Rotational Motion

- Rotational motion is an important part of everyday life.
  - The rotation of the Earth creates the cycle of day and night,
  - the rotation of wheels enables easy vehicular motion,
  - and modern technology depends on circular motion in a variety of situations, from the tiny gears (عجلات مسننة) in a Swiss watch to the operation of lathes (مخارط) and other machinery
- In the study of previous linear motion,

The important concepts are displacement Δx, velocity v, and acceleration a.

Each of these concepts has its equivalent in rotational motion: angular displacement $\Delta\theta$, angular velocity ω, and angular acceleration α.

Linear Motion	Rotational Motion
$\Delta \boldsymbol{x}$	$\Delta\theta$
$\vec{\boldsymbol{v}}$	$\vec{\omega}$
$\vec{\boldsymbol{a}}$	$\vec{\alpha}$

- The radian, a unit of angular measure, is essential to the understanding of these concepts.
- Recall that the distance s around a circle is given by s = 2πr, where r is the radius of the circle.

$$(s = 2\pi r) \div r \quad \rightarrow \quad \frac{s}{r} = 2\pi \quad \text{Unit-less}$$

- However, the value 2π corresponds to a displacement around a circle.
- A half circle would give an answer of π, a quarter circle an answer of $\frac{\pi}{2}$.

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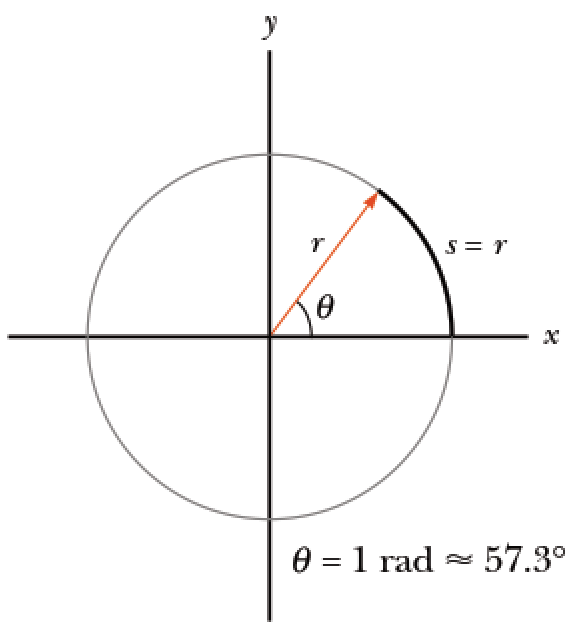
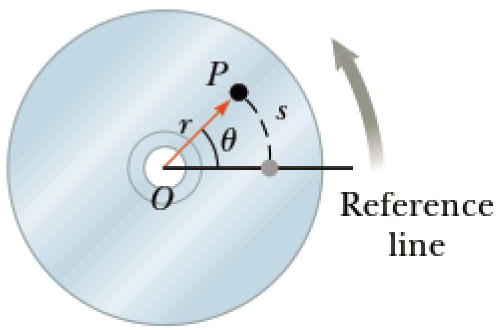
$Radians$	$Degrees$
2π	$360°$
π	$180°$
$\frac{\pi}{2}$	$90°$
$180° = \pi\, rad$	

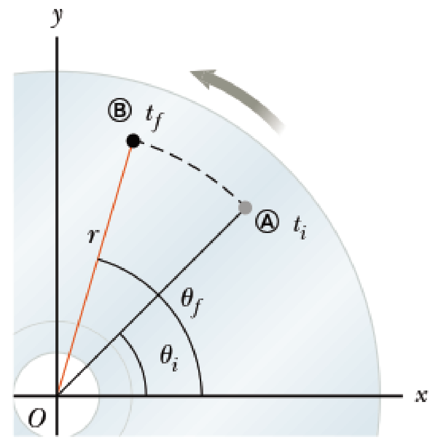
- The angle θ subtended by an arc length s along a circle of radius r, measured in radians counterclockwise from the positive x -axis, is

$$\theta = \frac{s}{r} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\,(1)$$



- The angle θ in equ.1 is actually an **angular displacement** from the positive *x-axis*, and s the corresponding displacement along the circular arc, again measured from the positive *x-axis*.
- **Angular quantities in physics must be expressed in radians.**





- In above figure as a point on the rotating compact disc moves from Ⓐ to Ⓑ in a time Δt , it starts at an angle θ_i and ends at an angle θ_f .
- The difference $\Delta\theta = \theta_f - \theta_i$ is called the **angular displacement.**

$$\Delta\theta = \theta_f - \theta_i \;\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

- The **average angular speed** ω_{av} of a rotating rigid object during the time interval Δt is the angular displacement $\Delta\theta$ divided by Δt:

$$\omega_{av} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} \quad \ldots\ldots\ldots.(3)$$

- **SI unit: radian per second (rad/s)**

- **The Instantaneous Angular speed ω of a rotating rigid object is the limit of the average speed $^{\Delta\theta}/_{\Delta t}$ as the time interval Δt approaches zero:**

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

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Counterclockwise		Positive angle
Clockwise		Negative angle

- **We take ω to be positive when θ is increasing (counterclockwise motion) and negative when θ is decreasing (clockwise motion).**
- If **angular speed** is constant ➔ $\boldsymbol{\omega = \omega_{av}}$

Example1:

The rotor on a helicopter turns at an angular speed of 3.2×10^2 rpm (revolutions per minute).

(a) Express this angular speed in radians per second.
(b) If the rotor has a radius of 2m, what arc-length does the tip of the blade trace out in 3.2×10^2 s?

SOLUTION

(a) Express this angular speed in radians per second.

Apply the conversion factors 1 rev = 2π rad and 60.0 s = 1 min:

$$\omega = 3.20 \times 10^2 \, \frac{\text{rev}}{\text{min}}$$

$$= 3.20 \times 10^2 \, \frac{\cancel{\text{rev}}}{\cancel{\text{min}}}\left(\frac{2\pi \text{ rad}}{1 \, \cancel{\text{rev}}}\right)\left(\frac{1.00 \, \cancel{\text{min}}}{60.0 \text{ s}}\right)$$

$$= \boxed{33.5 \text{ rad/s}}$$

$$\omega = \frac{\Delta\theta}{\Delta t}$$

(b) Multiply the angular speed by the time to obtain the angular displacement:

$$\Delta\theta = \omega t = (33.5 \text{ rad/s})(3.00 \times 10^2 \text{ s}) = 1.01 \times 10^4 \text{ rad}$$

Multiply the angular displacement by the radius to get the arc length:

$$\Delta s = r\Delta\theta = (2.00 \text{ m})(1.01 \times 10^4 \text{ rad}) = \boxed{2.02 \times 10^4 \text{ m}}$$

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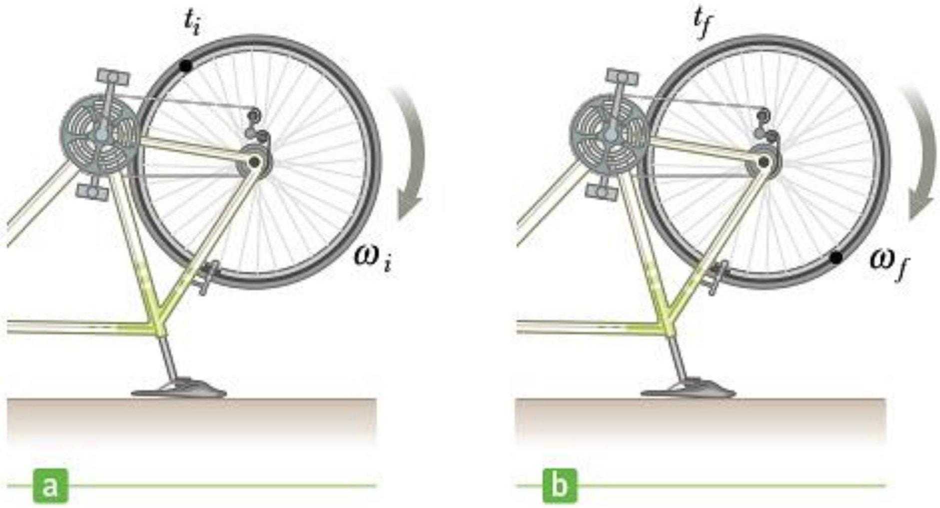


Now, this fig. shows a bicycle turned upside down so that a repair technician can work on the rear wheel. The bicycle pedals are turned so that at time, t_i the wheel has angular speed ω_i (fig.a) and at a later time, t_f it has angular speed ω_f (fig.b).

- Just as a changing speed leads to the concept of acceleration.

An object's average ***angular acceleration*** $\vec{\alpha}$ during the time interval Δt is the change in its angular speed $\Delta\vec{\omega}$ divided by Δt:

$$\alpha_{av} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \quad \text{… … … …} (5)$$

SI unit: radian per second squared ($\frac{\text{rad}}{\text{s}^2}$)

- If the angular speed goes from 15 rad/s to 9 rad/s in 3s, the average angular acceleration during that time interval is;

$$\alpha_{\text{av}} = \frac{\Delta\omega}{\Delta t} = \frac{9.0\ \text{rad/s} - 15\ \text{rad/s}}{3.0\ \text{s}} = -2.0\ \text{rad/s}^2$$

- The negative sign indicates that the angular acceleration is **clockwise** (although the angular speed, still positive but slowing down, is in the **counterclockwise** direction).
- There is also an **instantaneous** version of angular acceleration:

The instantaneous angular acceleration α is the limit of the average angular acceleration $\Delta\omega/\Delta t$ as the time interval Δt approaches zero:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(6)$$

- When a rigid object rotates about a fixed axis, as does the bicycle wheel, every portion of the object has the same angular speed and the same angular acceleration. This fact is what makes these variables so useful for describing rotational motion.
- **In contrast**, the tangential (linear) speed and acceleration of the object take different values that depend on the distance from a given point to the axis of rotation.

Rotational Motion under Constant Angular Acceleration:

- A number of parallels exist between the equations for **rotational** motion and those for **linear** motion. For example, compare the defining equation for the average angular speed,

$$\omega_{\text{av}} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

with that of the average linear speed,

$$v_{\text{av}} \equiv \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t}$$

The resulting equations of rotational kinematics, along with the corresponding equations for linear motion, are as follows:

Linear Motion with a Constant (Variables: x and v)	Rotational Motion About a Fixed Axis with α Constant (Variables: θ and ω)
$v = v_i + at$	$\omega = \omega_i + \alpha t$
$\Delta x = v_i t + \frac{1}{2}at^2$	$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$
$v^2 = v_i^2 + 2a\Delta x$	$\omega^2 = \omega_i^2 + 2\alpha\Delta\theta$

Example 2:

A wheel rotates with a constant angular acceleration of 3.5 rad/s^2. If the angular speed of the wheel is 2 rad/s at t=0, **(a)** through what angle does the wheel rotate between t=0 and t=2s? Give your answer in radians and in revolutions. **(b)** What is the angular speed of the wheel at t=2s? **(c)** What angular displacement (in revolutions) results while the angular speed found in part (b) doubles?

Solution:

(a) Find the angular displacement after 2.00 s, in both radians and revolutions.

Use Equation 7.8, setting $\omega_i = 2.00$ rad/s, $\alpha = 3.5$ rad/s^2, and $t = 2.00$ s:

$$\Delta\theta = \omega_i t + \tfrac{1}{2}\alpha t^2 = (2.00\text{ rad/s})(2.00\text{ s}) + \tfrac{1}{2}(3.50\text{ rad/s}^2)(2.00\text{ s})^2 = \boxed{11.0\text{ rad}}$$

Convert radians to revolutions.

$$\Delta\theta = (11.0\text{ rad})(1.00\text{ rev}/2\pi\text{ rad}) = \boxed{1.75\text{ rev}}$$

(b) What is the angular speed of the wheel at $t = 2.00$ s?

Substitute the same values into Equation 7.7:

$$\omega = \omega_i + \alpha t = 2.00\text{ rad/s} + (3.50\text{ rad/s}^2)(2.00\text{ s}) = \boxed{9.00\text{ rad/s}}$$

(c) What angular displacement (in revolutions) results during the time in which the angular speed found in part (b) doubles?

Apply the time-independent rotational kinematics equation:

$$\omega_f^{\,2} - \omega_i^{\,2} = 2\alpha\Delta\theta$$

Substitute values, noting that $\omega_f = 2\omega_i$:

$$(2\times 9.00\text{ rad/s})^2 - (9.00\text{ rad/s})^2 = 2(3.50\text{ rad/s}^2)\Delta\theta$$

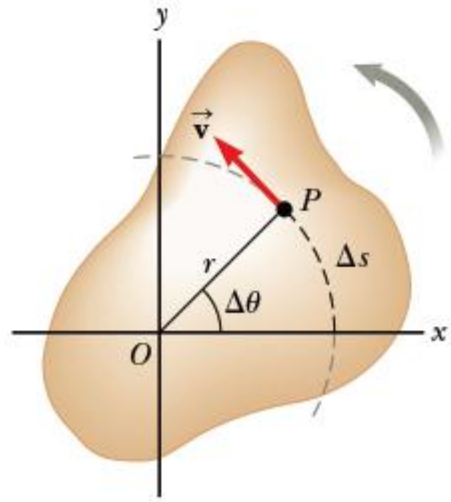
Solve for the angular displacement and convert to revolutions:

$$\Delta\theta = (34.7\text{ rad})\left(\frac{1\text{ rev}}{2\pi\text{ rad}}\right) = \boxed{5.52\text{ rev}}$$

Relations between Angular and Linear Quantities

- Angular variables are closely related to linear variables.
- Consider the arbitrarily shaped object in Active next fig. rotating about the z-axis through the point O.
- Assume the object rotates through the angle $\Delta\theta$, and hence point P moves through the arc length Δs, in the interval Δt.
- We know from the defining equation for radian measure that

$$\Delta\theta = \frac{\Delta s}{r} \ldots\ldots\ldots\ldots\ldots..(7)$$



- Dividing both sides of this equation by Δt, the time interval during which the rotation occurs, yields

$$\frac{\Delta\theta}{\Delta t} = \frac{1}{r}\ \frac{\Delta s}{\Delta t} \ldots\ldots\ldots\ldots\ldots..(8)$$

- When Δt is very small ➔ the angle $\Delta\theta$ is also small and ➔ $\Delta\theta/\Delta t$ is close to the **instantaneous angular speed** ω.
- On the other side of the equation, similarly, the ratio $\Delta s/\Delta t$ approaches the **instantaneous linear speed** v for small values of Δt. Hence, when Δt gets arbitrarily small, the preceding equation is equivalent to

$$\omega = \frac{v}{r} \ldots\ldots\ldots.\ldots\ldots\ldots.(9)$$

- In Active the fig., the point ***P*** traverses a distance Δs along a circular arc during the time interval Δt at a linear speed of v.
- The direction of ***P***'s velocity vector $\vec{V}$ is ***tangent (مماس) to the circular path***.
- The magnitude of $\vec{V}$ is the linear speed $v = v_t$, called the ***tangential speed of a particle*** moving in a circular path, written

$$v_t = \omega r \ldots\ldots\ldots.\ldots\ldots\ldots.(10)$$

- ***The tangential speed (v_t) of a point on a rotating object equals the distance of that point from the axis of rotation multiplied by the angular speed.***
- ***Every point on the rotating object has the same angular speed.***

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- Now, suppose the rotating object changes its angular speed by $\Delta\omega$ in the time interval Δt

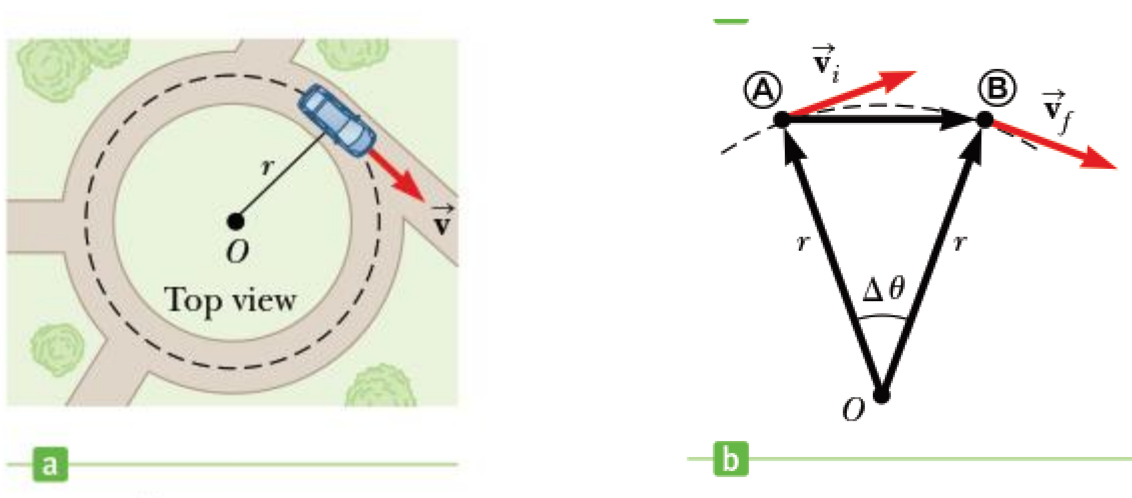
$$\Delta v_t = r\,\Delta\omega \quad \rightarrow \quad \frac{\Delta v_t}{\Delta t} = r\frac{\Delta\omega}{\Delta t} \qquad (\#)$$

- As the time interval Δt is taken to be arbitrarily small $\rightarrow$ $\frac{\Delta\omega}{\Delta t}$ approaches the **instantaneous angular acceleration**.
- On the left-hand side of the equation, note that the ratio $\frac{\Delta v_t}{\Delta t}$ tends to the instantaneous linear acceleration, called the **tangential acceleration** a_t of that point, given by

$$a_t = r\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

- ***The tangential acceleration of a point on a rotating object equals the distance of that point from the axis of rotation multiplied by the angular acceleration.***

Centripetal Acceleration:



Above Fig. a shows a car moving in a circular path with ***constant linear speed*** v. **Even though the car moves at a constant speed**, it still has acceleration. To understand this, consider the defining equation for average acceleration:

$$\vec{a}_{av} = \frac{\vec{V}_f - \vec{V}_i}{t_f - t_i} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

The numerator (البسط) represents the difference between the velocity vectors $\vec{V}_f - \vec{V}_i$. These vectors may have the same ***magnitude***, corresponding to the same speed, but if they have different *directions*, **their difference cannot equal zero (why?)**.

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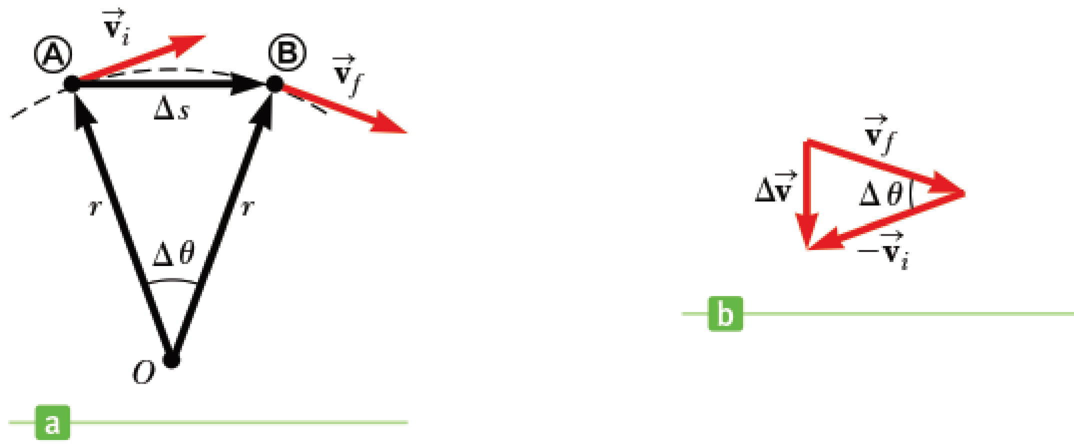


- For circular motion at constant speed, the acceleration vector always points toward the center of the circle. Such acceleration is called a ***centripetal acceleration*** (التعجيل المركزي). Its magnitude is given by

$$a_c = \frac{v^2}{r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13), \qquad \textit{drive it (H.W.)}$$

- Because the tangential speed is related to the angular speed through the relation $v_t = \omega r$, an alternate form of Equation

$$a_c = \frac{v^2}{r} = \frac{\omega^2 r^2}{r} = r\,\omega^2 \ldots\ldots(14)$$

- The foregoing derivations concern circular motion at constant speed. When an object moves in a circle but is ***speeding up or slowing down***,
- A tangential component of acceleration, $a_t = r\alpha$, is also present. Because the tangential and centripetal components of acceleration are **perpendicular** to each other, we can find the magnitude of the ***total acceleration*** with the Pythagorean theorem:

$$a = \sqrt{{a_t}^2 + {a_c}^2} \ldots\ldots\ldots(15)$$

Example 3:

A race car accelerates uniformly from a speed of 40 m/s to a speed of 60 m/s in 5s while traveling counterclockwise around a circular track of radius 4×10^2m. When the car reaches a speed of 50 m/s, **find** (a) the magnitude of the car's centripetal acceleration, (b) the angular speed, (c) the magnitude of the tangential acceleration, and (d) the magnitude of the total acceleration.

SOLUTION

(a) Find the magnitude of the centripetal acceleration when $v = 50.0$ m/s.

Substitute into Equation 7.13:

$$a_c = \frac{v^2}{r} = \frac{(50.0 \text{ m/s})^2}{4.00 \times 10^2 \text{ m}} = \boxed{6.25 \text{ m/s}^2}$$

(b) Find the angular speed.

Solve Equation 7.10 for ω and substitute:

$$\omega = \frac{v}{r} = \frac{50.0 \text{ m/s}}{4.00 \times 10^2 \text{ m}} = \boxed{0.125 \text{ rad/s}}$$

(c) Find the magnitude of the tangential acceleration.

Divide the change in linear speed by the time:

$$a_t = \frac{v_f - v_i}{\Delta t} = \frac{60.0 \text{ m/s} - 40.0 \text{ m/s}}{5.00 \text{ s}} = \boxed{4.00 \text{ m/s}^2}$$

(d) Find the magnitude of the total acceleration.

Substitute into Equation 7.18:

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{(4.00 \text{ m/s}^2)^2 + (6.25 \text{ m/s}^2)^2}$$

$$a = \boxed{7.42 \text{ m/s}^2}$$

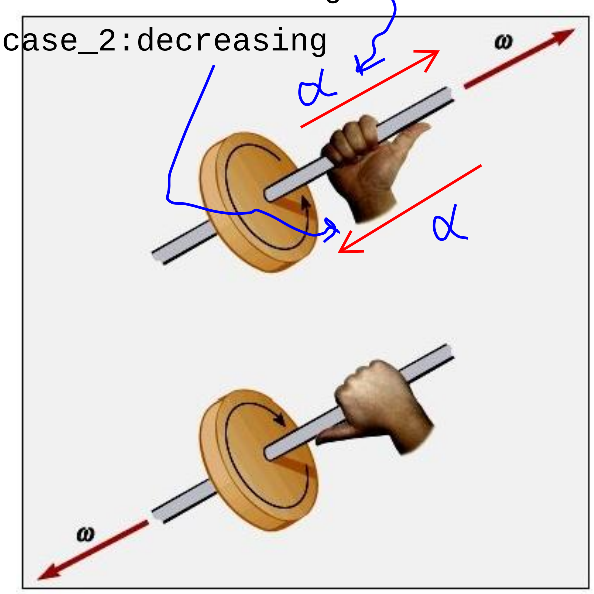
EXERCISE: Suppose the racecar now slows down uniformly from 60 m/s to 30 m/s in 4.5s to avoid an accident, while still traversing a circular path 4×10^2 m in radius. **Find** the car's (a) centripetal acceleration, (b) angular speed, (c) tangential acceleration, and (d) total acceleration when the speed is 40 m/s.

ANSWERS: (a) 4.00 m/s^2 (b) 0.100 rad/s (c) -6.67 m/s^2 (d) 7.78 m/s^2

Angular Quantities Are Vectors:

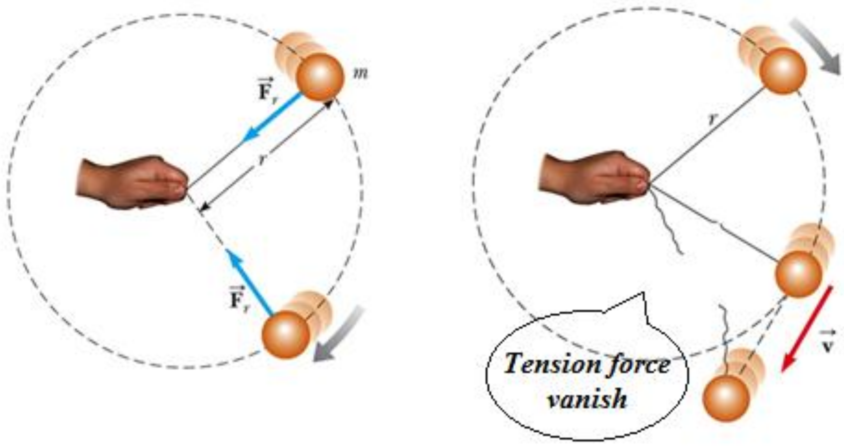
- The direction of the angular velocity vector $\vec{\omega}$ can be found with the **right-hand rule**, as illustrated in next figure.
- The directions of the angular acceleration $\vec{\alpha}$ and the angular velocity $\vec{\omega}$ are the **same** if the angular speed ω (the magnitude of $\vec{\omega}$) is **increasing** with time, and are opposite each other if the angular speed is **decreasing** with time.

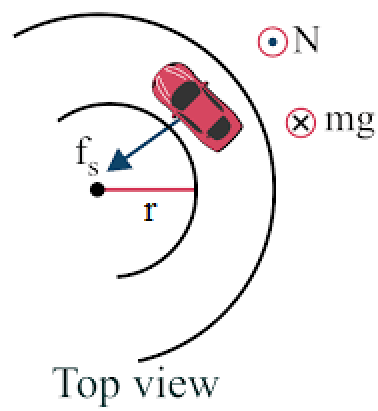
case_1: increasing
case_2:decreasing



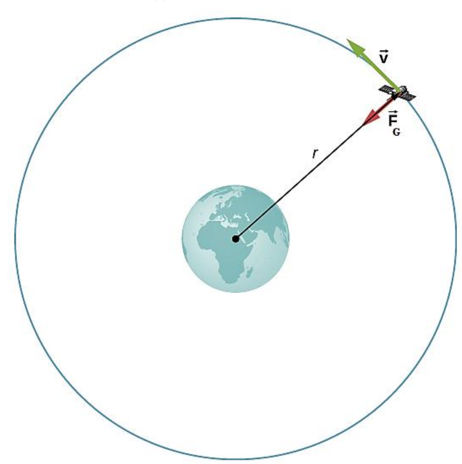
Forces Causing Centripetal Acceleration:

- **An object can have a centripetal acceleration only if some external force acts on it.**
  - For a ball rotating in a circle at the end of a string, that force is the **tension** in the string.



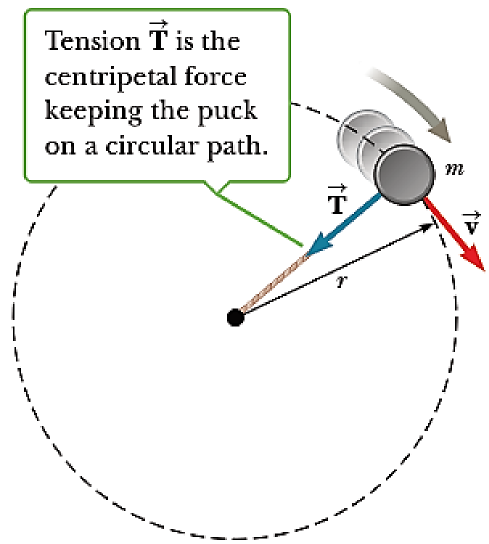


- In the case of a car moving on a flat circular road, the force is **frictional force** between the car and road.



- A satellite in circular orbit around Earth has a centripetal acceleration due to the **Gravitational force** between the satellite and Earth.

- **Note: Some books use the term "*centripetal force*," which can give the *mistaken impression* that it is a new force of nature. This is not the case: The adjective "centripetal" in "centripetal force" simply means that the force in inquiry *acts* toward a center.**



F = ma

- Consider a puck (قرص لعبة الهوكي) of mass *m* that is tied to a string of length *r* and is being rotated at constant speed in a horizontal circular path, as illustrated in the Fig.
- Its weight is supported by a frictionless table. Why does the puck move in a circle? Because of its inertia, the tendency of the puck is to move in a straight line; however, the string prevents motion along a straight line by **exerting a radial force** on the puck—a tension force—that makes it follow the circular path.
- The tension $\vec{T}$ is directed along the string toward the center of the circle, as shown in the figure.
- **In general**, converting Newton's second law to polar coordinates yields an equation relating the net centripetal force, F_c, The *magnitude* of the net centripetal force equals the **mass** times the magnitude of the **centripetal acceleration**:

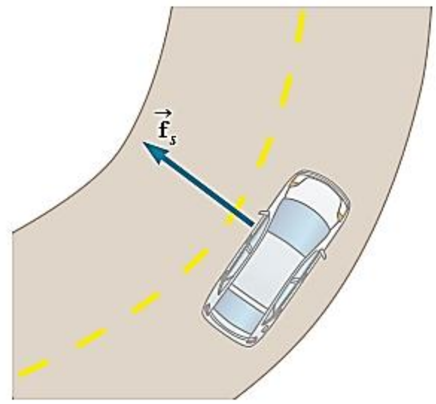
$$F_c = ma_c = m\frac{v^2}{r} \quad \dots\dots\dots\dots(16)$$

- A *net force causing a centripetal acceleration acts toward the center of the circular path* and effects a change in the direction of the velocity vector.

- If that *force should vanish (disappear)*, the object would immediately leave its circular path and move along a straight line tangent to the circle at the point where the force vanished.

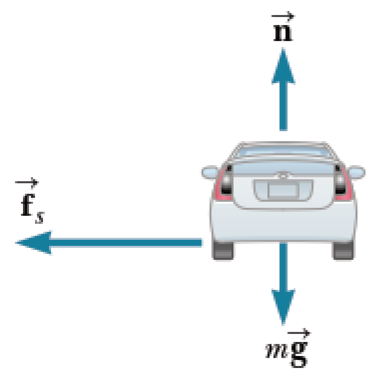
Example 4:

Car travels at a constant speed of 30 mi/h (13.4 m/s) on a level circular turn of radius 50m, as shown in the top view fig. What minimum coefficient of static friction, μ_s between the tires and roadway will allow the car to make the circular turn without sliding?



SOLUTION:

- Write the components of Newton's second law. The radial component involves only the maximum static friction force, $f_{s,max}$:
- In the vertical component of the second law, the gravity force and the normal force are in equilibrium:



$$m\frac{v^2}{r} = f_{s,\max} = \mu_s n$$

$$n - mg = 0 \quad \rightarrow \quad n = mg$$

$$m\frac{v^2}{r} = \mu_s mg$$

$$\mu_s = \frac{v^2}{rg} = \frac{(13.4\ \mathrm{m/s})^2}{(50.0\ \mathrm{m})(9.80\ \mathrm{m/s^2})} = \boxed{0.366}$$

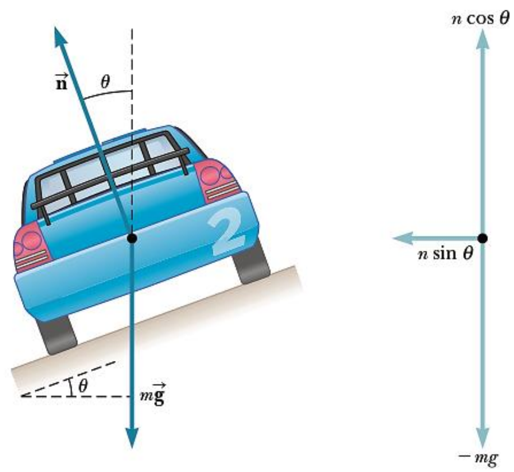
REMARKS: *The value of μ_s for rubber on dry concrete is very close to 1, so the car can negotiate the curve with ease. If the road were wet or icy, however, the value for μ_s could be 0.2 or lower. Under such conditions, the* *circular force provided by static friction would not be great enough to keep the car on the circular path, and it would slide off on a tangent, leaving the roadway.*

EXERCISE: At what maximum speed can a car cross a turn on a wet road with coefficient of static friction 0.23 without sliding out of control? The radius of the turn is 25m.

ANSWER: 7.51 m/s

Example 5:

The International Speedway Florida, is famous for its races, held every February. Both of its courses feature four-story, 31° banked curves, with maximum radius of 316m. If a car transfers the curve too slowly, it tends to slip down the incline (مضمار مائل) of the turn, whereas if it is going too fast, it may begin to slide up the incline. (a) Find the necessary centripetal acceleration on this banked curve so the car will not slip down or slide up the incline. (Neglect friction.) (b) Calculate the speed of the racecar.



(a) Find the centripetal acceleration.

Write Newton's second law for the car:

$$m\vec{a} = \sum \vec{F} = \vec{n} + m\vec{g}$$

Use the y-component of Newton's second law to solve for the normal force n:

$$n\cos\theta - mg = 0$$

$$n = \frac{mg}{\cos\theta}$$

Obtain an expression for the horizontal component of $\vec{n}$, which is the centripetal force F_c in this example:

$$F_c = n\sin\theta = \frac{mg\sin\theta}{\cos\theta} = mg\tan\theta$$

Substitute this expression for F_c into the radial component of Newton's second law and divide by m to get the centripetal acceleration:

$$ma_c = F_c$$

$$a_c = \frac{F_c}{m} = \frac{mg\tan\theta}{m} = g\tan\theta$$

$$a_c = (9.80\ \text{m/s}^2)(\tan 31.0°) = \boxed{5.89\ \text{m/s}^2}$$

(b) Find the speed of the race car.

$$\frac{v^2}{r} = a_c$$

$$v = \sqrt{ra_c} = \sqrt{(316\ \text{m})(5.89\ \text{m/s}^2)} = \boxed{43.1\ \text{m/s}}$$

km=1000m , hr=3600 s 155.16 km/hr

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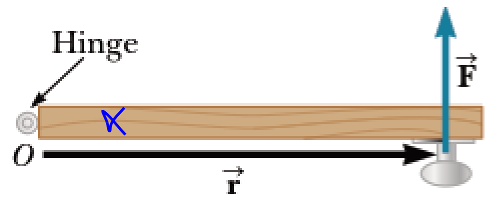
EXERCISE: A racetrack is to have a banked curve with radius of 245m. What should be the angle of the bank if the normal force alone is to allow safe travel around the curve at 58 m/s?

***ANSWER*:** 54.5°

Torque:

$\vec{\tau}$ العزم

- Forces cause accelerations; torques cause angular accelerations. There is a definite relationship, however, between the two concepts.
- If a force $\vec{F}$ is applied to the door, there are three factors that determine the effectiveness of the force in opening the door: the *magnitude* of the force, the *position* of application of the force, and the *angle* at which it is applied.



- For simplicity, we restrict our discussion to position and force vectors lying in a plane.

Let $\vec{F}$ be force acting on an object, and let $\vec{r}$ be a position vector from a chosen point O to the point of application of the force, with $\vec{F}$ perpendicular to $\vec{r}$. The magnitude of the torque $\vec{\tau}$ exerted by the force $\vec{F}$ is given by:

$$\vec{\tau} = rF$$

Where r is the length of the position vector and F is the magnitude of the force.

SI unit: Newton-meter (N. m)