2.10 Relative Minimums and Maximums

In this section we are going to extend one of the more important ideas from Calculus I into functions of two variables. We are going to start looking at trying to find minimums and maximums of functions. This in fact will be the topic of the following two sections as well.

In this section we are going to be looking at identifying relative minimums and relative maximums. Recall as well that we will often use the word extrema to refer to both minimums and maximums.

The definition of relative extrema for functions of two variables is identical to that for functions of one variable we just need to remember now that we are working with functions of two variables. So, for the sake of completeness here is the definition of relative minimums and relative maximums for functions of two variables.

Definition

1. A function $f(x,y)$ has a **relative minimum** at the point (a,b) if $f(x,y) \geq f(a,b)$ for all points (x,y) in some region around (a,b).
2. A function $f(x,y)$ has a **relative maximum** at the point (a,b) if $f(x,y) \leq f(a,b)$ for all points (x,y) in some region around (a,b).

Note that this definition does not say that a relative minimum is the smallest value that the function will ever take. It only says that in some region around the point (a,b) the function will always be larger than $f(a,b)$. Outside of that region it is completely possible for the function to be smaller. Likewise, a relative maximum only says that around (a,b) the function will always be smaller than $f(a,b)$. Again, outside of the region it is completely possible that the function will be larger.

Next, we need to extend the idea of **critical points** up to functions of two variables. Recall that a critical point of the function $f(x)$ was a number $x = c$ so that either $f'(c) = 0$ or $f'(c)$ doesn't exist. We have a similar definition for critical points of functions of two variables.

Definition

The point (a,b) is a **critical point** (or a **stationary point**) of $f(x,y)$ provided one of the following is true,

1. $\nabla f(a,b) = \vec{0}$ (this is equivalent to saying that $f_x(a,b) = 0$ and $f_y(a,b) = 0$),
2. $f_x(a,b)$ and/or $f_y(a,b)$ doesn't exist.

To see the equivalence in the first part let's start off with $\nabla f = \vec{0}$ and put in the definition of each part.

$$\nabla f(a,b) = \vec{0}$$
$$\langle f_x(a,b), f_y(a,b)\rangle = \langle 0,0\rangle$$

The only way that these two vectors can be equal is to have $f_x(a,b) = 0$ and $f_y(a,b) = 0$. In fact, we will use this definition of the critical point more than the gradient definition since it will be easier to find the critical points if we start with the partial derivative definition.

Note as well that BOTH of the first order partial derivatives must be zero at (a,b). If only one of the first order partial derivatives are zero at the point then the point will NOT be a critical point.

We now have the following fact that, at least partially, relates critical points to relative extrema.

Fact

If the point (a,b) is a relative extrema of the function $f(x,y)$ and the first order derivatives of $f(x,y)$ exist at (a,b) then (a,b) is also a critical point of $f(x,y)$ and in fact we'll have $\nabla f(a,b) = \vec{0}$.

Proof

This is a really simple proof that relies on the single variable version that we saw in Calculus I version, often called Fermat's Theorem.

Let's start off by defining $g(x) = f(x,b)$ and suppose that $f(x,y)$ has a relative extrema at (a,b). However, this also means that $g(x)$ also has a relative extrema (of the same kind as $f(x,y)$) at $x = a$. By Fermat's Theorem we then know that $g'(a) = 0$. But we also know that $g'(a) = f_x(a,b)$ and so we have that $f_x(a,b) = 0$.

If we now define $h(y) = f(a,y)$ and going through exactly the same process as above we will see that $f_y(a,b) = 0$.

So, putting all this together means that $\nabla f(a,b) = \vec{0}$ and so $f(x,y)$ has a critical point at (a,b).

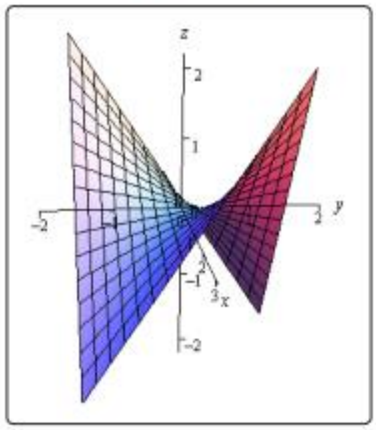
Note that this does NOT say that all critical points are relative extrema. It only says that relative extrema will be critical points of the function. To see this let's consider the function

$$f(x,y) = xy$$

The two first order partial derivatives are,

$$f_x(x,y) = y \qquad\qquad f_y(x,y) = x$$

The only point that will make both of these derivatives zero at the same time is $(0, 0)$ and so $(0, 0)$ is a critical point for the function. Here is a graph of the function.



Note that the axes are not in the standard orientation here so that we can see more clearly what is happening at the origin, *i.e.* at $(0, 0)$. If we start at the origin and move into either of the quadrants where both x and y are the same sign the function increases. However, if we start at the origin and move into either of the quadrants where x and y have the opposite sign then the function decreases. In other words, no matter what region you take about the origin there will be points larger than $f(0, 0) = 0$ and points smaller than $f(0, 0) = 0$. Therefore, there is no way that $(0, 0)$ can be a relative extrema.

Critical points that exhibit this kind of behavior are called **saddle points** The point (a, b) is a **critical point** (or a **stationary point**) of $f(x, y)$ provided one of the following is true, .

While we have to be careful to not misinterpret the results of this fact it is very useful in helping us to identify relative extrema. Because of this fact we know that if we have all the critical points of a function then we also have every possible relative extrema for the function. The fact tells us that all relative extrema must be critical points so we know that if the function does have relative extrema then they must be in the collection of all the critical points. Remember however, that it will be completely possible that at least one of the critical points won't be a relative extrema.

So, once we have all the critical points in hand all we will need to do is test these points to see if they are relative extrema or not. To determine if a critical point is a relative extrema (and in fact to determine if it is a minimum or a maximum) we can use the following fact.

Fact

Suppose that (a, b) is a critical point of $f(x, y)$ and that the second order partial derivatives are continuous in some region that contains (a, b). Next define,

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

We then have the following classifications of the critical point.

1. If $D > 0$ and $f_{xx}(a, b) > 0$ then there is a relative minimum at (a, b).
2. If $D > 0$ and $f_{xx}(a, b) < 0$ then there is a relative maximum at (a, b).
3. If $D < 0$ then the point (a, b) is a saddle point.
4. If $D = 0$ then the point (a, b) may be a relative minimum, relative maximum or a saddle point. Other techniques would need to be used to classify the critical point.

Note that if $D > 0$ then both $f_{xx}(a, b)$ and $f_{yy}(a, b)$ will have the same sign and so in the first two cases above we could just as easily replace $f_{xx}(a, b)$ with $f_{yy}(a, b)$. Also note that we aren't going to be seeing any cases in this class where $D = 0$ as these can often be quite difficult to classify. We will be able to classify all the critical points that we find.

Let's see a couple of examples.

Example 80

Find and classify all the critical points of $f(x, y) = 4 + x^3 + y^3 - 3xy$.

Solution

We first need all the first order (to find the critical points) and second order (to classify the critical points) partial derivatives so let's get those.

$$f_x = 3x^2 - 3y \qquad f_y = 3y^2 - 3x$$
$$f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = -3$$

Let's first find the critical points. Critical points will be solutions to the system of equations,

$$f_x = 3x^2 - 3y = 0$$
$$f_y = 3y^2 - 3x = 0$$

This is a non-linear system of equations and these can, on occasion, be difficult to solve. However, in this case it's not too bad. We can solve the first equation for y as follows,

$$3x^2 - 3y = 0 \quad \Rightarrow \quad y = x^2$$

Plugging this into the second equation gives,

$$3(x^2)^2 - 3x = 3x\left(x^3 - 1\right) = 0$$

From this we can see that we must have $x = 0$ or $x = 1$. Now use the fact that $y = x^2$ to get the critical points.

$$\begin{aligned} x = 0: \quad & y = 0^2 = 0 \;\Rightarrow \quad && (0,0) \\ x = 1: \quad & y = 1^2 = 1 \;\Rightarrow \quad && (1,1) \end{aligned}$$

So, we get two critical points. All we need to do now is classify them. To do this we will need D. Here is the general formula for D.

$$\begin{aligned} D(x,y) &= f_{xx}(x,y)\, f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2 \\ &= (6x)(6y) - (-3)^2 \\ &= 36xy - 9 \end{aligned}$$

To classify the critical points all that we need to do is plug in the critical points and use the fact above to classify them.

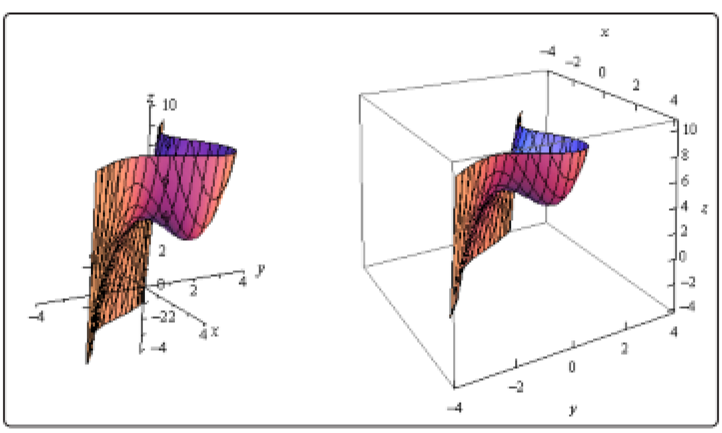
$(0,0): D = D(0,0) = -9 < 0$

So, for $(0,0)$ D is negative and so this must be a saddle point.

$(1,1): D = D(1,1) = 36 - 9 = 27 > 0 \qquad f_{xx}(1,1) = 6 > 0$

For $(1,1)$ D is positive and f_{xx} is positive and so we must have a relative minimum.

For the sake of completeness here is a graph of this function.



Notice that in order to get a better visual we used a somewhat nonstandard orientation. We can see that there is a relative minimum at $(1, 1)$ and (hopefully) it's clear that at $(0, 0)$ we do get a saddle point.

Example 81

Find and classify all the critical points for $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

Solution

As with the first example we will first need to get all the first and second order derivatives.

$$f_x = 6xy - 6x \qquad f_y = 3x^2 + 3y^2 - 6y$$
$$f_{xx} = 6y - 6 \qquad f_{yy} = 6y - 6 \qquad f_{xy} = 6x$$

We'll first need the critical points. The equations that we'll need to solve this time are,

$$6xy - 6x = 0$$
$$3x^2 + 3y^2 - 6y = 0$$

These equations are a little trickier to solve than the first set, but once you see what to do they really aren't terribly bad.

First, let's notice that we can factor out a 6 x from the first equation to get,

$$6x(y - 1) = 0$$

So, we can see that the first equation will be zero if $x = 0$ or $y = 1$. Be careful to not just cancel the x from both sides. If we had done that we would have missed $x = 0$.

To find the critical points we can plug these (individually) into the second equation and solve for the remaining variable.

$x = 0$:

$$3y^2 - 6y = 3y(y - 2) = 0 \qquad \Rightarrow \qquad y = 0,\ y = 2$$

$y = 1$:

$$3x^2 - 3 = 3\left(x^2 - 1\right) = 0 \qquad \Rightarrow \qquad x = -1,\ x = 1$$

So, if $x = 0$ we have the following critical points,

$$(0, 0) \qquad (0, 2)$$

and if $y = 1$ the critical points are,

$$(1, 1) \qquad (-1, 1)$$

Now all we need to do is classify the critical points. To do this we'll need the general formula for D.

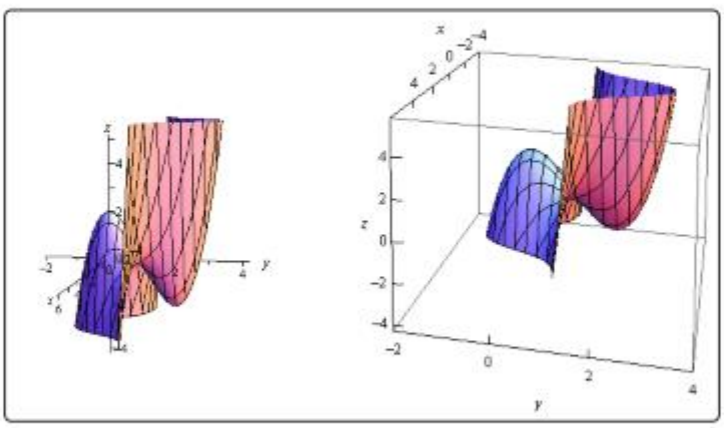
$$D(x,y) = (6y-6)(6y-6) - (6x)^2 = (6y-6)^2 - 36x^2$$

$(0,0)$: $D = D(0,0) = 36 > 0$ $f_{xx}(0,0) = -6 < 0$ $(0,2)$: $D = D(0,2) = 36 > 0$ $f_{xx}(0,2) = 6 > 0$ $(1,1)$: $D = D(1,1) = -36 < 0$ $(-1,1)$: $D = D(-1,1) = -36 < 0$

So, it looks like we have the following classification of each of these critical points.

$(0,0)$: Relative Maximum
$(0,2)$: Relative Minimum
$(1,1)$: Saddle Point
$(-1,1)$: Saddle Point

Here is a graph of the surface for the sake of completeness.



Let's do one more example that is a little different from the first two.

Example 82

Determine the point on the plane $4x-2y+z=1$ that is closest to the point $(-2,-1,5)$.

Solution

Note that we are NOT asking for the critical points of the plane. In order to do this example we are going to need to first come up with the equation that we are going to have to work with.

First, let's suppose that (x,y,z) is any point on the plane. The distance between this point and the point in question, $(-2,-1,5)$, is given by the formula,

$$d=\sqrt{(x+2)^2+(y+1)^2+(z-5)^2}$$

What we are then asked to find is the minimum value of this equation. The point (x,y,z) that gives the minimum value of this equation will be the point on the plane that is closest to $(-2,-1,5)$.

There are a couple of issues with this equation. First, it is a function of x, y and z and we can only deal with functions of x and y at this point. However, this is easy to fix. We can solve the equation of the plane to see that,

$$z=1-4x+2y$$

Plugging this into the distance formula gives,

$$\begin{aligned} d&=\sqrt{(x+2)^2+(y+1)^2+(1-4x+2y-5)^2}\\ &=\sqrt{(x+2)^2+(y+1)^2+(-4-4x+2y)^2}\end{aligned}$$

Now, the next issue is that there is a square root in this formula and we know that we're going to be differentiating this eventually. So, in order to make our life a little easier let's notice that finding the minimum value of d will be equivalent to finding the minimum value of d^2.

So, let's instead find the minimum value of

$$f(x,y)=d^2=(x+2)^2+(y+1)^2+(-4-4x+2y)^2$$

Now, we need to be a little careful here. We are being asked to find the closest point on the plane to $(-2,-1,5)$ and that is not really the same thing as what we've been doing in this section. In this section we've been finding and classifying critical points as relative minimums or maximums and what we are really asking is to find the smallest value the function will take, or the absolute minimum. Hopefully, it does make sense from a physical standpoint that there will be a closest point on the plane to $(-2,-1,5)$. This point should also be a relative minimum in addition to being an absolute minimum.

So, let's go through the process from the first and second example and see what we get as far as relative minimums go. If we only get a single relative minimum then we will be done since that point will also need to be the absolute minimum of the function and hence the point on the plane that is closest to $(-2,-1,5)$.

We'll need the derivatives first.

$$\begin{aligned} f_x &= 2(x+2) + 2(-4)(-4-4x+2y) = 36 + 34x - 16y \\ f_y &= 2(y+1) + 2(2)(-4-4x+2y) = -14 - 16x + 10y \\ f_{xx} &= 34 \\ f_{yy} &= 10 \\ f_{xy} &= -16 \end{aligned}$$

Now, before we get into finding the critical point let's compute D quickly.

$$D = 34(10) - (-16)^2 = 84 > 0$$

So, in this case D will always be positive and also notice that $f_{xx} = 34 > 0$ is always positive and so any critical points that we get will be guaranteed to be relative minimums.

Now let's find the critical point(s). This will mean solving the system.

$$\begin{aligned} 36 + 34x - 16y &= 0 \\ -14 - 16x + 10y &= 0 \end{aligned}$$

To do this we can solve the first equation for x.

$$x = \frac{1}{34}(16y - 36) = \frac{1}{17}(8y - 18)$$

Now, plug this into the second equation and solve for y.

$$-14 - \frac{16}{17}(8y - 18) + 10y = 0 \qquad \Rightarrow \qquad y = -\frac{25}{21}$$

Back substituting this into the equation for x gives $x = -\frac{34}{21}$.

So, it looks like we get a single critical point : $\left(-\frac{34}{21}, -\frac{25}{21}\right)$. Also, since we know this will be a relative minimum and it is the only critical point we know that this is also the x and y coordinates of the point on the plane that we're after. We can find the z coordinate by plugging into the equation of the plane as follows,

$$z = 1 - 4\left(-\frac{34}{21}\right) + 2\left(-\frac{25}{21}\right) = \frac{107}{21}$$

So, the point on the plane that is closest to $(-2,-1,5)$ is $\left(-\frac{34}{21}, -\frac{25}{21}, \frac{107}{21}\right)$.

2.11 Absolute Extrema

In this section we are going to extend the work from the previous section. In the previous section we were asked to find and classify all critical points as relative minimums, relative maximums and/or saddle points. In this section we want to optimize a function, that is identify the absolute minimum and/or the absolute maximum of the function, on a given region in $\mathbb{R}^2$. Note that when we say we are going to be working on a region in $\mathbb{R}^2$ we mean that we're going to be looking at some region in the *xy*-plane.

In order to optimize a function in a region we are going to need to get a couple of definitions out of the way and a fact. Let's first get the definitions out of the way.

Definitions

1. A region in $\mathbb{R}^2$ is called **closed** if it includes its boundary. A region is called **open** if it doesn't include any of its boundary points.
2. A region in $\mathbb{R}^2$ is called **bounded** if it can be completely contained in a disk. In other words, a region will be bounded if it is finite.

Let's think a little more about the definition of closed. We said a region is closed if it includes its boundary. Just what does this mean? Let's think of a rectangle. Below are two definitions of a rectangle, one is closed and the other is open.

Open	Closed
$-5 < x < 3$	$-5 \le x \le 3$
$1 < y < 6$	$1 \le y \le 6$

In this first case we don't allow the ranges to include the endpoints (*i.e.* we aren't including the edges of the rectangle) and so we aren't allowing the region to include any points on the edge of the rectangle. In other words, we aren't allowing the region to include its boundary and so it's open.

In the second case we are allowing the region to contain points on the edges and so will contain its entire boundary and hence will be closed.

This is an important idea because of the following fact.

Extreme Value Theorem

If $f(x, y)$ is continuous in some closed, bounded set D in $\mathbb{R}^2$ then there are points in D, (x_1, y_1) and (x_2, y_2) so that $f(x_1, y_1)$ is the absolute maximum and $f(x_2, y_2)$ is the absolute minimum of the function in D.

Note that this theorem does NOT tell us where the absolute minimum or absolute maximum will occur. It only tells us that they will exist. Note as well that the absolute minimum and/or absolute maximum may occur in the interior of the region or it may occur on the boundary of the region.

The basic process for finding absolute maximums is pretty much identical to the process that we used in Calculus I when we looked at finding absolute extrema of functions of single variables. There will however, be some procedural changes to account for the fact that we now are dealing with functions of two variables. Here is the process.

Finding Absolute Extrema

1. Find all the critical points of the function that lie in the region D and determine the function value at each of these points.
2. Find all extrema of the function on the boundary. This usually involves the Calculus I approach for this work.
3. The largest and smallest values found in the first two steps are the absolute minimum and the absolute maximum of the function.

The main difference between this process and the process that we used in Calculus I is that the "boundary" in Calculus I was just two points and so there really wasn't a lot to do in the second step. For these problems the majority of the work is often in the second step as we will often end up doing a Calculus I absolute extrema problem one or more times.

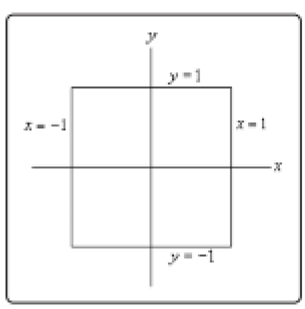
Let's take a look at an example or two.

Example 83

Find the absolute minimum and absolute maximum of $f(x, y) = x^2 + 4y^2 - 2x^2y + 4$ on the rectangle given by $-1 \le x \le 1$ and $-1 \le y \le 1$.

Solution

Let's first get a quick picture of the rectangle for reference purposes.



The boundary of this rectangle is given by the following conditions.

right side : $x = 1,\ -1 \le y \le 1$
left side : $x = -1,\ -1 \le y \le 1$
upper side : $y = 1,\ -1 \le x \le 1$
lower side : $y = -1,\ -1 \le x \le 1$

These will be important in the second step of our process.

We'll start this off by finding all the critical points that lie inside the given rectangle. To do this we'll need the two first order derivatives.

$$f_x = 2x - 4xy \qquad f_y = 8y - 2x^2$$

Note that since we aren't going to be classifying the critical points we don't need the second order derivatives. To find the critical points we will need to solve the system,

$$\begin{aligned} 2x - 4xy &= 0 \\ 8y - 2x^2 &= 0 \end{aligned}$$

We can solve the second equation for y to get,

$$y = \frac{x^2}{4}$$

Plugging this into the first equation gives us,

$$2x - 4x\left(\frac{x^2}{4}\right) = 2x - x^3 = x\left(2 - x^2\right) = 0$$

This tells us that we must have $x = 0$ or $x = \pm\sqrt{2} = \pm 1.414...$. Now, recall that we only want critical points in the region that we're given. That means that we only want critical points for which $-1 \le x \le 1$. The only value of x that will satisfy this is the first one so we can ignore the last two for this problem. Note however that a simple change to the boundary would include these two so don't forget to always check if the critical points are in the region (or on the boundary since that can also happen).

Plugging $x = 0$ into the equation for y gives us,

$$y = \frac{0^2}{4} = 0$$

The single critical point, in the region (and again, that's important), is $(0, 0)$. We now need to get the value of the function at the critical point.

$$f(0, 0) = 4$$

Eventually we will compare this to values of the function found in the next step and take the largest and smallest as the absolute extrema of the function in the rectangle.

Now we have reached the long part of this problem. We need to find the absolute extrema of the function along the boundary of the rectangle. What this means is that we're going to need to look at what the function is doing along each of the sides of the rectangle listed above.

Let's first take a look at the right side. As noted above the right side is defined by

$$x = 1, \ -1 \le y \le 1$$

Notice that along the right side we know that $x = 1$. Let's take advantage of this by defining a new function as follows,

$$g(y) = f(1, y) = 1^2 + 4y^2 - 2(1^2)y + 4 = 5 + 4y^2 - 2y$$

Now, finding the absolute extrema of $f(x, y)$ along the right side will be equivalent to finding the absolute extrema of $g(y)$ in the range $-1 \le y \le 1$. Hopefully you recall how to do this from Calculus I. We find the critical points of $g(y)$ in the range $-1 \le y \le 1$ and then evaluate $g(y)$ at the critical points and the end points of the range of y's.

Let's do that for this problem.

$$g'(y) = 8y - 2 \qquad \Rightarrow \qquad y = \frac{1}{4}$$

This is in the range and so we will need the following function evaluations.

$$g(-1) = 11 \qquad g(1) = 7 \qquad g\left(\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

Notice that, using the definition of $g(y)$ these are also function values for $f(x, y)$.

$$\begin{aligned} g(-1) &= f(1, -1) = 11 \\ g(1) &= f(1, 1) = 7 \\ g\left(\frac{1}{4}\right) &= f\left(1, \frac{1}{4}\right) = \frac{19}{4} = 4.75 \end{aligned}$$

We can now do the left side of the rectangle which is defined by,

$$x = -1, \ -1 \le y \le 1$$

Again, we'll define a new function as follows,

$$g(y) = f(-1, y) = (-1)^2 + 4y^2 - 2(-1)^2y + 4 = 5 + 4y^2 - 2y$$

Notice however that, for this boundary, this is the same function as we looked at for the right side. This will not always happen, but since it has let's take advantage of the fact that we've already done the work for this function. We know that the critical point is $y = \frac{1}{4}$ and we know that the function value at the critical point and the end points are,

$$g(-1) = 11 \qquad g(1) = 7 \qquad g\left(\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

The only real difference here is that these will correspond to values of $f(x, y)$ at different points than for the right side. In this case these will correspond to the following function values for $f(x, y)$.

$$\begin{aligned} g(-1) &= f(-1,-1) = 11 \\ g(1) &= f(-1,1) = 7 \\ g\left(\frac{1}{4}\right) &= f\left(-1,\frac{1}{4}\right) = \frac{19}{4} = 4.75 \end{aligned}$$

We can now look at the upper side defined by,

$$y = 1, \ -1 \le x \le 1$$

We'll again define a new function except this time it will be a function of x.

$$h(x) = f(x, 1) = x^2 + 4\left(1^2\right) - 2x^2(1) + 4 = 8 - x^2$$

We need to find the absolute extrema of $h(x)$ on the range $-1 \le x \le 1$. First find the critical point(s).

$$h'(x) = -2x \qquad \Rightarrow \qquad x = 0$$

The value of this function at the critical point and the end points is,

$$h(-1) = 7 \qquad h(1) = 7 \qquad h(0) = 8$$

and these in turn correspond to the following function values for $f(x, y)$

$$\begin{aligned} h(-1) &= f(-1,1) = 7 \\ h(1) &= f(1,1) = 7 \\ h(0) &= f(0,1) = 8 \end{aligned}$$

Note that there are several "repeats" here. The first two function values have already been computed when we looked at the right and left side. This will often happen.

Finally, we need to take care of the lower side. This side is defined by,

$$y = -1, \ -1 \le x \le 1$$

The new function we'll define in this case is,

$$h(x) = f(x,-1) = x^2 + 4(-1)^2 - 2x^2(-1) + 4 = 8 + 3x^2$$

The critical point for this function is,

$$h'(x) = 6x \qquad \Rightarrow \qquad x = 0$$

The function values at the critical point and the endpoint are,

$$h(-1) = 11 \qquad h(1) = 11 \qquad h(0) = 8$$

and the corresponding values for $f(x,y)$ are,

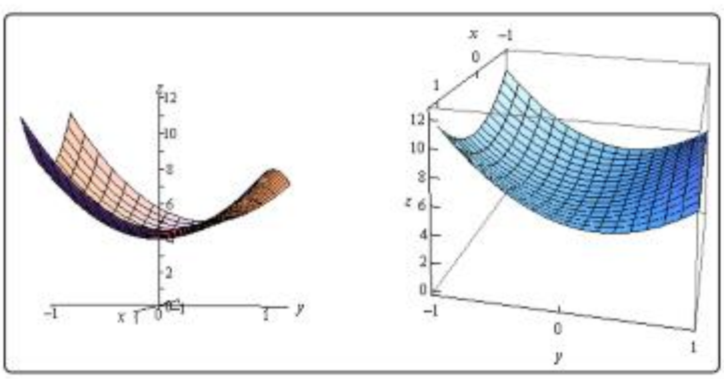
$$h(-1) = f(-1,-1) = 11$$
$$h(1) = f(1,-1) = 11$$
$$h(0) = f(0,-1) = 8$$

The final step to this (long...) process is to collect up all the function values for $f(x,y)$ that we've computed in this problem. Here they are,

$$f(0,0) = 4 \qquad f(1,-1) = 11 \qquad f(1,1) = 7$$
$$f\left(1,\frac{1}{4}\right) = 4.75 \qquad f(-1,1) = 7 \qquad f(-1,-1) = 11$$
$$f\left(-1,\frac{1}{4}\right) = 4.75 \qquad f(0,1) = 8 \qquad f(0,-1) = 8$$

The absolute minimum is at $(0,0)$ since gives the smallest function value and the absolute maximum occurs at $(1,-1)$ and $(-1,-1)$ since these two points give the largest function value.

Here is a sketch of the function on the rectangle for reference purposes.



As this example has shown these can be very long problems on occasion. Let's take a look at an easier, well shorter anyway, problem with a different kind of boundary.

Example 84

Find the absolute minimum and absolute maximum of $f(x, y) = 2x^2 - y^2 + 6y$ on the disk of radius 4, $x^2 + y^2 \leq 16$

Solution

First note that a disk of radius 4 is given by the inequality in the problem statement. The "less than" inequality is included to get the interior of the disk and the equal sign is included to get the boundary. Of course, this also means that the boundary of the disk is a circle of radius 4.

Let's first find the critical points of the function that lies inside the disk. This will require the following two first order partial derivatives.

$$f_x = 4x \qquad f_y = -2y + 6$$

To find the critical points we'll need to solve the following system.

$$\begin{aligned} 4x &= 0 \\ -2y + 6 &= 0 \end{aligned}$$

This is actually a fairly simple system to solve however. The first equation tells us that $x = 0$ and the second tells us that $y = 3$. So, the only critical point for this function is $(0, 3)$ and this is inside the disk of radius 4. The function value at this critical point is,

$$f(0, 3) = 9$$

Now we need to look at the boundary. This one will be somewhat different from the previous example. In this case we don't have fixed values of x and y on the boundary. Instead we have,

$$x^2 + y^2 = 16$$

We can solve this for x^2 and plug this into the x^2 in $f(x, y)$ to get a function of y as follows.

$$\begin{aligned} x^2 &= 16 - y^2 \\ g(y) &= 2\left(16 - y^2\right) - y^2 + 6y = 32 - 3y^2 + 6y \end{aligned}$$

We will need to find the absolute extrema of this function on the range $-4 \leq y \leq 4$ (this is the range of y's for the disk....). We'll first need the critical points of this function.

$$g'(y) = -6y + 6 \qquad \Rightarrow \qquad y = 1$$

The value of this function at the critical point and the endpoints are,

$$g(-4) = -40 \qquad g(4) = 8 \qquad g(1) = 35$$

Unlike the first example we will still need to find the values of x that correspond to these. We can do this by plugging the value of y into our equation for the circle and solving for x.

$$\begin{aligned} y = -4: &\qquad x^2 = 16 - 16 = 0 \quad \Rightarrow \quad x = 0 \\ y = 4: &\qquad x^2 = 16 - 16 = 0 \quad \Rightarrow \quad x = 0 \\ y = 1: &\qquad x^2 = 16 - 1 = 15 \quad \Rightarrow \quad x = \pm\sqrt{15} = \pm 3.87 \end{aligned}$$

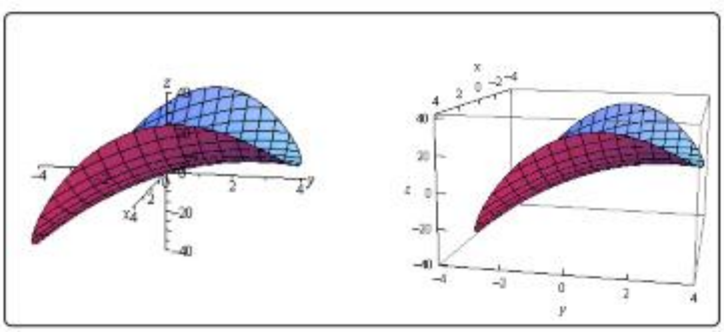
The function values for $g(y)$ then correspond to the following function values for $f(x, y)$.

$$\begin{aligned} g(-4) = -40 &\qquad \Rightarrow \quad f(0, -4) = -40 \\ g(4) = 8 &\qquad \Rightarrow \quad f(0, 4) = 8 \\ g(1) = 35 &\qquad \Rightarrow \quad f\left(-\sqrt{15}, 1\right) = 35 \text{ and } f\left(\sqrt{15}, 1\right) = 35 \end{aligned}$$

Note that the third one actually corresponded to two different values for $f(x, y)$ since that y also produced two different values of x.

So, comparing these values to the value of the function at the critical point of $f(x, y)$ that we found earlier we can see that the absolute minimum occurs at $(0, -4)$ while the absolute maximum occurs twice at $(-\sqrt{15}, 1)$ and $(\sqrt{15}, 1)$.

Here is a sketch of the region for reference purposes.



In both of these examples one of the absolute extrema actually occurred at more than one place. Sometimes this will happen and sometimes it won't so don't read too much into the fact that it happened in both examples given here.

Also note that, as we've seen, absolute extrema will often occur on the boundaries of these regions, although they don't have to occur at the boundaries. Had we given much more complicated examples with multiple critical points we would have seen examples where the absolute extrema occurred interior to the region and not on the boundary.