

التوزيعات الاحتمالية: م / مني طاهر غافل

التوقع الرياضي Mathematical Expectation

أولاً: إذا كان x متغير عشوائي متقطع وبدالة كتلة احتمالية probability mass function $p(x_i)$ فأن توقعه أو وسطه the mean of x (p.m.f) هو كالتالي:

$$E(x) = \mu = \sum_{i=0}^n x_i p(x_i), \quad x_i = 0, 1, 2, \dots, n$$

ثانياً: إذا كان x متغير عشوائي مستمر وبدالة كثافة احتمالية probability density function of x (p.d.f) $f(x_i)$ فأن توقعه أو وسطه the mean هو

$$E(x) = \mu = \int_{-\infty}^{\infty} x_i f(x_i) dx, \quad -\infty \leq x_i \leq \infty$$

Variance التباين

يرمز لتبابن المتغير العشوائي x ، $\text{Var}(x)$ أو δ^2 حيث أن:

$$\delta^2 = E(x^2) - [E(x)]^2$$

علما بأن: في حالة التوزيع المتقطع:

$$E(x^2) = \sum_{i=0}^n x_i^2 p(x_i), \quad x_i = 0, 1, 2, \dots, n$$

أما في حالة التوزيع المستمر:

$$E(x^2) = \int_{-\infty}^{\infty} x_i^2 f(x_i) dx, \quad -\infty \leq x_i \leq \infty$$

Example: Consider the distribution of x is

x_i	1	2	3	4	5	6
$p(x_i)$	1/36	3/36	5/36	7/36	9/36	11/36

Find the mean and variance of x.

The solution:

$$\text{The mean is } E(x) = \mu_x = \sum_{i=0}^n x_i p(x_i)$$

$$= \sum_{i=1}^6 x_i p(x_i)$$

$$= (1)\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{7}{36}\right) + 5\left(\frac{9}{36}\right) + 6\left(\frac{11}{36}\right)$$

$$= 4.47$$

$$\begin{aligned} E(x^2) &= \sum_{i=1}^6 x_i^2 p(x_i) \\ &= (1)^2 \left(\frac{1}{36}\right) + (2)^2 \left(\frac{3}{36}\right) + (3)^2 \left(\frac{5}{36}\right) + (4)^2 \left(\frac{7}{36}\right) + (5)^2 \left(\frac{9}{36}\right) \\ &\quad + (6)^2 \left(\frac{11}{36}\right) = 21.97 \end{aligned}$$

$$\begin{aligned} \text{The variance of } x &= \text{var}(x) = \delta_x^2 = E(x^2) - [E(x)]^2 \\ &= 21.97 - (4.47)^2 = 1.99 \end{aligned}$$

Example: Suppose the continuous distribution function is:

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{o.w.} \end{cases}$$

Find the mean and variance of x.

the solution:

$$\text{The mean of } x = \mu_x = E(x) = \int_0^1 x \, dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2} - \frac{0}{2} = \frac{1}{2}$$

$$E(x^2) = \int_0^1 x^2 \, dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$$

$$\text{The variance of } x = \sigma_x^2 = E(x^2) - [E(x)]^2 = \frac{1}{3} - \left[\frac{1}{2}\right]^2 = \frac{1}{12}$$

Example: Let the probability mass function of x be given by

x	-1	0	1	2
P(x)	1/8	3/8	1/4	1/4

Find the mean and variance of x.

The solution:

$$\text{The mean is } E(x) = \mu_x = \sum_{i=0}^n x_i p(x_i)$$

$$= \sum_{\text{all } x} x_i p(x_i)$$

$$= (-1)(1/8) + (0)(3/8) + (1)(1/4) + (2)(1/4) \\ = 5/8$$

$$E(x^2) = \sum_{i=0}^n x_i^2 p(x_i)$$

$$= (-1)^2 \left(\frac{1}{8}\right) + (0)^2 \left(\frac{3}{8}\right) + (1)^2 \left(\frac{1}{4}\right) + (2)^2 \left(\frac{1}{4}\right) = 11/8$$

$$\text{The variance of } x = \sigma_x^2 = E(x^2) - [E(x)]^2$$

$$= 11/8 - [5/8]^2 = 0.98$$

Example: Suppose the continuous distribution function is:

$$f(x) = \begin{cases} 1/10, & 20 < x < 30 \\ 0, & \text{o.w} \end{cases}$$

Find the mean and variance of x.

The solution:

$$\text{The mean of } x = \mu_x = E(x) = \int_{20}^{30} x * \frac{1}{10} dx = \left(\frac{1}{10}\right) * \left[\frac{x^2}{2}\right]_{20}^{30} \\ = \frac{1}{10} * \left[\frac{(30)^2 - (20)^2}{2}\right] = 25$$

$$E(x^2) = \int_{20}^{30} x^2 * \frac{1}{10} dx = \left(\frac{1}{10}\right) * \left[\frac{x^3}{3}\right]_{20}^{30} = \frac{1}{10} * \left[\frac{(30)^3 - (20)^3}{3}\right] = 633\frac{1}{3}$$

$$\text{The variance of } x = \sigma_x^2 = E(x^2) - [E(x)]^2 = 633\frac{1}{3} - [25]^2 = 8\frac{1}{3}$$

Example: A random variable X has the probability distribution shown in the table below

X	2	4	6	8
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$P(x)$	7k	5k	3k	k
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a) Calculate the value of the constant k.

b) Find $\text{Var}(x)$.

Solution:

a) To find k, the sum of the probability must be 1 so

$$7k + 5k + 3k + k = 1$$

$$16k = 1$$

$$\therefore k = 1/16$$

b) The probability table now is

X	2	4	6	8
$P(x)$	$7/16$	$5/16$	$3/16$	$1/16$

$$E(x) = 2 * 7/16 + 4 * 5/16 + 6 * 3/16 + 8 * 1/16 = 60/16$$

$$E(x^2) = 4 * 7/16 + 16 * 5/16 + 36 * 3/16 + 64 * 1/16 = 280/16$$

$$\therefore \text{var}(x) = \frac{280}{16} - \left(\frac{60}{16}\right)^2 = 3.44$$

Example: Let x have the p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0 & , \text{o.w} \end{cases}$$

$$\text{Show that } \mu_x = \frac{1}{3}, \sigma_x^2 = \frac{2}{9}$$

$$\text{Solution: } \mu_x = E(x) = \int_{-1}^1 x * \frac{1}{2}(x+1) dx$$

$$= \frac{1}{2} \int_{-1}^1 (x^2 + x) dx = \frac{1}{2} \left(\frac{x^3}{3} + \frac{x^2}{2} \right) \Big|_{-1}^1$$

$$= \frac{1}{2} \left[\left(\frac{1}{3} + \frac{1}{2} \right) - \left(\frac{-1}{3} + \frac{1}{2} \right) \right] = \frac{1}{2} \left[\frac{5}{6} - \frac{1}{6} \right] = \frac{1}{3}$$

$$E(x^2) = \int_{-1}^1 x^2 * \frac{1}{2}(x+1) dx = \frac{1}{2} \int_{-1}^1 (x^3 + x^2) dx = \frac{1}{2} \left(\frac{x^4}{4} + \frac{x^3}{3} \right) \Big|_{-1}^1$$

$$= \frac{1}{2} \left[\left(\frac{1}{4} + \frac{1}{3} \right) - \left(\frac{1}{4} + \frac{-1}{3} \right) \right] = \frac{1}{2} \left[\frac{7}{12} - \frac{-1}{12} \right] = \frac{1}{2} * \frac{8}{12} = \frac{1}{3}$$

$$\therefore \sigma^2 = \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{2}{9}$$

Moments العزوم

ليكن k عدد صحيح موجب

إذن العزم k هو $E(x^k)$ ، وبهذا يكون :

العزم الأول $E(x^1) = E(x) =$

العزم الثاني $E(x^2) =$

العزم الثالث $E(x^3) = \dots$ وهكذا

ولإيجاد العزم k إلى دالة كتلة احتمالية p.m.f.

$$E(x^k) = \sum_{all \ x} x_i^k p(x_i)$$

أما العزم k إلى دالة الكثافة الاحتمالية p.d.f.

$$E(x^k) = \int_{-\infty}^{\infty} x_i^k f(x_i) dx$$

Moment Generating Function m.g.f

الدالة المولدة للعزوم

The moment generating function (m.g.f) of a random variable X , is a function $M_x(t)$ defined as

$$M_x(t) = E[e^{tx}].$$

Then for discrete distribution $M_x(t) = E[e^{tx}] = \sum_{all \ x} e^{tx} p(x)$

And for continuous distribution $M_x(t) = E[e^{tx}] = \int_{-\infty}^{+\infty} e^{tx} f(x) dx$

Where $t \in real \ numbers$

We can obtain all moments of X from its m.g.f:

$$M_x(t) = E(e^{tx})$$

To find $E(x)$

$$\begin{aligned}\frac{dM_x(t)}{dt} &= M'_x(t) \\ \therefore E(x) &= M'_x(t=0)\end{aligned}$$

To find $E(x^2)$

$$\begin{aligned}\frac{d^2M_x(t)}{dt^2} &= M''_x(t) \\ \therefore E(x^2) &= M''_x(t=0)\end{aligned}$$

To find $E(x^k)$

$$\begin{aligned}\frac{d^kM_x(t)}{dt^k} &= M_x^k(t) \\ \therefore E(x^k) &= M_x^k(t=0)\end{aligned}$$

Example: If the function of x is:

$$p(x) = \begin{cases} \frac{1}{4}, & x = 2, 4, 8, 16 \\ 0, & o.w \end{cases}$$

1) Find the moment generating function of x .

2) Find the variance of x .

The solution:

$$\begin{aligned}1) M_x(t) &= E[e^{tx}] = \sum_{all\ x} e^{tx} p(x) = e^{t2} \left(\frac{1}{4}\right) + e^{t4} \left(\frac{1}{4}\right) + e^{t8} \left(\frac{1}{4}\right) + e^{t16} \left(\frac{1}{4}\right) \\ &= \left(\frac{1}{4}\right) [e^{2t} + e^{4t} + e^{8t} + e^{16t}]\end{aligned}$$

$$2) \frac{dM_x(t)}{dt} = M'_x(t) = \left(\frac{1}{4}\right) [2e^{2t} + 4e^{4t} + 8e^{8t} + 16e^{16t}]$$

$$E(x) = M'_x(t=0) = \left(\frac{1}{4}\right) [2 + 4 + 8 + 16] = \frac{30}{4} = 7.5 = \mu_x$$

$$\frac{d^2M_x(t)}{dt^2} = M''_x(t) = \left(\frac{1}{4}\right) [4e^{2t} + 16e^{4t} + 64e^{8t} + 256e^{16t}]$$

$$E(x^2) = M''_x(t=0) = \left(\frac{1}{4}\right) [4 + 16 + 64 + 256] = 85$$

The variance of $x = \sigma_x^2 = E(x^2) - [E(x)]^2 = 85 - (7.5)^2 = 28.75$

Example ممـ: Find the moment generating function of the random variables whose probability density is given by

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

And use it to find the variance of x.

The solution:

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} e^{-x} dx = \int_0^\infty e^{-x(1-t)} dx = \frac{-1}{(1-t)} e^{-x(1-t)} \Big|_0^\infty = 0 - \frac{-1}{(1-t)}$$

$$\therefore M_x(t) = \frac{1}{(1-t)}$$

$$\frac{dM_x(t)}{dt} = M'_x(t) = \frac{(1-t)(0) - (1)(-1)}{(1-t)^2} = \frac{1}{(1-t)^2} = (1-t)^{-2}$$

$$\therefore E(x) = M'_x(t=0) = (1-0)^{-2} = (1)^{-2} = 1$$

$$\frac{d^2M_x(t)}{dt^2} = M''_x(t) = (-2)(1-t)^{-3}(-1) = 2(1-t)^{-3}$$

$$\therefore E(x^2) = M''_x(t=0) = 2(1-0)^{-3} = 2(1)^{-3} = 2$$

The variance of $x = \sigma_x^2 = E(x^2) - [E(x)]^2 = 2 - (1)^2 = 1$

Some Special Discrete & Continuous Distributions:

1-Discrete Uniform Distribution التوزيع المنتظم المتقطع

$$P(x) = \frac{1}{N} \quad x = a, a+1, a+2, \dots, a+N-1$$

حيث: $N = \text{الحد الأعلى} - \text{الحد الأدنى} + 1$

$$1 + a - a + N - 1 = N$$

$$\mu = \frac{N+1}{2} \quad , \quad \sigma^2 = \frac{N^2-1}{12}$$

Example: Suppose that we numbered five balls from 1 to 5, and we selected one ball at random. Let x be the number of the selected ball. Find the Probability mass function of x?

the solution:

$$x = 1, 2, 3, 4, 5$$

N = 5-1+1

∴ The probability mass function of x is : $p(x) = \frac{1}{5}$ $x = 1,2,3,4,5$

2- Continuo Uniform Distribution: التوزيع المنتظم المستمر

$$f(x) = \frac{1}{b-a} , \quad a < x < b$$

$$\mu = \frac{a+b}{2} , \quad \sigma^2 = \frac{(b-a)^2}{12}$$

Example: If x is uniformly distribution over (0,10), calculate the $P(3 < x < 8)$.

The solution: $f(x) = \frac{1}{10}$, $0 < x < 10$

$$P(3 < x < 8) = \int_{3}^{8} \frac{1}{10} dx = \frac{x}{10} \Big|_3^8 = \frac{8-3}{10} = \frac{5}{10} = \frac{1}{2}$$

3-Poisson distribution:

The Poisson distribution is defined as follows:

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!} , \quad x = 0,1,2,\dots$$

where $\lambda > 0$ is some constant. This countably infinite distribution.

The mean & variance for Poisson distribution is, $\mu = \lambda$, $\sigma^2 = \lambda$

Example: The average number of homes sold in a specific city, is 2 homes per day. What is the probability that exactly 3 homes will be sold tomorrow?

The solution: $\lambda = 2$, $x = 3$

$$P(x) = \frac{e^{-2} 2^x}{x!} , \quad x = 0,1,2,\dots$$

$$P(3) = \frac{e^{-2} 2^3}{3!} = 0.180$$

4- Bernoulli Distribution: توزيع برنولي

$$P(x) = p^x (1-p)^{1-x} , \quad x = 0,1$$

$$\mu = p \quad \sigma^2 = pq$$

5- Binomial Distribution: توزيع باينوميل (توزيع ذي الحدين)

$$B(n, p) = P(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \dots, n$$

$$\mu = np \quad \sigma^2 = npq$$

Example: A fair coin is tossed once. Call the outcome a success if a head is rolled. Find the Probability mass function of x?

The Solution:

$$P(x) = p^x (1-p)^{1-x}, \quad x = 0, 1$$

$$P(x) = \left(\frac{1}{2}\right)^x \left(1 - \frac{1}{2}\right)^{1-x}, \quad x = 0, 1$$

$$x = 1, \quad P(x) = \left(\frac{1}{2}\right)^1 \left(1 - \frac{1}{2}\right)^{1-1} = \frac{1}{2}, \quad \text{نجاح المحاولة ظهور الصورة}$$

$$x = 0, \quad P(x) = \left(\frac{1}{2}\right)^0 \left(1 - \frac{1}{2}\right)^{1-0} = \frac{1}{2}, \quad \text{فشل المحاولة ظهور الكتابة}$$

ملاحظة: n من محاولات برنولي = باينوميل

Example: A fair coin is tossed 6 times. Call the outcome a success if a head is rolled.

a) Find the Probability mass function of x?

b) Find the probability that exactly two heads occur?

$$a) B(n, p) = P(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \dots, n$$

$$B\left(6, \frac{1}{2}\right) = P(x) = \binom{6}{x} \left(\frac{1}{2}\right)^x \left(1 - \frac{1}{2}\right)^{6-x}, \quad x = 0, 1, 2, 3, 4, 5, 6$$

$$b) B\left(6, \frac{1}{2}\right) = P(2) = \binom{6}{2} \left(\frac{1}{2}\right)^2 \left(1 - \frac{1}{2}\right)^{6-2} = \frac{15}{64}$$

يعطي توزيع بواسون تقريراً جيداً لتوزيع باينوميل (ذي الحدين) عندما تكون p صغيرة، حيث أن $\lambda = np$.

Example مهم: Suppose 300 misprints are distributed randomly throughout a book of 500 pages. Find the probability that a given page contains exactly 2 misprints?

The solution:

سوف ننظر إلى عدد الأخطاء في الصفحة على أنه عدد مرات النجاح في متتابعة من تجارب برنولي.

$$p = \frac{1}{500} \quad \text{احتمال ظهور خطأ في صفحة معينة} =$$

$$n = 300$$

حيث أن p صغيرة فسوف نستعمل تقرير بواسون للتوزيع بانوميل:

$$\lambda = np = 300 * \frac{1}{500} = 0.6$$

$$p(x) = \frac{(0.6)^x e^{-0.6}}{x!} \approx 0.1$$

Example: Suppose 2% of the items made by a factory are defective. Find the probability p that there are 3 defective items in a sample of 100 items.

The solution: The binomial distribution with $n = 100$ and $p = 0.02$. Since p is small, we use the poisson approximation with $\lambda = np = 2$.

$$p(x) = \frac{e^{-2}(2)^x}{x!} = 0.180$$

6- Exponential Distribution: التوزيع الأسوي

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$\mu = \frac{1}{\lambda}, \quad \sigma^2 = \frac{1}{\lambda^2}$$

Example: Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait a) more than 10 minutes. b) between 10 and 20 minutes.

أفترض أن طول المكالمة الهاتفية بالدقائق هو متغير عشوائي أسي مع المعلمة $\lambda = 1/10$. إذا وصل شخص ما أمامك مباشرة في كشك هاتف عام ، فابحث عن احتمال أن تضطر إلى الانتظار (أ) أكثر من ١٠ دقيقة. (ب) ما بين ١٠ و ٢٠ دقيقة.

The Solution:

$$a) p(x > 10) = \int_{10}^{\infty} \frac{1}{10} e^{-\frac{1}{10}x} dx = -e^{-\frac{1}{10}x} \Big|_{10}^{\infty} = -[e^{-\frac{10}{10}} - e^{-\frac{1}{10}}] = -[0 - e^{-1}] = e^{-1} = \frac{1}{e} = \frac{1}{2.71828} = 0.37$$

$$b) p(10 < x < 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{1}{10}x} dx = -e^{-\frac{1}{10}x} \Big|_{10}^{20} = -[e^{-\frac{20}{10}} - e^{-\frac{10}{10}}] = -[e^{-2} - e^{-1}] = [e^{-1} - e^{-2}] = \frac{1}{2.71828} - \frac{1}{(2.71828)^2} = 0.37 - 0.14 = 0.23$$

Example: The number of years a radio function is exponentially distribution with parameter $\lambda = \frac{1}{8}$. Jones buys a used radio, what is the probability that it will be working additional work after 8 years?

عدد سنوات التوزيع الأسوي للدالة الراديوية مع المعلمة $(\lambda = 1/8)$. جونز يشتري راديو مستعمل ، ما هو إحتمال أن يشتغل عملاً إضافياً بعد 8 سنوات؟

$$\text{The solution: } p(x > 8) = \int_8^\infty \lambda e^{-\lambda x} dx = \int_8^\infty \frac{1}{8} e^{-\frac{1}{8}x} dx = -\int_8^\infty \frac{1}{8} e^{-\frac{1}{8}x} dx = -e^{-\frac{1}{8}x} \Big|_8^\infty = -\left[e^{-\frac{8}{8}} - e^{-\frac{8}{8}}\right] = \left[e^{-1} - e^{-1}\right] = [e^{-1} - 0] = e^{-1}$$

Gamma Distribution: توزيع كاما

$$f(x) = \frac{\lambda^\alpha e^{-\lambda x} x^{\alpha-1}}{\Gamma_\alpha}, \quad 0 < x < \infty$$

$$\Gamma_\alpha = (\alpha - 1)!$$

$$\mu = \frac{\alpha}{\lambda}, \quad \sigma^2 = \frac{\alpha}{\lambda^2}$$

إذا كان المتغير العشوائي x يتوزع طبيعياً بوسط حسابي μ وتبان σ^2 فأن:

ويطلق على z التوزيع الطبيعي القياسي Standard Normal Distribution

فإذا كان المطلوب أيجاد: $P(a < x < b)$

$$\begin{aligned} P\left(\frac{a-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) &= P\left(\frac{a-\mu}{\sigma} < z < \frac{b-\mu}{\sigma}\right) = P\left(\frac{b-\mu}{\sigma}\right) - P\left(\frac{a-\mu}{\sigma}\right) \\ &= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right) \end{aligned}$$

Example: Let $x \sim N(2,25)$, Find $P(0 < x < 10)$

$$\text{The Solution: } P(0 < x < 10) = P\left(\frac{a-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) = P\left(\frac{0-2}{5} < \frac{x-2}{5} < \frac{10-2}{5}\right) =$$

$$P(-0.4 < z < 1.6) = P(z < 1.6) - P(z < -0.4) = P(z < 1.6) - (1 - P(z < 0.4)) = 0.9452 - (1 - 0.6554) = 0.6006$$

Example: Let $x \sim N(3,4)$, Find $P(x \leq 4)$

$$P\left(\frac{x-\mu}{\sigma} \leq \frac{4-3}{2}\right) = P\left(z \leq \frac{1}{2}\right) = 0.6915$$

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