

**For Check at n = 0**

$$C_0 = \frac{1}{\pi} \qquad ,C_0 = \frac{a_0}{2} = \frac{1}{2p}\int_{d}^{d+T} f(t)dt$$

$$C_0 = \frac{1}{\pi}\int_{0}^{\pi/2}\sin t\, dt = \frac{-1}{\pi}[\cos t]_0^{\pi/2} = \frac{1}{\pi}$$

**Example:** Find the complex Fourier series for the following function defined in one period.




$$f(t)=\begin{cases} t^2 & 0<t<1 \\ 0 & 1<t<2 \end{cases}$$

$$T=2,\ p=1,\ \omega_n=\frac{n\pi}{p}=n\pi$$

$$C_n=\frac{1}{2p}\int_d^{d+T} f(t)e^{-i\omega_n t}dt=\frac{1}{2}\int_0^1 t^2 e^{-in\pi t}dt+0$$

$$=\frac{1}{2}\left[t^2\frac{e^{-in\pi t}}{-in\pi}-2t\frac{e^{-in\pi t}}{(-in\pi)^2}+2\frac{e^{-in\pi t}}{(-in\pi)^3}\right]_0^1=\frac{1}{2}\left[t^2\frac{e^{-in\pi t}}{-in\pi}+2t\frac{e^{-in\pi t}}{n^2\pi^2}+2\frac{e^{-in\pi t}}{in^3\pi^3}\right]_0^1$$

$$=\frac{1}{2n\pi}\left\{\left[ie^{-in\pi}+2\frac{e^{-in\pi}}{n\pi}-2i\frac{e^{-in\pi}}{n^2\pi^2}\right]-\left(0+0-\frac{i2}{n^2\pi^2}\right)\right\}$$

$$=\frac{1}{2n\pi}\left\{\left[i\cos n\pi+\frac{2\cos n\pi}{n\pi}-\frac{i2\cos n\pi}{n^2\pi^2}\right]+\frac{i2}{n^2\pi^2}\right\}$$

$$\therefore C_n=\frac{1}{2n\pi}\left\{\cos n\pi\left(i+\frac{2}{n\pi}-\frac{2i}{n^2\pi^2}\right)+\frac{i2}{n^2\pi^2}\right\}$$

Note: $e^{-in\pi}=\left(\cos n\pi-i\sin n\pi^{=0}\right)=\cos n\pi$

$$\therefore f(t)=\sum_{n=-\infty}^{n=\infty}C_n e^{-i\omega_n t}=\sum_{n=-\infty}^{n=\infty}\frac{1}{2n\pi}\left\{\cos\left(i+\frac{2}{n\pi}-\frac{2i}{n^2\pi^2}\right)+\frac{i2}{n^2\pi^2}\right\}e^{-i\omega_n t}$$

**When n = 0, $C_0$ must be calculated individually:**

$$C_0=\frac{1}{2p}\int_d^{d+T} f(t)dt\ \frac{1}{2}\int_0^1 t^2 dt=\frac{1}{6}$$

**Hint:** Continuous integration by parts. For example

Find $\int t^2\cos\omega_n t dt$

| $\frac{d}{dt}$ | $\int dt$ | product | sign |
|---|---|---|---|
| $t^2$ | $\cos\omega_n t$ | | |
| $2t$ | $\frac{\sin\omega_n t}{\omega_n}$ | $t^2\,\frac{\sin\omega_n t}{\omega_n}$ | $(+)$ |
| $2$ | $\frac{-\cos\omega_n t}{\omega_n^2}$ | $2t\,\frac{-\cos\omega_n t}{\omega_n^2}$ | $(-)$ |
| | $\frac{-\sin\omega_n t}{\omega_n^3}$ | $2\times\frac{-\sin\omega_n t}{\omega_n^3}$ | $(+)$ |

$$so\ far: \int t^2\cos\omega_n t\,dt = t^2\,\frac{\sin\omega_n t}{\omega_n} + \frac{2t\cos\omega_n t}{\omega_n^2} - \frac{2\sin\omega_n t}{\omega_n^3}$$

# Fourier Transform

From the Complex Fourier series;

$$f(t)=\sum_{n=-\infty}^{n=\infty} C_n e^{i\omega_n t} \qquad \text{where:} \quad C_n=\frac{1}{2p}\int_d^{d+2p} f(t)e^{-i\omega_n t}dt$$

$$f(t)=\sum_{n=-\infty}^{n=\infty}\left[\frac{1}{2p}\int_d^{d+2p} f(t)e^{-i\omega_n t}dt\right]e^{i\omega_n t} \qquad \textbf{... (1)}$$

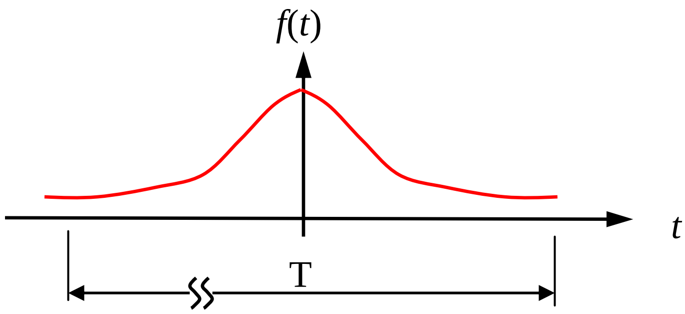
$$At \quad T=-\infty \to \infty$$

$$p \to \infty$$

$$d+2p \to \infty$$

$$\Delta\omega_n=\omega_{n+1}-\omega_n$$

$$=\frac{\pi(n+1)}{p}-\frac{\pi n}{p}=\frac{\pi}{p}$$




$$T=\infty,\ p=\infty,\ \omega_n=\frac{n\pi}{p}$$

Multiply and divided eq.(1) by $\Delta\omega_n$

$$f(t)=\sum_{-\infty}^{\infty}\left[\frac{1}{2p}\int_d^{d+2p} f(t)e^{-i\omega_n t}dt\right]e^{i\omega_n t}\ .\frac{\Delta\omega_n}{\Delta\omega_n}$$

$$=\sum_{-\infty}^{\infty}\frac{\not{p}}{2\not{p}\pi}\int_d^{d+2p} f(t)e^{-i\omega_n t}dt.e^{i\omega_n t}\ .\Delta\omega_n$$

$$as\ p\to\infty,\ \Delta\omega\to 0$$

$$then \quad \Delta\omega \to dw\ \ and\ \sum \ \to \int$$

$$also:\ \omega_n=\omega$$

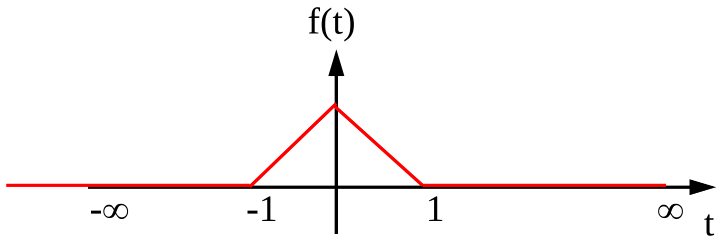
$$f(t)=\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right].e^{i\omega t}.d\omega$$

$$f(t)=\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}d\omega \quad , \quad G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

**"Fourier Integral Pair"**

**Example:** Find Fourier integral for the following function**:**

$$f(t)=\begin{cases}1+t & -1<t<0\\ 0 & otherwise\\ 1-t & 0<t<1\end{cases}$$




$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{-1}0.dt+\int_{-1}^{0}(1+t)e^{-i\omega t}dt+\int_{0}^{1}(1-t)e^{-i\omega t}dt+\int_{1}^{\infty}0.dt+\right]$$

$$=\frac{1}{2\pi}\left[\left(\frac{e^{-i\omega t}}{-i\omega}+\frac{te^{-i\omega t}}{-i\omega}-\frac{e^{-i\omega t}}{i^2\omega^2}\right)_{-1}^{0}+\left(\frac{e^{-i\omega t}}{-i\omega}-\frac{te^{-i\omega t}}{-i\omega}-\frac{e^{-i\omega t}}{-i^2\omega^2}\right)_{0}^{1}\right]$$

$$=\frac{1}{2\pi}\left[\left(\frac{1}{-i\omega}+\frac{0}{-i\omega}+\frac{1}{\omega^2}\right)-\left(\frac{e^{i\omega}}{-i\omega}+\frac{e^{i\omega}}{i\omega}+\frac{e^{i\omega}}{\omega^2}\right)\right.$$

$$\left.+\left(\frac{e^{-i\omega}}{-i\omega}+\frac{e^{-i\omega}}{i\omega}-\frac{e^{i\omega}}{\omega^2}\right)-\left(\frac{1}{-i\omega}+\frac{0}{i\omega}-\frac{1}{\omega^2}\right)\right]$$

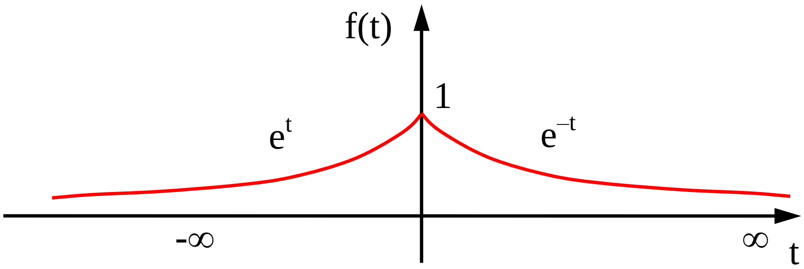
$$=\frac{1}{2\pi}\left[\frac{2}{\omega^2}+\cancel{\frac{e^{i\omega}}{i\omega}}-\cancel{\frac{e^{i\omega}}{i\omega}}-\frac{e^{i\omega}}{\omega^2}-\cancel{\frac{e^{-i\omega}}{i\omega}}+\cancel{\frac{e^{-i\omega}}{\omega}}-\frac{e^{-i\omega}}{\omega^2}\right]$$

$$G(\omega)=\frac{1}{2\pi}\left[\frac{2}{\omega^2}-\frac{2\cos\omega}{\omega^2}\right]=\frac{1}{\pi\omega^2}(1-\cos\omega)$$

$$f(t)=\int_{-\infty}^{\infty}G(\omega).e^{i\omega t}d\omega=\int_{-\infty}^{\infty}\frac{1}{\pi\omega^2}(1-\cos\omega).e^{i\omega t}d\omega$$

**Example:** Find Fourier integral for the following function: $f(t) = e^{-|t|}$

$$\therefore f(t) = \begin{cases} e^{t} & t < 0 \\ e^{-t} & t > 0 \end{cases}$$




$$f(t) = \int_{-\infty}^{\infty} G(\omega).e^{i\omega t} d\omega$$

$$G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t).e^{i\omega t} dt$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{t}.e^{-i\omega t} dt + \int_{0}^{\infty} e^{-t}.e^{-i\omega t} dt\right]$$

$$= \frac{1}{2\pi}\left[\left.\frac{e^{(1-i\omega)t}}{1-i\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right|_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-i\omega}(1-0) - \frac{1}{1+i\omega}(0-1)\right] = \frac{1}{2\pi}\left[\frac{1}{1-i\omega} + \frac{1}{1+i\omega}\right]$$

$$G(\omega) = \frac{1}{2\pi}\left[\frac{1+i\omega+1-i\omega}{1+\omega^2}\right] = \frac{1}{\pi\left(1+\omega^2\right)}$$

**Example:** Find the inverse Fourier integral for the following function:

$$G(\omega) = \begin{cases} \cos\omega & 0 < \omega < \pi/2 \\ 0 & otherwise \end{cases}$$

$$f(t) = \int_{-\infty}^{\infty} G(\omega).e^{i\omega t} d\omega$$

$$f(t) = \left[ \int_{-\infty}^{0} 0.e^{i\omega t} d\omega + \int_{0}^{\pi/2} \cos\omega . e^{i\omega t} d\omega + \int_{\pi/2}^{\infty} 0.e^{i\omega t} d\omega \right]$$

$$= \int_{0}^{\pi/2} \frac{e^{i\omega} + e^{-i\omega}}{2} . e^{i\omega t} d\omega$$

$$= \frac{1}{2} \int_{0}^{\pi/2} \left( e^{i\omega(t+1)} + e^{i\omega(t-1)} \right) d\omega$$

$$= \frac{1}{2} \left[ \frac{e^{i\omega(t+1)}}{t+1} + \frac{e^{i\omega(t-1)}}{t-1} \right]_{0}^{\pi/2} = \frac{1}{2} \left[ \frac{e^{i\pi(t+1)/2}}{t+1} + \frac{e^{i\pi(t-1)/2}}{t-1} - \left( \frac{1}{t+1} + \frac{1}{t-1} \right) \right]$$

$$= \frac{1}{2} \left[ \frac{e^{i\pi t/2} e^{i\pi/2}}{t+1} + \frac{e^{i\pi t/2} e^{-i\pi/2}}{t-1} - \left( \frac{2t}{t^2 - 1} \right) \right]$$

$$= \frac{1}{2} \left[ \frac{ie^{i\pi t/2}}{t+1} - \frac{ie^{i\pi t/2}}{t-1} - \left( \frac{2t}{t^2 - 1} \right) \right]$$

$$= \frac{1}{2} \left[ \frac{i2e^{i\pi t/2}}{t^2 - 1} - \left( \frac{2t}{t^2 - 1} \right) \right]$$

$$f(t) = \frac{1}{t^2 - 1} \left[ ie^{i\pi t/2} - t \right]$$