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Engineering Analysis

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Partial Differential Equations (PDEs)

The partial differential equation contains one or more partial derivative of a function with respect to more than one independent variable.

Examples:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$: Laplace eq. in three dimensions

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$: Wave equation is 1-dimension

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$: Diffuse equation (heat equation) in 1-dimension

u: dependent variable
x, y, z, t : Independent variables
Dimension of the equation: number of spatial independent variables.

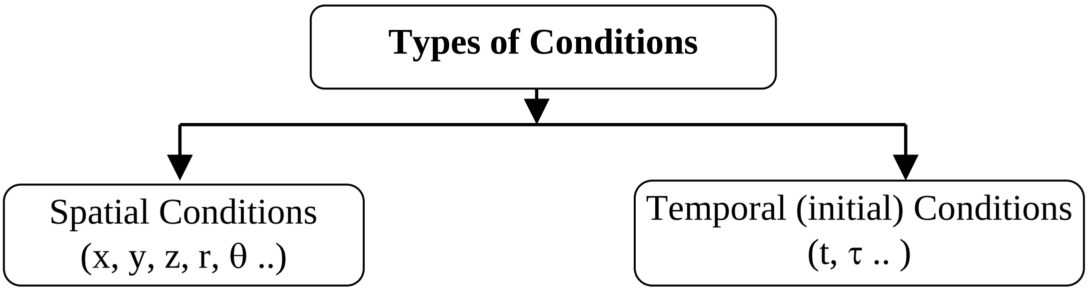
Types of Variables

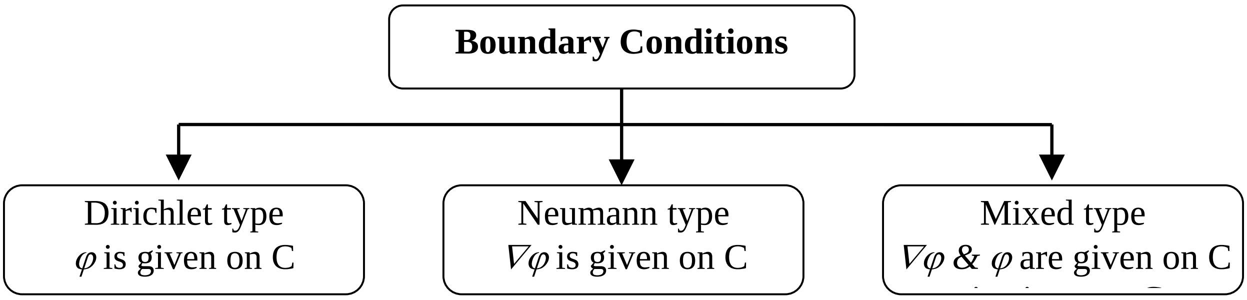
Spatial: x ,y ,z, r , θ (describe spaces or dimension)
Temporal: t, τ (describe time)

Short form:

$$\frac{\partial u}{\partial x}=u_x \quad , \quad \frac{\partial^2 u}{\partial x^2}=u_{xx} \qquad \Rightarrow example: u_{xx}+u_{yy}+u_{zz}=0$$

$$Or \quad \frac{\partial u}{\partial x}=D_x u \quad , \quad where\ D_x=\frac{\partial}{\partial x}$$

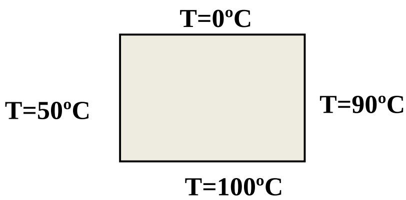


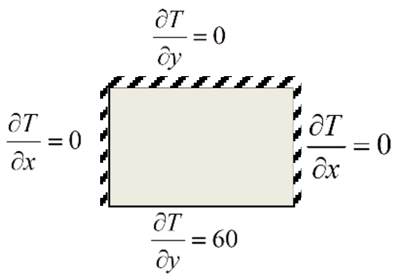


Examples

1. Dirichlet type

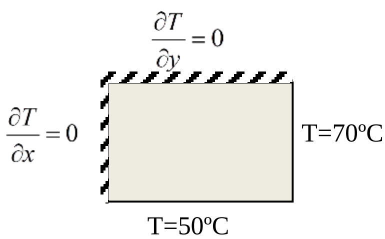
T is given on C





2. Neumann type

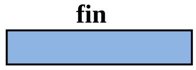
∇T is given in C



3. Mixed type

T & ∇T is given in C

Types of PDEs



1. Heat diffusion equation

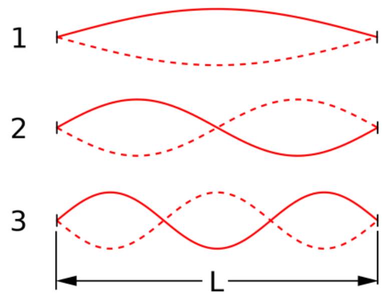
$$\frac{1}{\alpha}u_t = u_{xx}$$: heat diffuse equation in one dimension

Where; $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ and α : diffusivity

$$\frac{1}{\alpha}u_t = u_{xx} + u_{yy}$$: heat diffuse equation in two dimension

$$\frac{1}{\alpha}u_t = u_{xx} + u_{yy} + u_{zz}$$:heat diffuse equation in three dimension

2. Wave equation



$$\frac{1}{c^2}u_{tt} = u_{xx}$$: wave equation in 1D

where c is the speed of wave in the material

$$\frac{1}{c^2}u_{tt} = u_{xx} + u_{yy}$$: wave equation in 2D

$$\frac{1}{c^2}u_{tt} = u_{xx} + u_{yy} + u_{zz}$$: wave equation in 3D

3. Laplace equation

$u_{xx} + u_{yy} = 0$ Laplace equation in 2D

$u_{xx} + u_{yy} + u_{zz} = 0$ Laplace equation in 3D

Or $\nabla^2 u = 0$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

4. Poisson's equation

$u_{xx} + u_{yy} = f(x, y)$ Poisson's equation in 2D

$u_{xx} + u_{yy} + u_{zz} = f(x, y)$ Or $\nabla^2 u = f(x, y)$ Poisson's eq. in 3D

Conic Section Classification

Used for 2nd order PDE with two independent variables;

$$A(x,y)\phi_{xx} + B(x,y)\phi_{xy} + C(x,y)\phi_{yy} = F\left(x, y, \phi_x, \phi_y\right)$$

x, *y* are any two independent variables (Spatial or Temporal)

$$Z = B^2 - 4AC \begin{cases} < 0 & elliptic \quad eq. \\ = 0 & parabolic \quad eq. \\ > 0 & hyperbolic \quad eq. \end{cases}$$ Z:discrimination factor

Examples

Laplace equation in 2-Dim:

$$\phi_{xx} + \phi_{yy} = 0 \qquad \therefore A = 1, \quad B = 0, \quad C = 1$$

$$Z = B^2 - 4AC = 0 - 4\times1\times1 = -4 < 0 \quad \Rightarrow \quad elliptic \quad eq.$$

Poisson's eq. in 2-Dim: $\phi_{xx} + \phi_{yy} = f(x,y)$ elliptic eq.

Diffusion equation in 1-Dim:

$$\phi_{xx} = \frac{1}{\alpha}\phi_t \qquad \therefore A = 1, \quad B = 0, \quad C = 0$$

$$Z = B^2 - 4AC = 0 - 4\times1\times0 = 0 \qquad parabolic \quad eq.$$

Wave equation in 1-Dim:

$$u_{xx} - \frac{1}{c^2}u_{tt} = 0 \qquad \therefore A = 1, \quad B = 0, \quad C = \frac{-1}{c^2}$$

$$Z = B^2 - 4AC = 0 - 4\times1\times\left(\frac{-1}{c^2}\right) = \frac{4}{c^2} > 0 \qquad hyperbolic \quad eq.$$

Wave equation in 2-Dim:

$$\frac{1}{c^2}u_{tt} = u_{xx} + u_{yy}$$

elliptic equation in (x, y) & hyperbolic eq. in (x, t) and (y , t) coordinates.

Notes:

1. Elliptic equations usually describe **steady** state problem and require spatial B.C. only
2. Parabolic & hyperbolic equations represent **unsteady** state conditions, therefore require both spatial and initial conditions.

Solutions of PDEs

Note: The arbitrary parameters in the solution of PDE are function.

Separation of Variables

(1) Wave equation in 1-Dim

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

Assume $u(x,t) = X(x).G(t)$

$$u_{xx} = X''(x).G(t)$$
$$u_{tt} = X(x).\ddot{G}(t)$$

Substitute in the diff. eq.

$$X''(x).G(t) = \frac{1}{c^2} X(x).\ddot{G}(t) \qquad \text{dividing by } X(x).G(t)$$

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)}$$

The left side is a function of *x* only, meanwhile the right side is a function of *t* only. Therefore, each side of equation must be a constant;

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)} = \text{constant}\begin{cases} \lambda^2 & \textit{positive con.} \\ 0 & \textit{zero con.} \\ -\lambda^2 & \textit{negative con.} \end{cases}$$

Note:

1) $if \ (D-m)y = 0 \quad \Rightarrow \ y = Ce^{mx}$

2) $if \ (D-m_1)(D-m_2)y = 0 \quad \Rightarrow \ \therefore y = C_1e^{m_1x} + C_2e^{m_2x}$

(i) Taking positive constant

$$\frac{X''(x)}{X(x)} = \lambda^2 \qquad \Rightarrow X''(x) - \lambda^2 X(x) = 0 \quad \to \quad (D^2 - \lambda^2)X = 0$$

$$(D-\lambda)\left(D+\lambda\right)X = 0 \qquad \Rightarrow \; \therefore X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

$$\frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)} = \lambda^2 \qquad \Rightarrow \ddot{G}(t) - \lambda^2 c^2 G(t) = 0 \quad \to \quad (D^2 - \lambda^2 c^2)G = 0$$

$$(D-\lambda c)\left(D+\lambda c\right)G = 0 \qquad \Rightarrow \; \therefore G(t) = C_3 e^{\lambda ct} + C_4 e^{-\lambda ct}$$

But a solution of the form

$$u(x,t) = X(x).G(t) = \left(C_1 e^{\lambda x} + C_2 e^{-\lambda x}\right)\left(C_3 e^{\lambda ct} + C_4 e^{-\lambda ct}\right)$$

Cannot describe a periodic motion. Hence, the positive constant must be rejected.

(ii) Taking zero constant

$$\frac{X''(x)}{X(x)} = 0 \qquad \Rightarrow \; X''(x) = 0 \quad \to \quad \therefore X(x) = Ax + B$$

$$\frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)} = 0 \qquad \Rightarrow \ddot{G}(t) = 0 \quad \to \quad \therefore G(t) = Et + F$$

But a solution of the form

$$u(x,t) = X(x).T(t) = \left(Ax + B\right)\left(Et + F\right)$$

Cannot describe a periodic motion. Hence, the zero constant must be rejected.

Finally:

(ii) Taking negative constant

$$\frac{X''(x)}{X(x)} = -\lambda^2 \qquad \Rightarrow X''(x) + \lambda^2 X(x) = 0 \quad \rightarrow \; (D^2 + \lambda^2)X = 0$$

$$(D - i\lambda)\left(D + i\lambda\right)X = 0 \qquad \Rightarrow \; \therefore X(x) = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$$

$$X(x) = c_1\left(\cos\lambda x + i\sin\lambda x\right) + c_2\left(\cos\lambda x - i\sin\lambda x\right)$$

$$= \left(c_1 + c_2\right)^{=A}\cos\lambda x + \; i\left(c_1 - c_2\right)^{=B}\sin\lambda x$$

$$\therefore X(x) = A\cos\lambda x + B\sin\lambda x$$

$$\frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)} = -\lambda^2 \qquad \Rightarrow \ddot{G}(t) + \lambda^2 c^2 G(t) = 0 \;\; \Rightarrow \; (D^2 + \lambda^2 c^2)G = 0$$

$$(D - i\lambda c)\left(D + i\lambda c\right)G(t) = 0 \qquad \Rightarrow \; G(t) = c_3 e^{i\lambda ct} + c_4 e^{-i\lambda ct}$$

$$\therefore G(t) = C\cos\lambda ct + D\sin\lambda ct$$

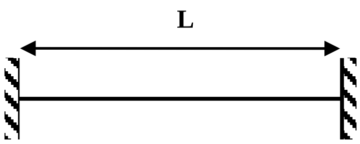
$$or \quad \therefore G(t) = C\cos kt + D\sin kt \qquad\qquad where \;\; k = \lambda c$$

In this case the solution

$$u(x,t) = X(x).G(t) = \left(A\cos\lambda x + B\sin\lambda x\right)\left(C\cos\lambda ct + D\sin\lambda ct\right)$$

is clearly periodic

To apply the boundary conditions:



B.C. $\quad u(0,t) = u(L,t) = 0$

$$u(x,t) = X(x).G(t) = \left(A\cos\lambda x + B\sin\lambda x\right)\left(C\cos\lambda ct + D\sin\lambda ct\right)$$

(1) apply the first B.C.: $u(0,t) = 0$

$$u(0,t)=0=\left(A\times 1+B\times 0\right)\left(C\cos\lambda ct+D\sin\lambda ct\right)$$
$$u(0,t)=0=A\left(C\cos\lambda ct+D\sin\lambda ct\right)$$

$but\ G(t)\neq 0$ since this will give trivial solution

$$\rightarrow \quad A=0$$

$$u(x,t)=B\sin\lambda x\left(C\cos\lambda ct+D\sin\lambda ct\right)$$

$$\therefore u(x,t)=\sin\lambda x\left(C^{*}\cos\lambda ct+D^{*}\sin\lambda ct\right)$$

where $C^{*}=B.C\ \ and\ \ D^{*}=B.D$

(2) apply the second B.C.: $u(L,t)=0$

$$\therefore u(l,t)=0=\sin\lambda l\left(C^{*}\cos\lambda ct+D^{*}\sin\lambda ct\right)$$

but $G(t)\neq 0$ since this will give trivial solution

$$\rightarrow \quad \sin\lambda L=0$$

$$\Rightarrow\ \lambda L=n\pi \qquad \rightarrow \lambda=\frac{n\pi}{L}$$

$$\Rightarrow \lambda_n=\frac{n\pi}{L} \qquad ,where\ n=1,\ 2,\ 3,\cdots$$

$$\therefore u_n(x,t)=\sin\lambda_n x\left(C_n^{\ *}\cos\lambda_n ct+D_n^{\ *}\sin\lambda_n ct\right)$$
$$=\sin\frac{n\pi x}{L}\left(C_n^{\ *}\cos\frac{n\pi ct}{L}+D_n^{\ *}\sin\frac{n\pi ct}{L}\right)$$

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}\sin\lambda_n x\left(C_n^{\ *}\cos\lambda_n ct+D_n^{\ *}\sin\lambda_n ct\right)}$$

Applying the initial conditions: $u(x,0)=f(x) \quad and \quad u_t(x,0)=v(x)$

(3) Apply the initial displacement condition: $u(x,0)=f(x)$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \left(C_n^* \times 1 + D_n^* \times 0\right) \sin\frac{n\pi x}{L}$$

$$\therefore\ f(x) = \sum_{n=1}^{\infty} C_n^* \sin\frac{n\pi x}{L} \qquad\qquad \omega_n = \frac{n\pi}{p}$$

The above expansion is a Fourier series expansion with periodic function of period 2*L (or p = L)*. Applying the half-range sine series will give:

$$\boxed{C_n^* = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx}$$

(4) Apply the initial velocity condition: $u_t(x,0) = v(x)$

$$u_t(x,t) = \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \left(-C_n^* \sin\frac{n\pi ct}{L} + D_n^* \cos\frac{n\pi ct}{L}\right) \frac{n\pi c}{L}$$

$$u_t(x,0) = v(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \left(-C_n^* \times 0 + D_n^* \times 1\right) \frac{n\pi c}{L}$$

$$v(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L} D_n^*\right) \sin\frac{n\pi x}{L}$$

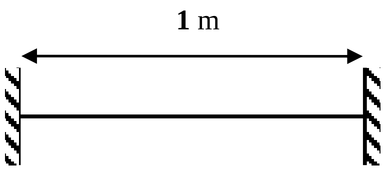
Apply the half-range sine series will give:

$$\frac{n\pi c}{L} D_n^* = \frac{2}{L}\int_0^L v(x) \sin\frac{n\pi x}{L} dx$$

$$\boxed{D_n^* = \frac{2}{n\pi c}\int_0^L v(x) \sin\frac{n\pi x}{L} dx}$$

<u>Example-1</u>

Find the subsequent motion for a string of length 1 m fixed at both ends and subjected to the following initial conditions, given that c = 2 m/s:



$$f(x)=\begin{cases}0 & 0<x<0.25\\ 2x-0.5 & 0.25<x<0.5\\ 0 & 0.5<x<1\end{cases}$$

$$v(x)=\begin{cases}1 & 0.25<x<0.5\\ 0 & otherwise\end{cases}$$

Solution:

$$\lambda_n=\frac{n\pi}{L}=n\pi$$

$$C_n^*=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx=2\int_{0.25}^{0.5}(2x-0.5)\sin n\pi x dx$$

$$=4\left[\frac{-x\cos n\pi x}{n\pi}+\frac{\sin n\pi x}{n^2\pi^2}+\frac{0.25\cos n\pi x}{n\pi}\right]_{0.25}^{0.5}$$

$$=\frac{4}{n\pi}\left[\left(-0.5\cos 0.5n\pi+\frac{\sin 0.5n\pi}{n\ \pi}+0.25\cos 0.5n\pi\right)-\right.$$

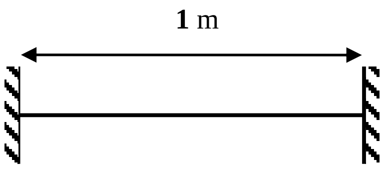
$$\left.\left(-0.25\cos 0.25n\pi+\frac{\sin 0.25n\pi}{n\ \pi}+0.25\cos 0.25n\pi\right)\right]$$

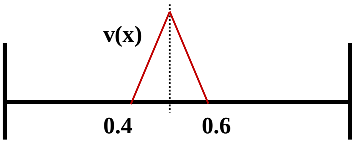
$$D_n^*=\frac{2}{n\pi c}\int_0^L v(x)\sin\frac{n\pi x}{L}dx=\frac{1}{n\pi}\int_{0.25}^{0.5}1\times\sin n\pi x dx=\frac{1}{n\pi}\left[\frac{-\cos n\pi x}{n\pi}\right]_{0.25}^{0.5}$$

$$=\frac{-1}{n^2\pi^2}\left[\cos\frac{n\pi}{2}-\cos\frac{n\pi}{4}\right]$$

Example-2

A string of length 1 meter is fixed at both ends and subjected to zero initial displacement and to the following initial velocity, determine its subsequent motion using separation of variables method.





$$v(x) = \begin{cases} 10\left(0.1 - |x - 0.5|\right) & 0.4 \le x \le 0.6 \\ 0 & elsewhere \end{cases}$$

Solution:

v(x) , for the sake of integration, can be re-written as:

$$v(x) = \begin{cases} 0 & 0 \le x < 0.4 \\ 10x - 4 & 0.4 \le x < 0.5 \\ 6 - 10x & 0.5 \le x \le 0.6 \\ 0 & 0.6 < x \le 1.0 \end{cases}$$

The solution is given by; $u(x,t) = \sum_{n=1}^{\infty} \sin \lambda_n x \left(C_n \cos \lambda_n ct + D_n \sin \lambda_n ct\right)$

$$where\ \lambda_n = \frac{n\pi}{L} = n\pi, \Rightarrow\ c\lambda_n = n\pi c$$

$$C_n = \frac{2}{L}\int_0^l h(x) \sin \lambda_n x\, dx \quad and \quad D_n = \frac{2}{n\pi c}\int_0^l v(x) \sin \lambda_n x\, dx$$

Therefore; $C_n = 0$

$$D_n = \frac{2}{n\pi c}\int_0^1 v(x) \sin n\pi x\, dx$$

$$= \frac{2}{n\pi c}\left[\int_{0.4}^{0.5} (10x - 4) \sin n\pi x\, dx + \int_{0.5}^{0.6} (6 - 10x) \sin n\pi x\, dx\right]$$

$$= \frac{2}{n\pi c}\left[\frac{-10x \cos n\pi x}{n\pi} + \frac{10 \sin n\pi x}{n^2\pi^2} + \frac{4 \cos n\pi x}{n\pi}\right]_{0.4}^{0.5}$$

$$+ \frac{2}{n\pi c}\left[\frac{-6 \cos n\pi x}{n\pi} + \frac{10x \cos n\pi x}{n\pi} - \frac{10 \sin n\pi x}{n^2\pi^2}\right]_{0.5}^{0.6}$$

$$=\frac{2}{n^2\pi^2 c}\left[(4-10x)\cos n\pi x+\frac{10\sin n\pi x}{n\pi}\right]_{0.4}^{0.5}+$$

$$\frac{2}{n^2\pi^2 c}\left[(10x-6)\cos n\pi x-\frac{10\sin n\pi x}{n\pi}\right]_{0.5}^{0.6}$$

$$=\frac{2}{n^2\pi^2 c}\left[-\cos 0.5n\pi+\frac{10\sin 0.5n\pi}{n\pi}-\frac{10\sin 0.4n\pi}{n\pi}\right]+$$

$$\frac{2}{n^2\pi^2 c}\left[-\frac{10\sin 0.6n\pi}{n\pi}-\cos 0.5n\pi+\frac{10\sin 0.5n\pi}{n\pi}\right]$$

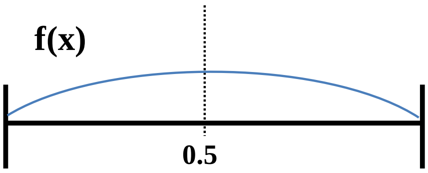
$$=\frac{2}{n^2\pi^2 c}\left[-2\cos 0.5n\pi+\frac{10}{n\pi}\left(2\sin 0.5n\pi-\sin 0.4n\pi-\sin 0.6n\pi\right)\right]$$

So, $u(x,t)=\sum_{n=1}^{\infty} D_n \sin n\pi ct \sin n\pi x$

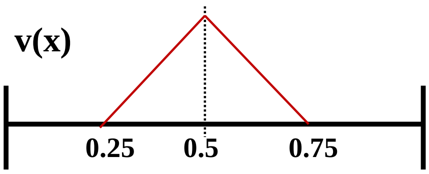
Example-3

A string of length (L) is fixed at both ends, write the equation of motion for the string and describe it. Solve the equation using separation of variables method to find $u(x,t)$. If the string length is 1 m and it is subjected to the following initial conditions, find the subsequent motion $u(x,t)$. Use c = 5 m/s

$$f(x)=\sin \pi x \qquad 0\le x\le 1$$



$$v(x)=\begin{cases}1-4\left|x-0.5\right| & \left|x-0.5\right|<0.25\\ 0 & Otherwise\end{cases}$$



Solution:

Starting with

$$u_{xx}=\frac{1}{c^2}u_{tt}$$

Assume $u(x,t) = X(x).G(t)$

$$u_{xx} = X''(x).G(t)$$

$$u_{tt} = X(x).\ddot{G}(t)$$

Substitute in the diff. eq.

$$X''(x).G(t) = \frac{1}{c^2} X(x).\ddot{G}(t)$$ dividing by $X(x).G(t)$

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)}$$

The left side is a function of *x*, meanwhile the right side is a function of t only.

Therefore, each side of equation must be a constant;

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)} = -\lambda^2$$

And continue …. Till we get:

$C_n^* = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$
$D_n^* = \frac{2}{n\pi c}\int_0^L v(x)\sin\frac{n\pi x}{L}dx$

Applying initial displacement:

$$C_n^* = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = 2\int_0^1 \sin\pi x.\sin n\pi x dx$$

This will give zero for all values of *n* except when *n* = 1:

$$C_1^* = 2\int_0^1 \sin \pi x . \sin \pi x dx = 2\int_0^1 \sin^2 \pi x dx$$

$$= 2\int_0^1 \frac{1-\cos 2\pi x}{2} dx = \left[x - \frac{\overbrace{(\sin 2\pi x)}^{=0}}{2\pi} \right]_0^1$$

$$= \quad 1$$

Applying initial velocity:

$$v(x) = \begin{cases} 0 & 0 < x < 0.25 \\ 4x - 1 & 0.25 \le x \le 0.5 \\ 3 - 4x & 0.5 < x \le 0.75 \\ 0 & 0.75 < x < 1 \end{cases}$$

$$D_n^* = \frac{2}{n\pi c} \int_0^L v(x) \sin \frac{n\pi x}{L} dx =$$

$$\frac{1}{5n\pi} \left[\int_{0.25}^{0.5} (4x-1) \sin n\pi x dx + \int_{0.5}^{0.75} (3-4x) \sin n\pi x dx \right]$$

$$= \frac{1}{5n\pi} \left[\frac{-4x \cos n\pi x}{n\pi} + \frac{4 \sin n\pi x}{n^2 \pi^2} + \frac{\cos n\pi x}{n\pi} \right]_{0.25}^{0.5} +$$

$$\frac{1}{5n\pi} \left[\frac{3 \cos n\pi x}{n\pi} + \frac{4x \cos n\pi x}{n\pi} - \frac{4 \sin n\pi x}{n^2 \pi^2} \right]_{0.5}^{0.75}$$

$$= \frac{1}{5n\pi} \left[\left(\frac{-\cos 0.5n\pi}{n\pi} + \frac{4 \sin 0.5n\pi}{n^2 \pi^2} \right) - \left(\frac{4 \sin 0.25n\pi}{n^2 \pi^2} \right) \right] +$$

$$\frac{1}{5n\pi} \left[\left(\frac{6 \cos 0.75n\pi}{n\pi} - \frac{4 \sin 0.75n\pi}{n^2 \pi^2} \right) - \left(\frac{5 \cos 0.5n\pi}{n\pi} - \frac{4 \sin 0.5n\pi}{n^2 \pi^2} \right) \right]$$

$$= \frac{1}{5n\pi}\left[\frac{-6\cos 0.5n\pi}{n\pi} + \frac{8\sin 0.5n\pi}{n^2\pi^2} - \frac{4\sin 0.25n\pi}{n^2\pi^2} + \frac{6\cos 0.75n\pi}{n\pi} - \frac{4\sin 0.75n\pi}{n^2\pi^2} \right]$$

(2) Separation of Variables for 2-Dim Laplace equation

$$u_{xx} + u_{yy} = 0$$

Assume $u(x, y) = X(x).Y(y)$

$$u_{xx} = X''.Y$$
$$u_{yy} = X.Y''$$

Substitute in the diff. eq.

$X''.Y + X.Y'' = 0$ dividing by $X(x).Y(y)$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{constant}\begin{cases} \lambda^2 & \text{positive constant} \\ 0 & \textit{zero} \\ -\lambda^2 & \text{negative constant} \end{cases}$$

Note: Select the constant dependent on B.C

(1) If the symmetric towards x, then use negative constant $= -\lambda^2$

(2) If the symmetric towards y, then use positive constant $= \lambda^2$

(1) Consider negative constant

$$\frac{X''(x)}{X(x)} = -\lambda^2 \quad \Rightarrow X''(x) + \lambda^2 X(x) = 0 \;\; \rightarrow \;\; (D^2 + \lambda^2)X = 0$$

$$(D - i\lambda)(D + i\lambda)X = 0 \quad \Rightarrow \;\therefore\; X(x) = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$$

$$X(x) = c_1(\cos\lambda x + i\sin\lambda x) + c_2(\cos\lambda x - i\sin\lambda x)$$

$$= \underbrace{(c_1 + c_2)}_{=A}\cos\lambda x + \underbrace{i(c_1 - c_2)}_{=B}\sin\lambda x$$

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

$$-\frac{Y''(y)}{Y(y)} = -\lambda^2$$

$$\Rightarrow Y''(y) - \lambda^2 Y(y) = 0 \;\; \rightarrow \;\; (D^2 - \lambda^2)Y = 0$$

$$(D - \lambda)(D + \lambda)Y = 0$$

$$Y(y) = c_3 e^{\lambda y} + c_4 e^{-\lambda y}$$

$$Y(y) = c_3(\cosh\lambda y + \sinh\lambda y) + c_4(\cosh\lambda y - \sinh\lambda y)$$

$$= \cancel{(c_3 + c_4)}^{=C} \cosh\lambda y + \cancel{(c_3 - c_4)}^{=D} \sinh\lambda y$$

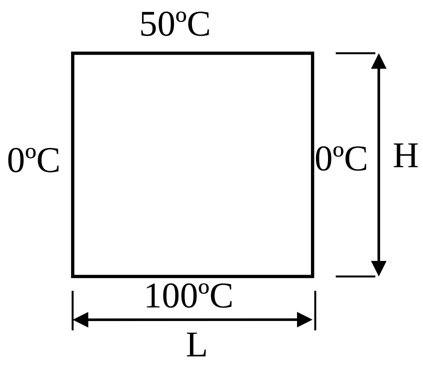
$$Y(y) = C\cosh\lambda y + D\sinh\lambda y$$

In this case the solution is given by:

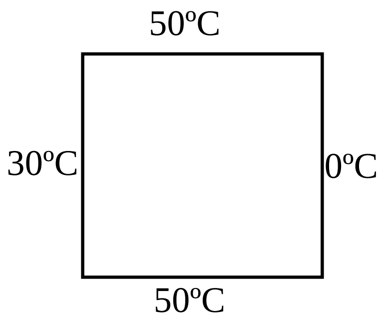
$$u(x,t) = X(x).Y(y) = (A\cos\lambda x + B\sin\lambda x)(C\cosh\lambda y + D\sinh\lambda y)$$

H.W. if considering the positive constant $= \lambda^2$, show that the solution is:

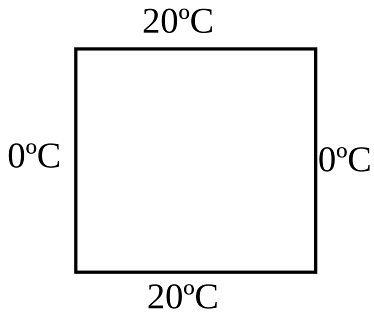
$$u(x,t) = (A\cosh\lambda x + B\sinh\lambda x)(C\cos\lambda y + D\sin\lambda y)$$



Symmetric towards x



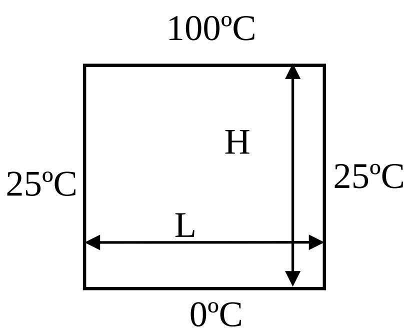
Symmetric towards y



towards x or y

<u>Example</u>

Find the steady state temperature distribution for the plate shown.



$$T_{xx} + T_{yy} = 0$$

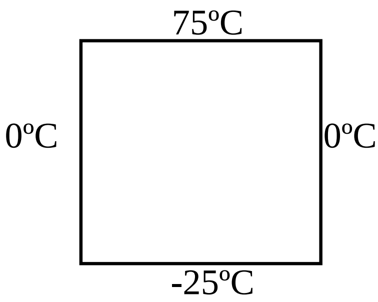
Let $\phi(x, y) = T(x, y) - 25$ satisfy Laplace eq.

$$\phi_{xx} + \phi_{yy} = 0$$

<u>B.C.</u>

(1) $\phi(0,y)=0$, (2) $\phi(L,y)=0$

(3) $\phi(x,0)=-25$, (4) $\phi(x,H)=75$



Assume $\phi(x,t)=X(x).Y(y)$

$$\phi_{xx}=X''.Y \quad , \quad \phi_{yy}=X.Y''$$

$$X''.Y+X.Y''=0 \qquad \% \, X.Y$$

$$\frac{X''}{X}=-\frac{Y''}{Y}=\text{constant}=-\lambda^2$$ (Because symmetric towards *x*)

$$X''+\lambda^2 X=0 \quad \Rightarrow \quad X(x)=A\cos\lambda x+B\sin\lambda x$$

$$Y''-\lambda^2 Y=0 \quad \Rightarrow \quad Y(y)=C\cosh\lambda y+D\sinh\lambda y$$

$$\therefore \phi(x,t)=\left(A\cos\lambda x+B\sin\lambda x\right)\left(C\cosh\lambda y+D\sinh\lambda y\right)$$

Apply the B.C.

$$\phi(0,y)=0$$

$$0=X(0).Y(y) \qquad but\ Y(y)\neq 0$$

$$\therefore X(0)=0=A\underbrace{\cos(0)}_{=1}+B\underbrace{\sin(0)}_{=0}$$

$$\Rightarrow \quad A=0$$

$$\phi(x,y)=B\sin\lambda x\left(C\cosh\lambda y+D\sinh\lambda y\right)$$

$$\therefore \phi(x,y)=\sin\lambda x\left(C^*\cosh\lambda y+D^*\sinh\lambda y\right)$$

$$where\ C^*=B.C\ ,\ D^*=B.D$$

The second Boundary condition: $\phi(L,y)=0$

$$0=\sin\lambda L\left(C^*\cosh\lambda y+D^*\sinh\lambda y\right) \quad ,$$

$$but\left(C^*\cosh\lambda y+D^*\sinh\lambda y\right)\neq 0$$

$$\Rightarrow \sin\lambda L=0 \quad \Rightarrow \quad \lambda L=n\pi \quad \Rightarrow \quad \lambda=\frac{n\pi}{L}$$

$$\therefore \lambda_n=\frac{n\pi}{L}$$ where n= 1, 2, 3, ...

$$\therefore \phi(x,t) = \sum_{n=1}^{\infty} \sin\lambda_n x\left(C_n^{\ *}\cosh\lambda_n y + D_n^{\ *}\sinh\lambda_n y\right)$$

Applying third boundary condition: $\phi(x,0) = -25$

$$-25 = \sum_{n=1}^{\infty} \sin\lambda_n x\left(C_n^{\ *}\cancel{\cosh(0)}^{=1} + D_n^{\ *}\cancel{\sinh(0)}^{=0}\right)$$

$$-25 = \sum_{n=1}^{\infty} C_n^* \sin\frac{n\pi}{L}x \qquad \frac{n\pi}{L} = \omega_n = \frac{n\pi}{p} \quad \rightarrow \therefore\ p = L \quad \text{(Fourier series)}$$

$$\therefore C_n^* = \frac{2}{L}\int_0^L (-25)\sin\frac{n\pi}{L}x\,dx = \frac{-50}{\cancel{L}}\left[\frac{-\cos\frac{n\pi}{L}x}{\frac{n\pi}{\cancel{L}}}\right]_0^L = \frac{50}{n\pi}\left[\cos n\pi - 1\right]$$

$$\therefore C_n^* = \begin{cases} 0 & n: even \\ \dfrac{-100}{n\pi} & n: odd \end{cases}$$

Applying forth boundary condition: $\phi(x,H) = 75$

$$75 = \sum_{n=1}^{\infty}\left(C_n^{\ *}\cosh\lambda_n H + D_n^{\ *}\sinh\lambda_n H\right)\sin\frac{n\pi}{L}x$$

Apply Fourier series

$$\left(C_n^{\ *}\cosh\lambda_n H + D_n^{\ *}\sinh\lambda_n H\right) = \frac{2}{L}\int_0^L (75)\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{150}{n\pi}\left(1 - \cos n\pi\right)$$

$$\therefore D_n^{\ *} = \frac{1}{\sinh\lambda_n H}\left[\frac{150}{n\pi}\left(1-\cos n\pi\right) - \overbrace{\frac{50}{n\pi}\left[\cos n\pi - 1\right]}^{C_n^*}\cosh\lambda_n H\right]$$

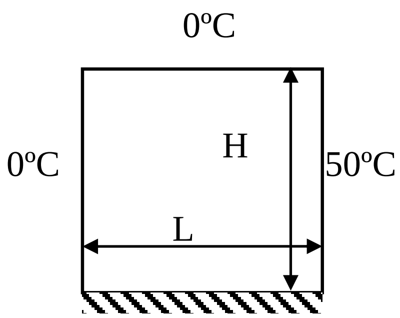
$$\therefore D_n^{\ *} = \begin{cases} 0 & n: even \\ \dfrac{1}{\sinh\lambda_n H}\left[\dfrac{300}{n\pi} + \dfrac{100}{n\pi}\cosh\lambda_n H\right] & n: odd \end{cases}$$

$$\therefore \phi(x,t)=\sum_{n=1}^{\infty}\sin\lambda_n x\,\frac{1}{n\pi}\left(-100\cosh\lambda_n y+\frac{1}{\sinh\lambda_n H}\left(300+100\cosh\lambda_n H\right)\sinh\lambda_n y\right)$$

$$T(x,y)=\phi(x,y)+25$$

Example:

Find the steady state temperature distribution for a thin square plate $0\le x\le L\ ,\ \ 0\le y\le H$. The edges $x=0,\ \ y=H,$ are maintained at zero temperature. The edge $y=0$ is insulated and the edge $x=L$ is kept at a temperature of 50ºC. (An insulated edge implies that the normal derivative of temperature is zero there).



Solution:

$$T_{xx}+T_{yy}=0$$

B.C:

(1) $T(0,y)=0$

(2) $T(L,y)=50$

(3) $\dfrac{\partial T(x,0)}{\partial y}=0$

(4) $T(x,H)=0$

Use the following solution because symmetry is towards y-axis:

$$T(x,t)=\left(A\cosh\lambda x+B\sinh\lambda x\right)\left(C\cos\lambda y+D\sin\lambda y\right)$$

Applying the B.C.:

The first boundary condition: $T(0,y)=0$

$$0=X(0).G(y)$$

$$but\ \ G(y)\neq 0$$

$$\therefore X(0)=0=A\underbrace{\cosh(0)}^{=1}+B\underbrace{\sinh(0)}^{=0}\quad\Rightarrow\quad\therefore A=0$$

$$T\ (x,y)=B\sinh\lambda x\left(C\cos\lambda y+D\sin\lambda y\right)$$

$$T\ (x,y)=\sinh\lambda x\left(C^{*}\cos\lambda y+D^{*}\sin\lambda y\right)\qquad where\ C^{*}=BC\ ,\ D^{*}=BD$$

The second boundary condition: $T_y(x,0)=0$

$$\frac{\partial T}{\partial y} = \sinh\lambda x\left(-\lambda C^* \sin\lambda y + \lambda D^* \cos\lambda y\right)$$

$$0 = \sinh\lambda x\left(-\lambda C^* \cancel{\sin(0)}^{=0} + \lambda D^* \cancel{\cos(0)}^{=1}\right)$$

$but\ \sinh\lambda x \neq 0$

$\therefore \lambda D^* = 0$,

$but\ \lambda \neq 0 \qquad \Rightarrow D^* = 0$

$\therefore T(x,y) = C^* \sinh\lambda x . \cos\lambda y$

The third boundary condition: $T(x,H) = 0$

$0 = C^* \sinh\lambda x . \cos\lambda H$

$C^* \neq 0, \qquad \sinh\lambda x \neq 0,$

$so\ \ far : \cos\lambda H = 0 \quad \rightarrow \lambda H = \dfrac{n\pi}{2} \qquad n : odd$

$\therefore \lambda_n = \dfrac{n\pi}{2H} \qquad n : odd\ \ only$

$\therefore T(x,t) = \sum_{n=1}^{\infty} C^* \sinh\lambda_n x\ \ \cos\lambda_n y$ n:odd only

The forth boundary condition: $T(L,y) = 50$

$50 = \sum_{n=1}^{\infty}\left(C^* \sinh\lambda_n L\right)\ \cos\dfrac{n\pi}{2H}\, y \qquad \omega_n = \dfrac{n\pi}{p} = \dfrac{n\pi}{2H} \qquad \therefore p = 2H$

Applying Fourier series:

$C_n^* \sinh\lambda_n L = \dfrac{\cancel{2}}{\cancel{2}H}\int_0^{2H} (50)\cos\lambda_n y\, dy \qquad Note : \int_0^{2H} = 2\int_0^{H}$

$$C_n^* \sinh \lambda_n L = \frac{2\times 50}{H}\int_0^H \cos\lambda_n y\, dy = \frac{100}{\lambda_n H}\left(\sin\lambda_n H - 0\right) = \frac{200}{n\pi}\sin\frac{n\pi}{2}$$

$$C_n^* = \frac{1}{\sinh\lambda_n L}\frac{200}{n\pi}\sin\frac{n\pi}{2} \qquad n : odd$$

$$\therefore T(x,t) = \sum_{n=1}^{\infty}\frac{200}{n\pi}\sin\frac{n\pi}{2}\frac{\sinh\lambda_n x}{\sinh\lambda_n L}\cos\lambda_n y$$

(3) Separation of Variables for 1D Diffuse Equation

$$u_{xx} = \frac{1}{\alpha} u_t$$

Assume $u(x,t) = X(x).T(t)$

$$u_{xx} = X''.T$$
$$u_t = X.T'$$

Substitute in the differential equation

$X''.T + X.T' = 0$ dividing by $X.T$

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha}\frac{T'(t)}{T(t)} = \text{constant}\begin{cases} \lambda^2 \\ 0 \\ -\lambda^2 \end{cases}$$

(i) Taking positive constant

$$\frac{X''(x)}{X(x)} = \lambda^2 \qquad \Rightarrow X''(x) - \lambda^2 X(x) = 0 \quad \rightarrow \ (D^2 - \lambda^2)X = 0$$

$$(D-\lambda)\left(D+\lambda\right)X = 0 \qquad \Rightarrow \ \therefore X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

$$\frac{1}{\alpha}\frac{T'(t)}{T(t)} = \lambda^2 \qquad \Rightarrow T' - \alpha\lambda^2 T = 0 \quad \rightarrow \ (D - \alpha\lambda^2)T = 0$$

$$\Rightarrow T(t) = ce^{\alpha\lambda^2 t}$$

But a solution of the form

$$u(x,t) = X(x).T(t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(ce^{\alpha\lambda^2 t}\right)$$

But this indicates that the temperature u=X.T increases as time increases. Hence, we reject the positive constant.

(ii) Taking zero constant

$$\frac{X''}{X} = 0 \qquad \Rightarrow \ X'' = 0 \quad \rightarrow \ \therefore X(x) = Ax + B$$

$$\frac{1}{\alpha}\frac{T'}{T}=0 \qquad \Rightarrow T'=0 \;\; \rightarrow \;\; \therefore T(t)=C$$

But a solution of the form

$u(x,t)=X.T=A^{*}x+B^{*}$ does not contain time. Hence, must be rejected.

Finally:

(ii) Taking negative constant

$$\frac{X''}{X}=-\lambda^{2} \qquad \Rightarrow X''+\lambda^{2}X=0 \;\; \rightarrow \;\; (D^{2}+\lambda^{2})X=0$$

$$(D-i\lambda)(D+i\lambda)X=0 \qquad \Rightarrow \;\; \therefore X(x)=c_{1}e^{i\lambda x}+c_{2}e^{-i\lambda x}$$

$$\therefore X(x)=A\cos\lambda x+B\sin\lambda x$$

$$\frac{1}{\alpha}\frac{T'}{T}=-\lambda^{2} \qquad \Rightarrow T'+\lambda^{2}\alpha T=0 \;\; \rightarrow \;\; (D+\alpha\lambda^{2})T=0 \;\; \rightarrow \therefore T(t)=Ce^{-\alpha\lambda^{2}t}$$

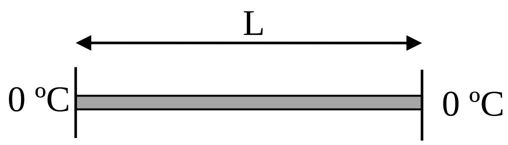
In this case the solution

$$u(x,t)=X.T=\left(A\cos\lambda x+B\sin\lambda x\right)\left(Ce^{-\alpha\lambda^{2}t}\right)$$

$$\boxed{u(x,t)=\left(A^{*}\cos\lambda x+B^{*}\sin\lambda x\right)e^{-\alpha\lambda^{2}t}}$$

Example

A bar of length L is subjected to the following initial conditions, find the unsteady state temperature distribution.



$$u(x,0)=f(x)=\begin{cases}50 & \dfrac{L}{4}<x<\dfrac{L}{2}\\ 0 & otherwise\end{cases}$$

$$u_{xx}=\frac{1}{\alpha}u_{t}$$

From the previous section get the analytic solution as;

$$\therefore u(x,t) = \left(A^* \cos\lambda x + B^* \sin\lambda x\right) e^{-\alpha\lambda^2 t}$$

Applying the B.C.:

$$(1)\ u(0,t) = 0 = \left(A^* \cos(0) + B^* \sin(0)\right) e^{-\alpha\lambda^2 t} = A^* e^{-\alpha\lambda^2 t}$$

$$e^{-\alpha\lambda^2 t} \neq 0 \quad \Rightarrow \quad \therefore A^* = 0$$

$$\therefore u(x,t) = B^* \sin\lambda x . e^{-\alpha\lambda^2 t}$$

$$(2)\, u(L,t) = 0 = B^* \sin\lambda L . e^{-\alpha\lambda^2 t} \quad \rightarrow B^* \neq 0 \quad , e^{-\alpha\lambda^2 t} \neq 0$$

$$\therefore \sin\lambda L = 0 \quad \rightarrow \quad \lambda L = n\pi \quad \Rightarrow \therefore \lambda = \frac{n\pi}{L} \quad \rightarrow \lambda_n = \frac{n\pi}{L} \quad n = 1,\ 2, \cdots$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} B_n^{\ *} \sin\lambda_n x . e^{-\alpha\lambda_n^2 t}$$

Applying the I.C.:

$$\therefore u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n^{\ *} \sin\lambda_n x . e^{0}$$

$$f(x) = \sum_{n=1}^{\infty} B_n^{\ *} \sin\frac{n\pi}{L} x.$$

Applying Fourier series:

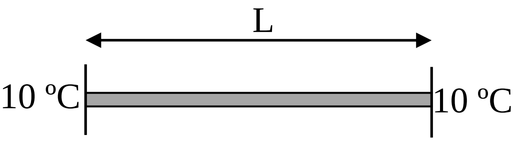
$$B_n^* = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx$$

$$B_n^* = \frac{100}{L}\int_{L/4}^{L/2} \sin\frac{n\pi x}{L} dx = \frac{-100}{L}\left[\frac{\cos\frac{n\pi x}{L}}{n\pi / L}\right]_{L/4}^{L/2} = \frac{100}{n\pi}\left[\cos\frac{n\pi}{4} - \cos\frac{n\pi}{2}\right]$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{100}{n\pi}\left[\cos\frac{n\pi}{4} - \cos\frac{n\pi}{2}\right] \sin\lambda_n x . e^{-\alpha\lambda_n^2 t}$$

Example-2

A bar of length L is subjected to the following initial conditions, find the unsteady state temperature distribution.



$$u(x,0) = f(x) = 10 + 50\sin\frac{\pi x}{L}$$

$$u_{xx} = \frac{1}{\alpha}u_t$$

Let $v(x,t) = u(x,t) - 10$, therefore: $v_{xx} = \frac{1}{\alpha}v_t$ with

B.C: $v(0,t) = v(L,t) = 0$ and initial conditions: $v(x,0) = f(x) - 10 = 50\sin\frac{\pi x}{L}$

From the previous section get the analytic solution as;

$$\therefore v(x,t) = \left(A^* \cos\lambda x + B^* \sin\lambda x\right)e^{-\alpha\lambda^2 t}$$

Applying the B.C.:

$$(1)\ v(0,t) = 0 = \left(A^* \cos(0) + B^* \sin(0)\right)e^{-\alpha\lambda^2 t} = A^* e^{-\alpha\lambda^2 t}$$

$$e^{-\alpha\lambda^2 t} \neq 0 \quad \Rightarrow \quad \therefore A^* = 0$$

$$\therefore v(x,t) = B^* \sin\lambda x . e^{-\alpha\lambda^2 t}$$

$$(2)\, v(L,t) = 0 = B^* \sin\lambda L . e^{-\alpha\lambda^2 t} \quad \rightarrow B^* \neq 0 \quad , e^{-\alpha\lambda^2 t} \neq 0$$

$$\therefore \sin\lambda L = 0 \quad \rightarrow \quad \lambda L = n\pi \quad \Rightarrow \therefore \lambda = \frac{n\pi}{L} \quad \rightarrow \lambda_n = \frac{n\pi}{L} \quad n = 1,\ 2, \cdots$$

$$\therefore v(x,t) = \sum_{n=1}^{\infty} B_n^* \sin\lambda_n x . e^{-\alpha\lambda_n^2 t}$$

Applying the I.C.:

$$\therefore v(x,0) = h(x) = \sum_{n=1}^{\infty} B_n^{\,*} \sin \lambda_n x . e^0$$

$$h(x) = \sum_{n=1}^{\infty} B_n^{\,*} \sin \frac{n\pi}{L} x$$

Applying Fourier series:

$$B_n^* = \frac{2}{L}\int_0^L h(x)\sin\frac{n\pi x}{L}\,dx$$

$$B_n^* = \frac{100}{L}\int_0^L \sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

Therefore: $B_n^* = 0 \;\; for \;\; n \neq 1$, and

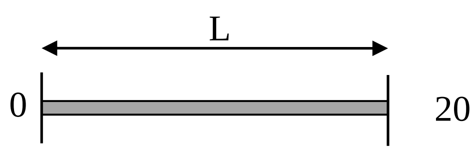
$$B_1^* = \frac{100}{L}\int_0^L \sin\frac{\pi x}{L}\sin\frac{\pi x}{L}\,dx = \frac{100}{L}\int_0^L \left(\sin\frac{\pi x}{L}\right)^2 dx$$

$$= \frac{100}{L}\int_0^L \frac{1-\cos\frac{2\pi x}{L}}{2}\,dx = \frac{50}{L}\left[x - \cancel{\sin\frac{2\pi x}{L}}\right]_0^L = 50$$

$$so\ far: u(x,t) = 10 + 50\sin\lambda_1 x . e^{-\alpha\lambda_1^2 t}$$

Asymmetric Conditions

Consider the case with the following conditions



Initial conditions: $u(x,0) = f(x)$

$$u_{xx} = \frac{1}{\alpha} u_t$$

B.C:

$$u(0,t) = 0 \;\; , \;\; u(L,t) = 20$$

Step-1:

The steady state solution after equilibrium is simply: $u_s(x) = \frac{20x}{L}$

Step-2:

Perform variable transformation by using $v(x,t) = u(x,t) - u_s$

Therefore, the diffuse equation still applicable to the new variable: $v_{xx} = \frac{1}{\alpha} v_t$ with

B.C: $v(0,t) = v(L,t) = 0$ and

initial conditions: $v(x,0) = f(x) - u_s = f(x) - \frac{20x}{L}$

Step-3: Solve the new problem as before.

Step-4: evaluate $u(x,t) = v(x,t) + u_s = v(x,t) + \frac{20x}{L}$

(4) Separation of Variables for 3-Dim Laplace equation:

$$u_{xx} + u_{yy} + u_{zz} = 0$$

Assume $u(x,y) = X(x)Y(y)Z(z)$

$$u_{xx} = X''.Y.Z$$
$$u_{yy} = X.Y''.Z$$
$$u_{zz} = X.Y.Z''$$

Substitute in the diff. eq. **dividing by u**

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0$$

Which can be re-arranged in the following (possible) arrangement:

$$\frac{Z''(z)}{Z(z)} = -\left(\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\right) = \lambda^2$$

Considering the first part:

$$\frac{Z''(z)}{Z(z)} = \lambda^2$$

$$Z(z) = A\cosh\lambda z + B\sinh\lambda z$$

Now considering the second part:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda^2$$

This equation cannot be satisfied unless both $\frac{X''(x)}{X(x)}$ and $\frac{Y''(y)}{Y(y)}$ are constants, let's say:

$$\frac{X''(x)}{X(x)} = -p^2 \text{ and } \frac{Y''(y)}{Y(y)} = -q^2$$

Where: $\lambda^2 = p^2 + q^2$

This gives:

$$X(x) = C\cos px + D\sin px \quad \text{and} \quad Y(y) = E\cos qy + F\sin qy$$

Note: Selection of the constant is dependent on B.C

(1) If two constants are negative, the third one must be positive.

(2) There are many other arrangements depending on the B.C.

Example

Solve

$$u_{xx} + u_{yy} + u_{zz} = 0$$

Subjected to B.C.

(1) $u(0, y, z) = u(a, y, z) = 0$

(2) $u(x, 0, z) = u(x, b, z) = 0$

(3) $u(x, y, 0) = 0$, $u(x, y, c) = f(x, y)$

Solution:

The B.C. in x will provide:

$$X(0) = 0 = C$$

$$X(a) = 0 = D\sin pa \Rightarrow p_m = \frac{m\pi}{a}$$

Similarly, the B.C. in y will provide:

The B.C. in x will provide:

$$Y(0) = 0 = E$$

$$Y(b) = 0 = F\sin qb \Rightarrow q_n = \frac{n\pi}{b}$$

Therefore: $\lambda_{m,n}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

$$u_{m,n}(x,y) = X_m(x)Y_n(y)Z_{m,n}(z)$$

$$= \left(\sin\frac{m\pi}{a}x\right)\left(\sin\frac{n\pi}{b}y\right)\left(A_{m,n}\cos\lambda_{m,n}z + B_{m,n}\sin\lambda_{m,n}z\right)$$

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\sin\frac{m\pi}{a}x\right)\left(\sin\frac{n\pi}{b}y\right)\left(A_{m,n}\cosh\lambda_{m,n}z + B_{m,n}\sinh\lambda_{m,n}z\right)$$

Applying the B.C. in z will provide:

$$Z(0) = 0 = A$$

$$u(x,y,c) = f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\sin\frac{m\pi}{a}x\right)\left(\sin\frac{n\pi}{b}y\right)\left(B_{m,n}\sinh\lambda_{m,n}c\right)$$