2 Partial Derivatives

To this point, with the exception of the occasional section in the last chapter, we've been working almost exclusively with functions of a single variable. It is now time to formally start multi-variable Calculus, *i.e.* Calculus involving functions of two or more variables. We will be covering the same basic topics as we do with single variable Calculus. Namely, limits, derivatives and integrals.

In this chapter we will open up with a quick section discussing taking limits of multi-variable functions. We will only be covering limits of multi-variable functions with a single chapter because, as we'll see, many of the concepts from single variable limits still hold, with some natural extensions of course. However, as we'll also see the work will often be significantly longer/harder and so we won't be spending a lot of time discussing limits of multi-variable functions. Luckily enough for us we also won't need to worry all that much about limits of multi-variable functions so the quick discussion of limits in this chapter will suffice.

The rest of the chapter will be discussing how to take derivatives of multi-variable functions. We want to keep the "main" interpretation of derivatives, namely the derivative will still give the rate of change of the function. The issue here is that because we have multiple variables the function can have differing rates of change depending on how we allow the various variables to change.

So, to start out the derivative discussion we will start by defining the partial derivative. These will restrict just how we allow the various variables to change. We will eventually introduce the directional derivative which will allow the variables to change in any arbitrary manner. In the process of introducing the idea of a directional derivative we'll also introduce the concept of a gradient of a function. The gradient will arise in quite a few sections throughout the rest of this multi-variable Calculus material, including integrals.

Finally, as we'll see, if you can take derivatives of single variable functions then you have the majority of the knowledge that you need to take derivatives of multi-variable functions. There are, however, some subtleties that we'll need to remember to deal with. Those subtleties are, generally, the issues that most students run into when taking derivatives of multi-variable functions.

2.1 Limits

In this section we will take a look at limits involving functions of more than one variable. In fact, we will concentrate mostly on limits of functions of two variables, but the ideas can be extended out to functions with more than two variables.

Before getting into this let's briefly recall how limits of functions of one variable work. We say that,

$$\lim_{x \to a} f(x) = L$$

provided,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

Also, recall that,

$$\lim_{x \to a^+} f(x)$$

is a right hand limit and requires us to only look at values of x that are greater than a. Likewise,

$$\lim_{x \to a^-} f(x)$$

is a left hand limit and requires us to only look at values of x that are less than a.

In other words, we will have $\lim_{x \to a} f(x) = L$ provided $f(x)$ approaches L as we move in towards $x = a$ (without letting $x = a$) from both sides.

Now, notice that in this case there are only two paths that we can take as we move in towards $x = a$. We can either move in from the left or we can move in from the right. Then in order for the limit of a function of one variable to exist the function must be approaching the same value as we take each of these paths in towards $x = a$.

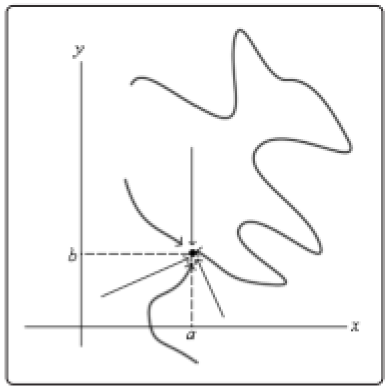
With functions of two variables we will have to do something similar, except this time there is (potentially) going to be a lot more work involved. Let's first address the notation and get a feel for just what we're going to be asking for in these kinds of limits.

We will be asking to take the limit of the function $f(x, y)$ as x approaches a and as y approaches b. This can be written in several ways. Here are a couple of the more standard notations.

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) \qquad \lim_{(x,y) \to (a,b)} f(x, y)$$

We will use the second notation more often than not in this course. The second notation is also a little more helpful in illustrating what we are really doing here when we are taking a limit. In taking a limit of a function of two variables we are really asking what the value of $f(x, y)$ is doing as we move the point (x, y) in closer and closer to the point (a, b) without actually letting it be (a, b).

Just like with limits of functions of one variable, in order for this limit to exist, the function must be approaching the same value regardless of the path that we take as we move in towards (a, b). The problem that we are immediately faced with is that there are literally an infinite number of paths that we can take as we move in towards (a, b). Here are a few examples of paths that we could take.



We put in a couple of straight line paths as well as a couple of "stranger" paths that aren't straight line paths. Also, we only included 6 paths here and as you can see simply by varying the slope of the straight line paths there are an infinite number of these and then we would need to consider paths that aren't straight line paths.

In other words, to show that a limit exists we would technically need to check an infinite number of paths and verify that the function is approaching the same value regardless of the path we are using to approach the point.

Luckily for us however we can use one of the main ideas from Calculus I limits to help us take limits here.

Definition

A function $f(x, y)$ is **continuous** at the point (a, b) if,

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

From a graphical standpoint this definition means the same thing as it did when we first saw continuity in Calculus I. A function will be continuous at a point if the graph doesn't have any holes or breaks at that point.

How can this help us take limits? Well, just as in Calculus I, if you know that a function is continuous at (a, b) then you also know that

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

must be true. So, if we know that a function is continuous at a point then all we need to do to take the limit of the function at that point is to plug the point into the function.

All the standard functions that we know to be continuous are still continuous even if we are plugging in more than one variable now. We just need to watch out for division by zero, square roots of negative numbers, logarithms of zero or negative numbers, *etc.*

Note that the idea about paths is one that we shouldn't forget since it is a nice way to determine if a limit doesn't exist. If we can find two paths upon which the function approaches different values as we get near the point then we will know that the limit doesn't exist.

Let's take a look at a couple of examples.

Example 53

Determine if the following limits exist or not. If they do exist give the value of the limit.

(a) $\lim\limits_{(x,y,z)\to(2,1,-1)} \left(3x^2z + yx\cos(\pi x - \pi z)\right)$

(b) $\lim\limits_{(x,y)\to(5,1)} \dfrac{xy}{x+y}$

Solution

(a) $\lim\limits_{(x,y,z)\to(2,1,-1)} \left(3x^2z + yx\cos(\pi x - \pi z)\right)$

Okay, in this case the function is continuous at the point in question and so all we need to do is plug in the values and we're done.

$$\lim_{(x,y,z)\to(2,1,-1)} \left(3x^2z + yx\cos(\pi x - \pi z)\right) = 3(2)^2(-1) + (1)(2)\cos(2\pi + \pi) = -14$$

(b) $\lim\limits_{(x,y)\to(5,1)} \dfrac{xy}{x+y}$

In this case the function will not be continuous along the line $y = -x$ since we will get division by zero when this is true. However, for this problem that is not something that we will need to worry about since the point that we are taking the limit at isn't on this line.

Therefore, all that we need to do is plug in the point since the function is continuous at this point.

$$\lim_{(x,y)\to(5,1)} \frac{xy}{x+y} = \frac{5}{6}$$

In the previous example there wasn't really anything to the limits. The functions were continuous at the point in question and so all we had to do was plug in the point. That, of course, will not always be the case so let's work a few examples that are more typical of those you'll see here.

Example 54

Determine if the following limit exist or not. If they do exist give the value of the limit.

$$\lim_{(x,y)\to(1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2}$$

Solution

In this case the function is not continuous at the point in question (clearly division by zero). However, that does not mean that the limit can't be done. We saw many examples of this in Calculus I where the function was not continuous at the point we were looking at and yet the limit did exist.

In the case of this limit notice that we can factor both the numerator and denominator of the function as follows,

$$\lim_{(x,y)\to(1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2} = \lim_{(x,y)\to(1,1)} \frac{(2x + y)(x - y)}{(x - y)(x + y)} = \lim_{(x,y)\to(1,1)} \frac{2x + y}{x + y}$$

So, just as we saw in many examples in Calculus I, upon factoring and canceling common factors we arrive at a function that in fact we can take the limit of. So, to finish out this example all we need to do is actually take the limit.

Taking the limit gives,

$$\lim_{(x,y)\to(1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2} = \lim_{(x,y)\to(1,1)} \frac{2x + y}{x + y} = \frac{3}{2}$$

Before we move on to the next set of examples we should note that the situation in the previous example is what generally happened in many limit examples/problems in Calculus I. In Calculus III however, this tends to be the exception in the examples/problems as the next set of examples will show. In other words, do not expect most of these types of limits to just factor and then exist as they did in Calculus I.

Example 55

Determine if the following limits exist or not. If they do exist give the value of the limit.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^4+3y^4}$

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^6+y^2}$

Solution

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^4+3y^4}$

In this case the function is not continuous at the point in question and so we can't just plug in the point. Also, note that, unlike the previous example, we can't factor this function and do some canceling so that the limit can be taken.

Therefore, since the function is not continuous at the point and because there is no factoring we can do, there is at least a chance that the limit doesn't exist. If we could find two different paths to approach the point that gave different values for the limit then we would know that the limit didn't exist. Two of the more common paths to check are the x and y-axis so let's try those.

Before actually doing this we need to address just what exactly do we mean when we say that we are going to approach a point along a path. When we approach a point along a path we will do this by either fixing x or y or by relating x and y through some function. In this way we can reduce the limit to just a limit involving a single variable which we know how to do from Calculus I.

So, let's see what happens along the x-axis. If we are going to approach $(0,0)$ along the x-axis we can take advantage of the fact that that along the x-axis we know that $y=0$. This means that, along the x-axis, we will plug in $y=0$ into the function and then take the limit as x approaches zero.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(x,0)\to(0,0)} \frac{x^2(0)^2}{x^4+3(0)^4} = \lim_{(x,0)\to(0,0)} 0 = 0$$

So, along the x-axis the function will approach zero as we move in towards the origin.

Now, let's try the y-axis. Along this axis we have $x=0$ and so the limit becomes,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(0,y)\to(0,0)} \frac{(0)^2y^2}{(0)^4+3y^4} = \lim_{(0,y)\to(0,0)} 0 = 0$$

So, the same limit along two paths. Don't misread this. This does NOT say that the limit exists and has a value of zero. This only means that the limit happens to have the same value along two paths.

Let's take a look at a third fairly common path to take a look at. In this case we'll move in towards the origin along the path $y = x$. This is what we meant previously about relating x and y through a function.

To do this we will replace all the y's with x's and then let x approach zero. Let's take a look at this limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(x,x)\to(0,0)} \frac{x^2x^2}{x^4+3x^4} = \lim_{(x,x)\to(0,0)} \frac{x^4}{4x^4} = \lim_{(x,x)\to(0,0)} \frac{1}{4} = \frac{1}{4}$$

So, a different value from the previous two paths and this means that the limit can't possibly exist.

Note that we can use this idea of moving in towards the origin along a line with the more general path $y = mx$ if we need to.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$

Okay, with this last one we again have continuity problems at the origin and again there is no factoring we can do that will allow the limit to be taken. So, again let's see if we can find a couple of paths that give different values of the limit.

First, we will use the path $y = x$. Along this path we have,

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^3x}{x^6+x^2} = \lim_{(x,x)\to(0,0)} \frac{x^4}{x^6+x^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^4+1} = 0$$

Now, let's try the path $y = x^3$. Along this path the limit becomes,

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim_{(x,x^3)\to(0,0)} \frac{x^3x^3}{x^6+(x^3)^2} = \lim_{(x,x^3)\to(0,0)} \frac{x^6}{2x^6} = \lim_{(x,x^3)\to(0,0)} \frac{1}{2} = \frac{1}{2}$$

We now have two paths that give different values for the limit and so the limit doesn't exist.

As this limit has shown us we can, and often need, to use paths other than lines like we did in the first part of this example.

So, as we've seen in the previous example limits are a little different here from those we saw in Calculus I. Limits in multiple variables can be quite difficult to evaluate and we've shown several examples where it took a little work just to show that the limit does not exist.

2.2 Partial Derivatives

Now that we have the brief discussion on limits out of the way we can proceed into taking derivatives of functions of more than one variable. Before we actually start taking derivatives of functions of more than one variable let's recall an important interpretation of derivatives of functions of one variable.

Recall that given a function of one variable, $f(x)$, the derivative, $f'(x)$, represents the rate of change of the function as x changes. This is an important interpretation of derivatives and we are not going to want to lose it with functions of more than one variable. The problem with functions of more than one variable is that there is more than one variable. In other words, what do we do if we only want one of the variables to change, or if we want more than one of them to change? In fact, if we're going to allow more than one of the variables to change there are then going to be an infinite amount of ways for them to change. For instance, one variable could be changing faster than the other variable(s) in the function. Notice as well that it will be completely possible for the function to be changing differently depending on how we allow one or more of the variables to change.

We will need to develop ways, and notations, for dealing with all of these cases. In this section we are going to concentrate exclusively on only changing one of the variables at a time, while the remaining variable(s) are held fixed. We will deal with allowing multiple variables to change in a later section.

Because we are going to only allow one of the variables to change taking the derivative will now become a fairly simple process. Let's start off this discussion with a fairly simple function.

Let's start with the function $f(x,y) = 2x^2y^3$ and let's determine the rate at which the function is changing at a point, (a,b), if we hold y fixed and allow x to vary and if we hold x fixed and allow y to vary.

We'll start by looking at the case of holding y fixed and allowing x to vary. Since we are interested in the rate of change of the function at (a,b) and are holding y fixed this means that we are going to always have $y = b$ (if we didn't have this then eventually y would have to change in order to get to the point...). Doing this will give us a function involving only x's and we can define a new function as follows,

$$g(x) = f(x,b) = 2x^2b^3$$

Now, this is a function of a single variable and at this point all that we are asking is to determine the rate of change of $g(x)$ at $x = a$. In other words, we want to compute $g'(a)$ and since this is a function of a single variable we already know how to do that. Here is the rate of change of the function at (a,b) if we hold y fixed and allow x to vary.

$$g'(a) = 4ab^3$$

We will call $g'(a)$ the **partial derivative** of $f(x,y)$ with respect to x at (a,b) and we will denote it in the following way,

$$f_x(a,b) = 4ab^3$$

Now, let's do it the other way. We will now hold x fixed and allow y to vary. We can do this in a similar way. Since we are holding x fixed it must be fixed at $x = a$ and so we can define a new function of y and then differentiate this as we've always done with functions of one variable.

Here is the work for this,

$$h(y) = f(a, y) = 2a^2y^3 \quad \Rightarrow \quad h'(b) = 6a^2b^2$$

In this case we call $h'(b)$ the **partial derivative** of $f(x, y)$ with respect to y at (a, b) and we denote it as follows,

$$f_y(a, b) = 6a^2b^2$$

Note that these two partial derivatives are sometimes called the **first order partial derivatives**. Just as with functions of one variable we can have derivatives of all orders. We will be looking at higher order derivatives in a later section.

Note that the notation for partial derivatives is different than that for derivatives of functions of a single variable. With functions of a single variable we could denote the derivative with a single prime. However, with partial derivatives we will always need to remember the variable that we are differentiating with respect to and so we will subscript the variable that we differentiated with respect to. We will shortly be seeing some alternate notation for partial derivatives as well.

Note as well that we usually don't use the (a, b) notation for partial derivatives as that implies we are working with a specific point which we usually are not doing. The more standard notation is to just continue to use (x, y). So, the partial derivatives from above will more commonly be written as,

$$f_x(x, y) = 4xy^3 \quad \text{and} \quad f_y(x, y) = 6x^2y^2$$

Now, as this quick example has shown taking derivatives of functions of more than one variable is done in pretty much the same manner as taking derivatives of a single variable. To compute $f_x(x, y)$ all we need to do is treat all the y's as constants (or numbers) and then differentiate the x's as we've always done. Likewise, to compute $f_y(x, y)$ we will treat all the x's as constants and then differentiate the y's as we are used to doing.

Before we work any examples let's get the formal definition of the partial derivative out of the way as well as some alternate notation.

Since we can think of the two partial derivatives above as derivatives of single variable functions it shouldn't be too surprising that the definition of each is very similar to the definition of the derivative for single variable functions. Here are the formal definitions of the two partial derivatives we looked at above.

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \qquad f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

If you recall the Calculus I definition of the limit these should look familiar as they are very close to the Calculus I definition with a (possibly) obvious change.

Now let's take a quick look at some of the possible alternate notations for partial derivatives. Given the function $z = f(x, y)$ the following are all equivalent notations,

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\Big(f(x, y)\Big) = z_x = \frac{\partial z}{\partial x} = D_x f$$
$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\Big(f(x, y)\Big) = z_y = \frac{\partial z}{\partial y} = D_y f$$

For the fractional notation for the partial derivative notice the difference between the partial derivative and the ordinary derivative from single variable calculus.

$$f(x) \quad \Rightarrow \quad f'(x) = \frac{df}{dx}$$
$$f(x, y) \quad \Rightarrow \quad f_x(x, y) = \frac{\partial f}{\partial x} \text{ \& } f_y(x, y) = \frac{\partial f}{\partial y}$$

Okay, now let's work some examples. When working these examples always keep in mind that we need to pay very close attention to which variable we are differentiating with respect to. This is important because we are going to treat all other variables as constants and then proceed with the derivative as if it was a function of a single variable. If you can remember this you'll find that doing partial derivatives are not much more difficult that doing derivatives of functions of a single variable as we did in Calculus I.

Example 56

Find all of the first order partial derivatives for the following functions.

(a) $f(x, y) = x^4 + 6\sqrt{y} - 10$

(b) $w = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$

(c) $h(s, t) = t^7 \ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4}$

(d) $f(x, y) = \cos\left(\frac{4}{x}\right) \mathbf{e}^{x^2y - 5y^3}$

Solution

(a) $f(x, y) = x^4 + 6\sqrt{y} - 10$

Let's first take the derivative with respect to x and remember that as we do so all the y's will be treated as constants. The partial derivative with respect to x is,

$$f_x(x, y) = 4x^3$$

Notice that the second and the third term differentiate to zero in this case. It should be clear why the third term differentiated to zero. It's a constant and we know that constants always differentiate to zero. This is also the reason that the second term differentiated to zero. Remember that since we are differentiating with respect to x here we are going to treat all y's as constants. That means that terms that only involve y's will be treated as constants and hence will differentiate to zero.

Now, let's take the derivative with respect to y. In this case we treat all x's as constants and so the first term involves only x's and so will differentiate to zero, just as the third term will. Here is the partial derivative with respect to y.

$$f_y(x,y) = \frac{3}{\sqrt{y}}$$

(b) $w = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$

With this function we've got three first order derivatives to compute. Let's do the partial derivative with respect to x first. Since we are differentiating with respect to x we will treat all y's and all z's as constants. This means that the second and fourth terms will differentiate to zero since they only involve y's and z's.

This first term contains both x's and y's and so when we differentiate with respect to x the y will be thought of as a multiplicative constant and so the first term will be differentiated just as the third term will be differentiated.

Here is the partial derivative with respect to x.

$$\frac{\partial w}{\partial x} = 2xy + 43$$

Let's now differentiate with respect to y. In this case all x's and z's will be treated as constants. This means the third term will differentiate to zero since it contains only x's while the x's in the first term and the z's in the second term will be treated as multiplicative constants. Here is the derivative with respect to y.

$$\frac{\partial w}{\partial y} = x^2 - 20yz^3 - 28\sec^2(4y)$$

Finally, let's get the derivative with respect to z. Since only one of the terms involve z's this will be the only non-zero term in the derivative. Also, the y's in that term will be treated as multiplicative constants. Here is the derivative with respect to z.

$$\frac{\partial w}{\partial z} = -30y^2z^2$$

(c) $h(s,t) = t^7 \ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4}$

With this one we'll not put in the detail of the first two. Before taking the derivative let's rewrite the function a little to help us with the differentiation process.

$$h(s,t) = t^7 \ln(s^2) + 9t^{-3} - s^{\frac{4}{7}}$$

Now, the fact that we're using s and t here instead of the "standard" x and y shouldn't be a problem. It will work the same way. Here are the two derivatives for this function.

$$h_s(s,t) = \frac{\partial h}{\partial s} = t^7\left(\frac{2s}{s^2}\right) - \frac{4}{7}s^{-\frac{3}{7}} = \frac{2t^7}{s} - \frac{4}{7}s^{-\frac{3}{7}}$$
$$h_t(s,t) = \frac{\partial h}{\partial t} = 7t^6 \ln(s^2) - 27t^{-4}$$

Remember how to differentiate natural logarithms.

$$\frac{d}{dx}\Big(\ln(g)(x)\Big) = \frac{g'(x)}{g(x)}$$

(d) $f(x,y) = \cos\left(\frac{4}{x}\right) \mathbf{e}^{x^2y-5y^3}$

Now, we can't forget the product rule with derivatives. The product rule will work the same way here as it does with functions of one variable. We will just need to be careful to remember which variable we are differentiating with respect to.

Let's start out by differentiating with respect to x. In this case both the cosine and the exponential contain x's and so we've really got a product of two functions involving x's and so we'll need to product rule this up. Here is the derivative with respect to x.

$$f_x(x,y) = -\sin\left(\frac{4}{x}\right)\left(-\frac{4}{x^2}\right)\mathbf{e}^{x^2y-5y^3} + \cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}(2xy)$$
$$= \frac{4}{x^2}\sin\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3} + 2xy\cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}$$

Do not forget the chain rule for functions of one variable. We will be looking at the chain rule for some more complicated expressions for multivariable functions in a later section. However, at this point we're treating all the y's as constants and so the chain rule will continue to work as it did back in Calculus I.

Also, don't forget how to differentiate exponential functions,

$$\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\,\mathbf{e}^{f(x)}$$

Now, let's differentiate with respect to y. In this case we don't have a product rule to worry about since the only place that the y shows up is in the exponential. Therefore, since x's are considered to be constants for this derivative, the cosine in the front will also be thought of as a multiplicative constant. Here is the derivative with respect to y.

$$f_y(x,y) = \left(x^2 - 15y^2\right)\cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}$$

Example 57

Find all of the first order partial derivatives for the following functions.

(a) $z = \dfrac{9u}{u^2+5v}$

(b) $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$

(c) $z = \sqrt{x^2 + \ln(5x - 3y^2)}$

Solution

(a) $z = \dfrac{9u}{u^2+5v}$

We also can't forget about the quotient rule. Since there isn't too much to this one, we will simply give the derivatives.

$$z_u = \frac{9\left(u^2+5v\right) - 9u(2u)}{\left(u^2+5v\right)^2} = \frac{-9u^2+45v}{\left(u^2+5v\right)^2}$$
$$z_v = \frac{(0)\left(u^2+5v\right) - 9u(5)}{\left(u^2+5v\right)^2} = \frac{-45u}{\left(u^2+5v\right)^2}$$

In the case of the derivative with respect to v recall that u's are constant and so when we differentiate the numerator we will get zero!

(b) $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$

Now, we do need to be careful however to not use the quotient rule when it doesn't need to be used. In this case we do have a quotient, however, since the x's and y's only appear in the numerator and the z's only appear in the denominator this really isn't a quotient rule problem.

Let's do the derivatives with respect to x and y first. In both these cases the z's are constants and so the denominator in this is a constant and so we don't really need to worry too much about it. Here are the derivatives for these two cases.

$$g_x(x,y,z) = \frac{\sin(y)}{z^2} \qquad g_y(x,y,z) = \frac{x\cos(y)}{z^2}$$

Now, in the case of differentiation with respect to z we can avoid the quotient rule with a quick rewrite of the function. Here is the rewrite as well as the derivative with respect to z.

$$\begin{aligned} g(x,y,z) &= x\sin(y)\,z^{-2} \\ g_z(x,y,z) &= -2x\sin(y)\,z^{-3} = -\frac{2x\sin(y)}{z^3} \end{aligned}$$

We went ahead and put the derivative back into the "original" form just so we could say that we did. In practice you probably don't really need to do that.

(c) $z = \sqrt{x^2 + \ln(5x - 3y^2)}$

In this last part we are just going to do a somewhat messy chain rule problem. However, if you had a good background in Calculus I chain rule this shouldn't be all that difficult of a problem. Here are the two derivatives,

$$\begin{aligned} z_x &= \frac{1}{2}\left(x^2 + \ln(5x - 3y^2)\right)^{-\frac{1}{2}} \frac{\partial}{\partial x}\left(x^2 + \ln(5x - 3y^2)\right) \\ &= \frac{1}{2}\left(x^2 + \ln(5x - 3y^2)\right)^{-\frac{1}{2}} \left(2x + \frac{5}{5x - 3y^2}\right) \\ &= \left(x + \frac{5}{2(5x - 3y^2)}\right)\left(x^2 + \ln(5x - 3y^2)\right)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} z_y &= \frac{1}{2}\left(x^2 + \ln(5x - 3y^2)\right)^{-\frac{1}{2}} \frac{\partial}{\partial y}\left(x^2 + \ln(5x - 3y^2)\right) \\ &= \frac{1}{2}\left(x^2 + \ln(5x - 3y^2)\right)^{-\frac{1}{2}} \left(\frac{-6y}{5x - 3y^2}\right) \\ &= -\frac{3y}{5x - 3y^2}\left(x^2 + \ln(5x - 3y^2)\right)^{-\frac{1}{2}} \end{aligned}$$

So, there are some examples of partial derivatives. Hopefully you will agree that as long as we can remember to treat the other variables as constants these work in exactly the same manner that derivatives of functions of one variable do. So, if you can do Calculus I derivatives you shouldn't have too much difficulty in doing basic partial derivatives.

There is one final topic that we need to take a quick look at in this section, implicit differentiation. Before getting into implicit differentiation for multiple variable functions let's first remember how implicit differentiation works for functions of one variable.

Example 58

Find $\dfrac{dy}{dx}$ for $3y^4 + x^7 = 5x$.

Solution

Remember that the key to this is to always think of y as a function of x, or $y = y(x)$ and so whenever we differentiate a term involving y's with respect to x we will really need to use the chain rule which will mean that we will add on a $\frac{dy}{dx}$ to that term.

The first step is to differentiate both sides with respect to x.

$$12y^3\frac{dy}{dx} + 7x^6 = 5$$

The final step is to solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{5 - 7x^6}{12y^3}$$

Now, we did this problem because implicit differentiation works in exactly the same manner with functions of multiple variables. If we have a function in terms of three variables x, y, and z we will assume that z is in fact a function of x and y. In other words, $z = z(x, y)$. Then whenever we differentiate z's with respect to x we will use the chain rule and add on a $\frac{\partial z}{\partial x}$. Likewise, whenever we differentiate z's with respect to y we will add on a $\frac{\partial z}{\partial y}$.

Let's take a quick look at a couple of implicit differentiation problems.

Example 59

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for each of the following functions.

(a) $x^3z^2 - 5xy^5z = x^2 + y^3$

(b) $x^2 \sin(2y - 5z) = 1 + y\cos(6zx)$

Solution

(a) $x^3z^2 - 5xy^5z = x^2 + y^3$

Let's start with finding $\frac{\partial z}{\partial x}$. We first will differentiate both sides with respect to x and remember to add on a $\frac{\partial z}{\partial x}$ whenever we differentiate a z from the chain rule.

$$3x^2z^2 + 2x^3z\frac{\partial z}{\partial x} - 5y^5z - 5xy^5\frac{\partial z}{\partial x} = 2x$$

Remember that since we are assuming $z = z(x, y)$ then any product of x's and z's will be a product and so will need the product rule!

Now, solve for $\frac{\partial z}{\partial x}$.

$$\begin{aligned}\left(2x^3z - 5xy^5\right)\frac{\partial z}{\partial x} &= 2x - 3x^2z^2 + 5y^5z \\ \frac{\partial z}{\partial x} &= \frac{2x - 3x^2z^2 + 5y^5z}{2x^3z - 5xy^5}\end{aligned}$$

Now we'll do the same thing for $\frac{\partial z}{\partial y}$ except this time we'll need to remember to add on a $\frac{\partial z}{\partial y}$ whenever we differentiate a z from the chain rule.

$$\begin{aligned}2x^3z\frac{\partial z}{\partial y} - 25xy^4z - 5xy^5\frac{\partial z}{\partial y} &= 3y^2 \\ \left(2x^3z - 5xy^5\right)\frac{\partial z}{\partial y} &= 3y^2 + 25xy^4z \\ \frac{\partial z}{\partial y} &= \frac{3y^2 + 25xy^4z}{2x^3z - 5xy^5}\end{aligned}$$

(b) $x^2\sin(2y - 5z) = 1 + y\cos(6zx)$

We'll do the same thing for this function as we did in the previous part. First let's find $\frac{\partial z}{\partial x}$.

$$2x\sin(2y - 5z) + x^2\cos(2y - 5z)\left(-5\frac{\partial z}{\partial x}\right) = -y\sin(6zx)\left(6z + 6x\frac{\partial z}{\partial x}\right)$$

Don't forget to do the chain rule on each of the trig functions and when we are differentiating the inside function on the cosine we will need to also use the product rule. Now let's solve for $\frac{\partial z}{\partial x}$.

$$\begin{aligned}2x\sin(2y - 5z) - 5\frac{\partial z}{\partial x}x^2\cos(2y - 5z) &= -6zy\sin(6zx) - 6yx\sin(6zx)\frac{\partial z}{\partial x} \\ 2x\sin(2y - 5z) + 6zy\sin(6zx) &= \left(5x^2\cos(2y - 5z) - 6yx\sin(6zx)\right)\frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial x} &= \frac{2x\sin(2y - 5z) + 6zy\sin(6zx)}{5x^2\cos(2y - 5z) - 6yx\sin(6zx)}\end{aligned}$$

Now let's take care of $\frac{\partial z}{\partial y}$. This one will be slightly easier than the first one.

$$\begin{aligned} x^2 \cos(2y-5z)\left(2-5\frac{\partial z}{\partial y}\right) &= \cos(6zx) - y\sin(6zx)\left(6x\frac{\partial z}{\partial y}\right) \\ 2x^2\cos(2y-5z) - 5x^2\cos(2y-5z)\frac{\partial z}{\partial y} &= \cos(6zx) - 6xy\sin(6zx)\frac{\partial z}{\partial y} \\ \left(6xy\sin(6zx) - 5x^2\cos(2y-5z)\right)\frac{\partial z}{\partial y} &= \cos(6zx) - 2x^2\cos(2y-5z) \\ \frac{\partial z}{\partial y} &= \frac{\cos(6zx) - 2x^2\cos(2y-5z)}{6xy\sin(6zx) - 5x^2\cos(2y-5z)} \end{aligned}$$

There's quite a bit of work to these. We will see an easier way to do implicit differentiation in a later section.