1.15 Velocity and Acceleration

In this section we need to take a look at the velocity and acceleration of a moving object.

From Calculus I we know that given the position function of an object that the velocity of the object is the first derivative of the position function and the acceleration of the object is the second derivative of the position function.

So, given this it shouldn't be too surprising that if the position function of an object is given by the vector function $\vec{r}(t)$ then the velocity and acceleration of the object is given by,

$$\vec{v}(t) = \vec{r}'(t) \qquad \vec{a}(t) = \vec{r}''(t)$$

Notice that the velocity and acceleration are also going to be vectors as well.

In the study of the motion of objects the acceleration is often broken up into a **tangential component**, a_T, and a **normal component**, a_N. The tangential component is the part of the acceleration that is tangential to the curve and the normal component is the part of the acceleration that is normal (or orthogonal) to the curve. If we do this we can write the acceleration as,

$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$

where $\vec{T}$ and $\vec{N}$ are the unit tangent and unit normal for the position function.

If we define $v = \|\vec{v}(t)\|$ then the tangential and normal components of the acceleration are given by,

Tangential and Normal Acceleration

$$a_T = v' = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}'(t)\|} \qquad a_N = \kappa v^2 = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|}$$

where κ is the curvature for the position function.

There are two formulas to use here for each component of the acceleration and while the second formula may seem overly complicated it is often the easier of the two. In the tangential component, v, may be messy and computing the derivative may be unpleasant. In the normal component we will already be computing both of these quantities in order to get the curvature and so the second formula in this case is definitely the easier of the two.

Let's take a quick look at a couple of examples.

Example 48

If the acceleration of an object is given by $\vec{a} = \vec{i} + 2\vec{j} + 6t\vec{k}$ find the object's velocity and position functions given that the initial velocity is $\vec{v}(0) = \vec{j} - \vec{k}$ and the initial position is $\vec{r}(0) = \vec{i} - 2\vec{j} + 3\vec{k}$.

Solution

We'll first get the velocity. To do this all (well almost all) we need to do is integrate the acceleration.

$$\begin{aligned}\vec{v}(t) &= \int \vec{a}(t)\, dt \\ &= \int \vec{i} + 2\vec{j} + 6t\vec{k}\, dt \\ &= t\,\vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k} + \vec{c}\end{aligned}$$

To completely get the velocity we will need to determine the "constant" of integration. We can use the initial velocity to get this.

$$\vec{j} - \vec{k} = \vec{v}(0) = \vec{c}$$

The velocity of the object is then,

$$\begin{aligned}\vec{v}(t) &= t\,\vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k} + \vec{j} - \vec{k} \\ &= t\,\vec{i} + (2t+1)\,\vec{j} + (3t^2-1)\,\vec{k}\end{aligned}$$

We will find the position function by integrating the velocity function.

$$\begin{aligned}\vec{r}(t) &= \int \vec{v}(t)\, dt \\ &= \int t\,\vec{i} + (2t+1)\,\vec{j} + (3t^2-1)\,\vec{k}\, dt \\ &= \frac{1}{2}t^2\,\vec{i} + (t^2+t)\,\vec{j} + (t^3-t)\,\vec{k} + \vec{c}\end{aligned}$$

Using the initial position gives us,

$$\vec{i} - 2\vec{j} + 3\vec{k} = \vec{r}(0) = \vec{c}$$

So, the position function is,

$$\vec{r}(t) = \left(\frac{1}{2}t^2 + 1\right)\vec{i} + (t^2 + t - 2)\,\vec{j} + (t^3 - t + 3)\,\vec{k}$$

Example 49

For the object in the previous example determine the tangential and normal components of the acceleration.

Solution

There really isn't much to do here other than plug into the formulas. To do this we'll need to notice that,

$$\vec{r}'(t) = t\,\vec{i} + (2t+1)\,\vec{j} + \left(3t^2-1\right)\vec{k}$$
$$\vec{r}''(t) = \vec{i} + 2\vec{j} + 6t\vec{k}$$

Let's first compute the dot product and cross product that we'll need for the formulas.

$$\vec{r}'(t)\cdot\vec{r}''(t) = t + 2\,(2t+1) + 6t\left(3t^2-1\right) = 18t^3 - t + 2$$

$$\begin{aligned}\vec{r}'(t)\times\vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & 2t+1 & 3t^2-1 \\ 1 & 2 & 6t \end{vmatrix} \\ &= (6t)\,(2t+1)\,\vec{i} + \left(3t^2-1\right)\vec{j} + 2t\vec{k} - 6t^2\vec{j} - 2\left(3t^2-1\right)\vec{i} - (2t+1)\,\vec{k} \\ &= \left(6t^2+6t+2\right)\vec{i} - \left(3t^2+1\right)\vec{j} - \vec{k}\end{aligned}$$

Next, we also need a couple of magnitudes.

$$\left\|\vec{r}'(t)\right\| = \sqrt{t^2 + (2t+1)^2 + \left(3t^2-1\right)^2} = \sqrt{9t^4 - t^2 + 4t + 2}$$
$$\left\|\vec{r}'(t)\times\vec{r}''(t)\right\| = \sqrt{\left(6t^2+6t+2\right)^2 + \left(3t^2+1\right)^2 + 1} = \sqrt{45t^4 + 72t^3 + 66t^2 + 24t + 6}$$

The tangential component of the acceleration is then,

$$a_T = \frac{18t^3 - t + 2}{\sqrt{9t^4 - t^2 + 4t + 2}}$$

The normal component of the acceleration is,

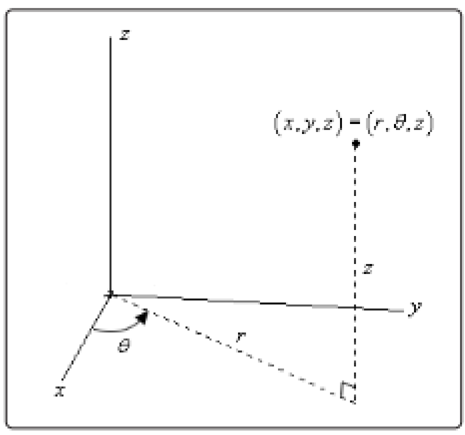
$$a_N = \frac{\sqrt{45t^4 + 72t^3 + 66t^2 + 24t + 6}}{\sqrt{9t^4 - t^2 + 4t + 2}} = \sqrt{\frac{45t^4 + 72t^3 + 66t^2 + 24t + 6}{9t^4 - t^2 + 4t + 2}}$$

1.16 Cylindrical Coordinates

As with two dimensional space the standard (x, y, z) coordinate system is called the Cartesian coordinate system. In the last two sections of this chapter we'll be looking at some alternate coordinate systems for three dimensional space.

We'll start off with the cylindrical coordinate system. This one is fairly simple as it is nothing more than an extension of polar coordinates into three dimensions. Not only is it an extension of polar coordinates, but we extend it into the third dimension just as we extend Cartesian coordinates into the third dimension. All that we do is add a z on as the third coordinate. The r and θ are the same as with polar coordinates.

Here is a sketch of a point in $\mathbb{R}^3$.



The conversions for x and y are the same conversions that we used back when we were looking at polar coordinates. So, if we have a point in cylindrical coordinates the Cartesian coordinates can be found by using the following conversions.

Cylindrical to Cartesian Converstion Formulas

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

The third equation is just an acknowledgement that the z-coordinate of a point in Cartesian and polar coordinates is the same.

Likewise, if we have a point in Cartesian coordinates the cylindrical coordinates can be found by using the following conversions.

Cartesian to Cylindrical Converstion Formulas

$$r = \sqrt{x^2 + y^2} \qquad \text{OR} \qquad r^2 = x^2 + y^2$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

Let's take a quick look at some surfaces in cylindrical coordinates.

Example 50

Identify the surface for each of the following equations.

(a) $r = 5$

(b) $r^2 + z^2 = 100$

(c) $z = r$

Solution

(a) $r = 5$

In two dimensions we know that this is a circle of radius 5. Since we are now in three dimensions and there is no z in equation this means it is allowed to vary freely. So, for any given z we will have a circle of radius 5 centered on the z-axis.

In other words, we will have a cylinder of radius 5 centered on the z-axis.

(b) $r^2 + z^2 = 100$

This equation will be easy to identify once we convert back to Cartesian coordinates.

$$r^2 + z^2 = 100$$
$$x^2 + y^2 + z^2 = 100$$

So, this is a sphere centered at the origin with radius 10.

(c) $z = r$

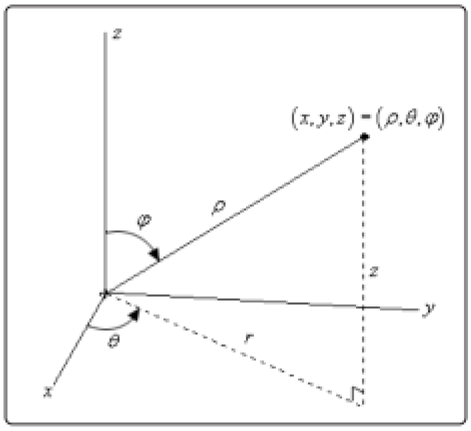
Again, this one won't be too bad if we convert back to Cartesian. For reasons that will be apparent eventually, we'll first square both sides, then convert.

$$z^2 = r^2$$
$$z^2 = x^2 + y^2$$

From the section on quadric surfaces we know that this is the equation of a cone.

1.17 Spherical Coordinates

In this section we will introduce spherical coordinates. Spherical coordinates can take a little getting used to. It's probably easiest to start things off with a sketch.



Spherical coordinates consist of the following three quantities.

First there is ρ. This is the distance from the origin to the point and we will require $\rho \geq 0$.

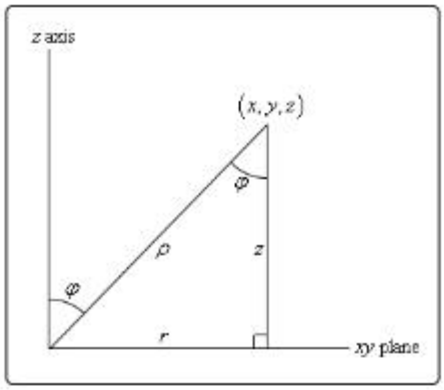
Next there is θ. This is the same angle that we saw in polar/cylindrical coordinates. It is the angle between the positive x-axis and the line above denoted by r (which is also the same r as in polar/cylindrical coordinates). There are no restrictions on θ.

Finally, there is φ. This is the angle between the positive z-axis and the line from the origin to the point. We will require $0 \leq \varphi \leq \pi$.

In summary, ρ is the distance from the origin to the point, φ is the angle that we need to rotate down from the positive z-axis to get to the point and θ is how much we need to rotate around the z-axis to get to the point.

We should first derive some conversion formulas. Let's first start with a point in spherical coordinates and ask what the cylindrical coordinates of the point are. So, we know (ρ, θ, φ) and want to find (r, θ, z). Of course, we really only need to find r and z since θ is the same in both coordinate systems.

If we look at the sketch above from directly in front of the triangle we get the following sketch,



We know that the angle between the z-axis and ρ is φ and with a little geometry we also know that the angle between ρ and the vertical side of the right triangle is also φ.

Then, with a little right triangle trig we get,

$$z = \rho\cos(\varphi)$$
$$r = \rho\sin(\varphi)$$

and these are exactly the formulas that we were looking for. So, given a point in spherical coordinates the cylindrical coordinates of the point will be,

Spherical to Cylindrical Conversion Formulas

$$r = \rho\sin(\varphi)$$
$$\theta = \theta$$
$$z = \rho\cos(\varphi)$$

Note as well from the Pythagorean theorem we also get,

Fact

$$\rho^2 = r^2 + z^2$$

Next, let's find the Cartesian coordinates of the same point. To do this we'll start with the cylindrical conversion formulas from the previous section.

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

Now all that we need to do is use the formulas from above for r and z to get,

Spherical to Cartesian Conversion Formulas

$$x = \rho\sin(\varphi)\cos(\theta)$$
$$y = \rho\sin(\varphi)\sin(\theta)$$
$$z = \rho\cos(\varphi)$$

Also note that since we know that $r^2 = x^2 + y^2$ we get,

Fact

$$\rho^2 = x^2 + y^2 + z^2$$

Converting points from Cartesian or cylindrical coordinates into spherical coordinates is usually done with the same conversion formulas. To see how this is done let's work an example of each.

Example 51

Perform each of the following conversions.

(a) Convert the point $\left(\sqrt{6}, \frac{\pi}{4}, \sqrt{2}\right)$ from cylindrical to spherical coordinates.

(b) Convert the point $(-1, 1, -\sqrt{2})$ from Cartesian to spherical coordinates.

Solution

(a) Convert the point $\left(\sqrt{6}, \frac{\pi}{4}, \sqrt{2}\right)$ from cylindrical to spherical coordinates.

We'll start by acknowledging that θ is the same in both coordinate systems and so we don't need to do anything with that.

Next, let's find ρ.

$$\rho = \sqrt{r^2 + z^2} = \sqrt{6+2} = \sqrt{8} = 2\sqrt{2}$$

Finally, let's get φ. To do this we can use either the conversion for r or z. We'll use the conversion for z.

$$z = \rho\cos(\varphi) \quad \Rightarrow \quad \cos(\varphi) = \frac{z}{\rho} = \frac{\sqrt{2}}{2\sqrt{2}} \quad \Rightarrow \quad \varphi = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Notice that there are many possible values of φ that will give $\cos(\varphi) = \frac{1}{2}$, however, we have restricted φ to the range $0 \le \varphi \le \pi$ and so this is the only possible value in that range.

So, the spherical coordinates of this point will are $(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3})$.

(b) Convert the point $(-1, 1, -\sqrt{2})$ from Cartesian to spherical coordinates.

The first thing that we'll do here is find ρ.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1+1+2} = 2$$

Now we'll need to find φ. We can do this using the conversion for z.

$$z = \rho\cos(\varphi) \quad \Rightarrow \quad \cos(\varphi) = \frac{z}{\rho} = \frac{-\sqrt{2}}{2} \quad \Rightarrow \quad \varphi = \cos^{-1}\left(\frac{-\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

As with the last parts this will be the only possible φ in the range allowed.

Finally, let's find θ. To do this we can use the conversion for x or y. We will use the conversion for y in this case.

$$\sin(\theta) = \frac{y}{\rho\sin(\varphi)} = \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \Rightarrow \quad \theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4}$$

Now, we actually have more possible choices for θ but all of them will reduce down to one of the two angles above since they will just be one of these two angles with one or more complete rotations around the unit circle added on.

We will however, need to decide which one is the correct angle since only one will be. To do this let's notice that, in two dimensions, the point with coordinates $x = -1$ and $y = 1$ lies in the second quadrant. This means that θ must be angle that will put the point into the second quadrant. Therefore, the second angle, $\theta = \frac{3\pi}{4}$, must be the correct one.

The spherical coordinates of this point are then $(2, \frac{3\pi}{4}, \frac{3\pi}{4})$.

Now, let's take a look at some equations and identify the surfaces that they represent.

Example 52

Identify the surface for each of the following equations.

(a) $\rho = 5$

(b) $\varphi = \dfrac{\pi}{3}$

(c) $\theta = \dfrac{2\pi}{3}$

(d) $\rho\sin(\varphi) = 2$

Solution

(a) $\rho = 5$

There are a couple of ways to think about this one.

First, think about what this equation is saying. This equation says that, no matter what θ and φ are, the distance from the origin must be 5. So, we can rotate as much as we want away from the z-axis and around the z-axis, but we must always remain at a fixed distance from the origin. This is exactly what a sphere is. So, this is a sphere of radius 5 centered at the origin.

The other way to think about it is to just convert to Cartesian coordinates.

$$\begin{aligned} \rho &= 5 \\ \rho^2 &= 25 \\ x^2 + y^2 + z^2 &= 25 \end{aligned}$$

Sure enough a sphere of radius 5 centered at the origin.

(b) $\varphi = \dfrac{\pi}{3}$

In this case there isn't an easy way to convert to Cartesian coordinates so we'll just need to think about this one a little. This equation says that no matter how far away from the origin that we move and no matter how much we rotate around the z-axis the point must always be at an angle of $\frac{\pi}{3}$ from the z-axis.

This is exactly what happens in a cone. All of the points on a cone are a fixed angle from the z-axis. So, we have a cone whose points are all at an angle of $\frac{\pi}{3}$ from the z-axis.

(c) $\theta = \dfrac{2\pi}{3}$

As with the last part we won't be able to easily convert to Cartesian coordinates here. In this case no matter how far from the origin we get or how much we rotate down from the positive z-axis the points must always form an angle of $\frac{2\pi}{3}$ with the x-axis.

Points in a vertical plane will do this. So, we have a vertical plane that forms an angle of $\frac{2\pi}{3}$ with the positive x-axis.

(d) $\rho \sin(\varphi) = 2$

In this case we can convert to Cartesian coordinates so let's do that. There are actually two ways to do this conversion. We will look at both since both will be used on occasion.

Solution 1

In this solution method we will convert directly to Cartesian coordinates. To do this we will first need to square both sides of the equation.

$$\rho^2 \sin^2(\varphi) = 4$$

Now, for no apparent reason add $\rho^2 \cos^2(\varphi)$ to both sides.

$$\begin{aligned}\rho^2 \sin^2(\varphi) + \rho^2 \cos^2(\varphi) &= 4 + \rho^2 \cos^2(\varphi)\\ \rho^2 \left(\sin^2(\varphi) + \cos^2(\varphi)\right) &= 4 + \rho^2 \cos^2(\varphi)\\ \rho^2 &= 4 + (\rho \cos(\varphi))^2\end{aligned}$$

Now we can convert to Cartesian coordinates.

$$\begin{aligned}x^2 + y^2 + z^2 &= 4 + z^2\\ x^2 + y^2 &= 4\end{aligned}$$

So, we have a cylinder of radius 2 centered on the z-axis.

This solution method wasn't too bad, but it did require some not so obvious steps to complete.

Solution 2

This method is much shorter, but also involves something that you may not see the first time around. In this case instead of going straight to Cartesian coordinates we'll first convert to cylindrical coordinates.

This won't always work, but in this case all we need to do is recognize that $r = \rho \sin(\varphi)$ and we will get something we can recognize. Using this we get,

$$\begin{aligned}\rho \sin(\varphi) &= 2\\ r &= 2\end{aligned}$$

At this point we know this is a cylinder (remember that we're in three dimensions and so this isn't a circle!). However, let's go ahead and finish the conversion process out.

$$r^2 = 4$$
$$x^2 + y^2 = 4$$

So, as we saw in the last part of the previous example it will *sometimes* be easier to convert equations in spherical coordinates into cylindrical coordinates before converting into Cartesian coordinates. This won't always be easier, but it can make some of the conversions quicker and easier.

The last thing that we want to do in this section is generalize the first three parts of the previous example.

$\rho = a$	sphere of radius a centered at the origin
$\varphi = \alpha$	cone that makes an angle of α with the positive $z-$ axis
$\theta = \beta$	vertical plane that makes an angle of β with the positive $x-$ axis