1.12 Tangent, Normal and Binormal Vectors

In this section we want to look at an application of derivatives for vector functions. Actually, there are a couple of applications, but they all come back to needing the first one.

In the past we've used the fact that the derivative of a function was the slope of the tangent line. With vector functions we get exactly the same result, with one exception.

Given the vector function, $\vec{r}(t)$, we call $\vec{r}'(t)$ the **tangent vector** provided it exists and provided $\vec{r}'(t) \neq \vec{0}$. The tangent line to $\vec{r}(t)$ at P is then the line that passes through the point P and is parallel to the tangent vector, $\vec{r}'(t)$. Note that we really do need to require $\vec{r}'(t) \neq \vec{0}$ in order to have a tangent vector. If we had

$$\vec{r}'(t) = \vec{0}$$

we would have a vector that had no magnitude and so couldn't give us the direction of the tangent.

Also, provided $\vec{r}'(t) \neq \vec{0}$, the **unit tangent vector** to the curve is given by,

Unit Tangent Vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

While, the components of the unit tangent vector can be somewhat messy on occasion there are times when we will need to use the unit tangent vector instead of the tangent vector.

Example 40

Find the general formula for the tangent vector and unit tangent vector to the curve given by $\vec{r}(t) = t^2\,\vec{i} + 2\sin(t)\,\vec{j} + 2\cos(t)\,\vec{k}$.

Solution

First, by general formula we mean that we won't be plugging in a specific t and so we will be finding a formula that we can use at a later date if we'd like to find the tangent at any point on the curve. With that said there really isn't all that much to do at this point other than to do the work.

Here is the tangent vector to the curve.

$$\vec{r}'(t) = 2t\,\vec{i} + 2\cos(t)\,\vec{j} - 2\sin(t)\,\vec{k}$$

To get the unit tangent vector we need the length of the tangent vector.

$$\begin{aligned}\left\|\vec{r}'(t)\right\| &= \sqrt{4t^2 + 4\cos^2(t) + 4\sin^2(t)} \\ &= \sqrt{4t^2+4}\end{aligned}$$

The unit tangent vector is then,

$$\begin{aligned}\vec{T}(t) &= \frac{1}{\sqrt{4t^2+4}}\left(2t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin(t)\,\vec{k}\right) \\ &= \frac{2t}{\sqrt{4t^2+4}}\,\vec{i} + \frac{2\cos(t)}{\sqrt{4t^2+4}}\,\vec{j} - \frac{2\sin(t)}{\sqrt{4t^2+4}}\,\vec{k}\end{aligned}$$

Example 41

Find the vector equation of the tangent line to the curve given by $\vec{r}(t) = t^2\,\vec{i} + 2\sin(t)\,\vec{j} + 2\cos(t)\,\vec{k}$ at $t = \frac{\pi}{3}$.

Solution

First, we need the tangent vector and since this is the function we were working with in the previous example we can just reuse the tangent vector from that example and plug in $t = \frac{\pi}{3}$.

$$\vec{r}'\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}\,\vec{i} + 2\cos\left(\frac{\pi}{3}\right)\vec{j} - 2\sin\left(\frac{\pi}{3}\right)\vec{k} = \frac{2\pi}{3}\,\vec{i} + \vec{j} - \sqrt{3}\,\vec{k}$$

We'll also need the point on the line at $t = \frac{\pi}{3}$ so,

$$\vec{r}\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}\,\vec{i} + \sqrt{3}\,\vec{j} + \vec{k}$$

The vector equation of the line is then,

$$\vec{r}(t) = \left\langle \frac{\pi^2}{9}, \sqrt{3}, 1\right\rangle + t\left\langle \frac{2\pi}{3}, 1, -\sqrt{3}\right\rangle$$

Before moving on let's note a couple of things about the previous example. First, we could have used the unit tangent vector had we wanted to for the parallel vector. However, that would have made for a more complicated equation for the tangent line.

Second, notice that we used $\vec{r}(t)$ to represent the tangent line despite the fact that we used that as well for the function. Do not get excited about that. The $\vec{r}(t)$ here is much like y is with normal functions. With normal functions, y is the generic letter that we used to represent functions and

$\vec{r}(t)$ tends to be used in the same way with vector functions.

Next, we need to talk about the **unit normal** and the **binormal** vectors.

The unit normal vector is defined to be,

Unit Normal Vector

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}$$

The unit normal is orthogonal (or normal, or perpendicular) to the unit tangent vector and hence to the curve as well. We've already seen normal vectors when we were dealing with Equations of Planes. They will show up with some regularity in several Calculus III topics.

The definition of the unit normal vector always seems a little mysterious when you first see it. It follows directly from the following fact.

Fact

Suppose that $\vec{r}(t)$ is a vector such that $\|\vec{r}(t)\| = c$ for all t. Then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$.

Proof

To prove this fact is pretty simple. From the fact statement and the relationship between the magnitude of a vector and the dot product we have the following.

$$\vec{r}(t) \cdot \vec{r}(t) = \|\vec{r}(t)\|^2 = c^2 \qquad \text{for all } t$$

Now, because this is true for all t we can see that,

$$\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) = \frac{d}{dt}\left(c^2\right) = 0$$

Also, recalling the fact from the previous section about differentiating a dot product we see that,

$$\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) = \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 2\vec{r}'(t) \cdot \vec{r}(t)$$

Or, upon putting all this together we get,

$$2\vec{r}'(t) \cdot \vec{r}(t) = 0 \qquad \Rightarrow \qquad \vec{r}'(t) \cdot \vec{r}(t) = 0$$

Therefore $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$.

The definition of the unit normal then falls directly from this. Because $\vec{T}(t)$ is a unit vector we know that $\left\|\vec{T}(t)\right\| = 1$ for all t and hence by the Fact $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$. However, because $\vec{T}(t)$ is tangent to the curve, $\vec{T}'(t)$ must be orthogonal, or normal, to the curve as well and so be a normal vector for the curve. All we need to do then is divide by $\left\|\vec{T}'(t)\right\|$ to arrive at a unit normal vector.

Next, is the binormal vector. The binormal vector is defined to be,

Binormal Vector

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Because the binormal vector is defined to be the cross product of the unit tangent and unit normal vector we then know that the binormal vector is orthogonal to both the tangent vector and the normal vector.

Example 42

Find the normal and binormal vectors for $\vec{r}(t) = \langle t, 3\sin(t), 3\cos(t)\rangle$.

Solution

We first need the unit tangent vector so first get the tangent vector and its magnitude.

$$\vec{r}'(t) = \langle 1, 3\cos(t), -3\sin(t)\rangle$$
$$\left\|\vec{r}'(t)\right\| = \sqrt{1 + 9\cos^2(t) + 9\sin^2(t)} = \sqrt{10}$$

The unit tangent vector is then,

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos(t), -\frac{3}{\sqrt{10}}\sin(t)\right\rangle$$

The unit normal vector will now require the derivative of the unit tangent and its magnitude.

$$\vec{T}'(t) = \left\langle 0, -\frac{3}{\sqrt{10}}\sin(t), -\frac{3}{\sqrt{10}}\cos(t)\right\rangle$$
$$\left\|\vec{T}'(t)\right\| = \sqrt{\frac{9}{10}\sin^2(t) + \frac{9}{10}\cos^2(t)} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

The unit normal vector is then,

$$\vec{N}(t) = \frac{\sqrt{10}}{3}\left\langle 0, -\frac{3}{\sqrt{10}}\sin(t), -\frac{3}{\sqrt{10}}\cos(t)\right\rangle = \langle 0, -\sin(t), -\cos(t)\rangle$$

Finally, the binormal vector is,

$$\begin{aligned}
\vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos(t) & -\frac{3}{\sqrt{10}}\sin(t) \\ 0 & -\sin(t) & -\cos(t) \end{vmatrix} \\
&= -\frac{3}{\sqrt{10}}\cos^2(t)\,\vec{i} - \frac{1}{\sqrt{10}}\sin(t)\,\vec{k} + \frac{1}{\sqrt{10}}\cos(t)\,\vec{j} - \frac{3}{\sqrt{10}}\sin^2(t)\,\vec{i} \\
&= -\frac{3}{\sqrt{10}}\vec{i} + \frac{1}{\sqrt{10}}\cos(t)\,\vec{j} - \frac{1}{\sqrt{10}}\sin(t)\,\vec{k}
\end{aligned}$$

1.13 Arc Length with Vector Functions

In this section we'll recast an old formula into terms of vector functions. We want to determine the length of a vector function,

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

on the interval $a \le t \le b$.

We actually already know how to do this. Recall that we can write the vector function into the parametric form,

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

Also, recall that with two dimensional parametric curves the arc length is given by,

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

There is a natural extension of this to three dimensions. So, the length of the curve $\vec{r}(t)$ on the interval $a \le t \le b$ is,

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

There is a nice simplification that we can make for this. Notice that the integrand (the function we're integrating) is nothing more than the magnitude of the tangent vector,

$$\|\vec{r}'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Therefore, the arc length can be written as,

Arc Length

$$L = \int_a^b \|\vec{r}'(t)\|\, dt$$

Let's work a quick example of this.

Example 43

Determine the length of the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$ on the interval $0 \le t \le 2\pi$.

Solution

We will first need the tangent vector and its magnitude.

$$\vec{r}'(t) = \langle 2, 6\cos(2t), -6\sin(2t)\rangle$$
$$\|\vec{r}'(t)\| = \sqrt{4 + 36\cos^2(2t) + 36\sin^2(2t)} = \sqrt{4+36} = 2\sqrt{10}$$

The length is then,

$$\begin{aligned} L &= \int_a^b \|\vec{r}'(t)\|\,dt \\ &= \int_0^{2\pi} 2\sqrt{10}\,dt \\ &= 4\pi\sqrt{10} \end{aligned}$$

We need to take a quick look at another concept here. We define the **arc length function** as,

Arc Length Function

$$s(t) = \int_0^t \|\vec{r}'(u)\|\,du$$

Before we look at why this might be important let's work a quick example.

Example 44

Determine the arc length function for $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t)\rangle$.

Solution

From the previous example we know that,

$$\|\vec{r}'(t)\| = 2\sqrt{10}$$

The arc length function is then,

$$s(t) = \int_0^t 2\sqrt{10}\,du = \left(2\sqrt{10}\,u\right)\Big|_0^t = 2\sqrt{10}\,t$$

Okay, just why would we want to do this? Well let's take the result of the example above and solve it for t.

$$t = \frac{s}{2\sqrt{10}}$$

Now, taking this and plugging it into the original vector function and we can **reparametrize** the function into the form, $\vec{r}(t(s))$. For our function this is,

$$\vec{r}(t(s)) = \left\langle \frac{s}{\sqrt{10}}, 3\sin\left(\frac{s}{\sqrt{10}}\right), 3\cos\left(\frac{s}{\sqrt{10}}\right) \right\rangle$$

So, why would we want to do this? Well with the reparameterization we can now tell where we are on the curve after we've traveled a distance of s along the curve. Note as well that we will start the measurement of distance from where we are at $t = 0$.

Example 45

Where on the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$ are we after traveling for a distance of $\frac{\pi\sqrt{10}}{3}$?

Solution

To determine this we need the reparameterization, which we have from above.

$$\vec{r}(t(s)) = \left\langle \frac{s}{\sqrt{10}}, 3\sin\left(\frac{s}{\sqrt{10}}\right), 3\cos\left(\frac{s}{\sqrt{10}}\right) \right\rangle$$

Then, to determine where we are all that we need to do is plug in $s = \frac{\pi\sqrt{10}}{3}$ into this and we'll get our location.

$$\vec{r}\left(t\left(\frac{\pi\sqrt{10}}{3}\right)\right) = \left\langle \frac{\pi}{3}, 3\sin\left(\frac{\pi}{3}\right), 3\cos\left(\frac{\pi}{3}\right) \right\rangle = \left\langle \frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2} \right\rangle$$

So, after traveling a distance of $\frac{\pi\sqrt{10}}{3}$ along the curve we are at the point $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

1.14 Curvature

In this section we want to briefly discuss the **curvature** of a smooth curve (recall that for a smooth curve we require $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0$). The curvature measures how fast a curve is changing direction at a given point.

There are several formulas for determining the curvature for a curve. The formal definition of curvature is,

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

where $\vec{T}$ is the unit tangent and s is the arc length. Recall that we saw in a previous section how to reparametrize a curve to get it into terms of the arc length.

In general the formal definition of the curvature is not easy to use so there are two alternate formulas that we can use. Here they are.

Curvature

$$\kappa = \frac{\left\| \vec{T}'(t) \right\|}{\left\| \vec{r}'(t) \right\|} \qquad \kappa = \frac{\left\| \vec{r}'(t) \times \vec{r}''(t) \right\|}{\left\| \vec{r}'(t) \right\|^3}$$

These may not be particularly easy to deal with either, but at least we don't need to reparametrize the unit tangent.

Example 46

Determine the curvature for $\vec{r}(t) = \langle t, 3\sin(t), 3\cos(t) \rangle$.

Solution

Back in the section when we introduced the tangent vector we computed the tangent and unit tangent vectors for this function. These were,

$$\vec{r}'(t) = \langle 1, 3\cos(t), -3\sin(t) \rangle$$
$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos(t), -\frac{3}{\sqrt{10}}\sin(t) \right\rangle$$

The derivative of the unit tangent is,

$$\vec{T}'(t) = \left\langle 0, -\frac{3}{\sqrt{10}}\sin(t), -\frac{3}{\sqrt{10}}\cos(t) \right\rangle$$

The magnitudes of the two vectors are,

$$\left\|\vec{r}'(t)\right\| = \sqrt{1+9\cos^2(t)+9\sin^2(t)} = \sqrt{10}$$
$$\left\|\vec{T}'(t)\right\| = \sqrt{0+\frac{9}{10}\sin^2(t)+\frac{9}{10}\cos^2(t)} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

The curvature is then,

$$\kappa = \frac{\left\|\vec{T}'(t)\right\|}{\left\|\vec{r}'(t)\right\|} = \frac{3/\sqrt{10}}{\sqrt{10}} = \frac{3}{10}$$

In this case the curvature is constant. This means that the curve is changing direction at the same rate at every point along it. Recalling that this curve is a helix this result makes sense.

Example 47

Determine the curvature of $\vec{r}(t) = t^2\,\vec{i} + t\,\vec{k}$.

Solution

In this case the second form of the curvature would probably be easiest. Here are the first couple of derivatives.

$$\vec{r}'(t) = 2t\,\vec{i} + \vec{k} \qquad \vec{r}''(t) = 2\,\vec{i}$$

Next, we need the cross product.

$$\vec{r}'(t)\times\vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 2\vec{j}$$

The magnitudes are,

$$\left\|\vec{r}'(t)\times\vec{r}''(t)\right\| = 2 \qquad \left\|\vec{r}'(t)\right\| = \sqrt{4t^2+1}$$

The curvature at any value of t is then,

$$\kappa = \frac{2}{(4t^2+1)^{\frac{3}{2}}}$$

There is a special case that we can look at here as well. Suppose that we have a curve given by

$y = f(x)$ and we want to find its curvature.

As we saw when we first looked at vector functions we can write this as follows,

$$\vec{r}(x) = x\,\vec{i} + f(x)\,\vec{j}$$

If we then use the second formula for the curvature we will arrive at the following formula for the curvature.

$$\kappa = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{\frac{3}{2}}}$$