1.5 Three Dimensional Space

In this section we will start looking at three dimensional space (3-D space or $\mathbb{R}^3$).As with the last chapter this is preparation for multi-variable Calculus (which we'll be starting in the next chapter) as the vast majority of the multi-variable Calculus material assumes we are in three dimensional (or higher dimensional) space.

In this chapter we will discuss the equations of lines and planes in three dimensional space as well as the equations of many of the standard quadric surfaces (*i.e* equations with at least one quadratic term in it).

We will define a vector function and discuss how to perform basic Calculus operations on vector functions. We will also discuss how to get tangent vectors (a vector tangent to a curve), normal vectors (a vector orthogonal/perpendicular) and the curvature of a curve from the vector function that defines the curve. We'll also have a quick discussion of how to get the velocity and acceleration of an object as it travels along a curve defined by a vector function.

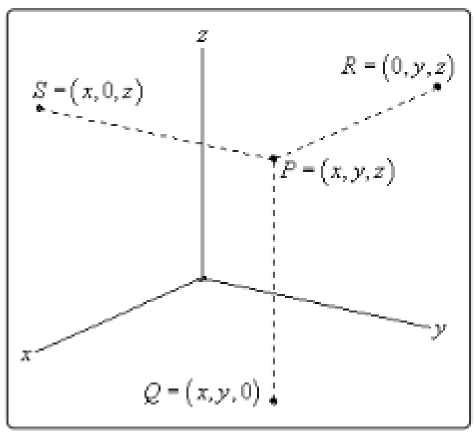
We will close out the chapter with a discussion a couple of alternative coordinates systems for three dimensional space, namely, cylindrical coordinates (a 3D extension of polar coordinates) and spherical coordinates.

1.5 The 3-D Coordinate System

We'll start the chapter off with a fairly short discussion introducing the 3-D coordinate system and the conventions that we'll be using. We will also take a brief look at how the different coordinate systems can change the graph of an equation.

Let's first get some basic notation out of the way. The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise, the 2-D coordinate system is often denoted by $\mathbb{R}^2$ and the 1-D coordinate system is denoted by $\mathbb{R}$. Also, as you might have guessed then a general n dimensional coordinate system is often denoted by $\mathbb{R}^n$.

Next, let's take a quick look at the basic coordinate system.



This is the standard placement of the axes in this class. It is assumed that only the positive directions are shown by the axes. If we need the negative axes for any reason we will put them in as needed.

Also note the various points on this sketch. The point P is the general point sitting out in 3-D space. If we start at P and drop straight down until we reach a z-coordinate of zero we arrive at the point Q. We say that Q sits in the xy-plane. The xy-plane corresponds to all the points which have a zero z-coordinate. We can also start at P and move in the other two directions as shown to get points in the xz-plane (this is S with a y-coordinate of zero) and the yz-plane (this is R with an x-coordinate of zero).

Collectively, the xy, xz, and yz-planes are sometimes called the coordinate planes. In the remainder of this class you will need to be able to deal with the various coordinate planes so make sure that you can.

Also, the point Q is often referred to as the projection of P in the xy-plane. Likewise, R is the projection of P in the yz-plane and S is the projection of P in the xz-plane.

Many of the formulas that you are used to working with in $\mathbb{R}^2$ have natural extensions in $\mathbb{R}^3$. For instance, the distance between two points in $\mathbb{R}^2$ is given by,

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

While the distance between any two points in $\mathbb{R}^3$ is given by,

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Likewise, the general equation for a circle with center (h, k) and radius r is given by,

$$(x - h)^2 + (y - k)^2 = r^2$$

and the general equation for a sphere with center (h, k, l) and radius r is given by,

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

With that said we do need to be careful about just translating everything we know about $\mathbb{R}^2$ into $\mathbb{R}^3$ and assuming that it will work the same way. A good example of this is in graphing to some extent. Consider the following example.

Example 17

Graph $x = 3$ in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

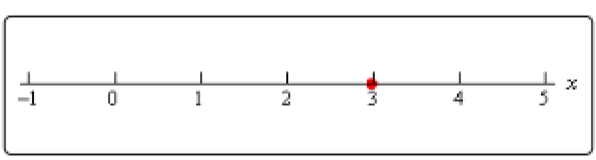
Solution

In $\mathbb{R}$ we have a single coordinate system and so $x = 3$ is a point in a 1-D coordinate system.

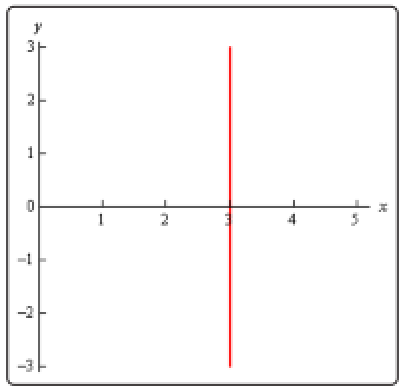
In $\mathbb{R}^2$ the equation $x = 3$ tells us to graph all the points that are in the form $(3, y)$. This is a vertical line in a 2-D coordinate system.

In $\mathbb{R}^3$ the equation $x = 3$ tells us to graph all the points that are in the form $(3, y, z)$. If you go back and look at the coordinate plane points this is very similar to the coordinates for the yz-plane except this time we have $x = 3$ instead of $x = 0$. So, in a 3-D coordinate system this is a plane that will be parallel to the yz-plane and pass through the x-axis at $x = 3$.

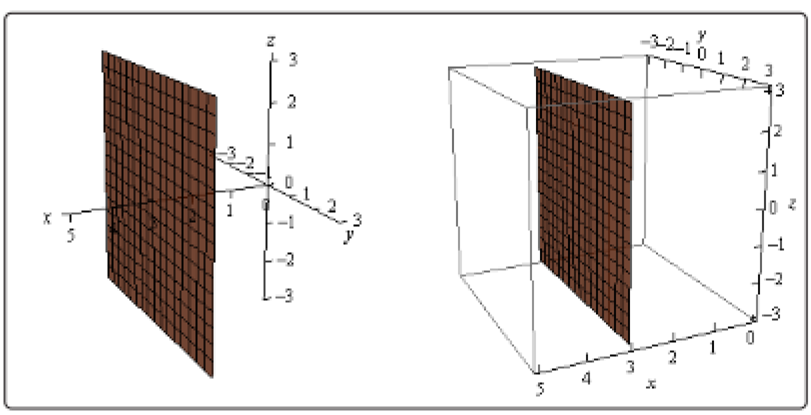
Here is the graph of $x = 3$ in $\mathbb{R}$.



Here is the graph of $x = 3$ in $\mathbb{R}^2$.



Finally, here is the graph of $x = 3$ in $\mathbb{R}^3$. Note that we've presented this graph in two different styles. On the left we've got the traditional axis system that we're used to seeing and on the right we've put the graph in a box. Both views can be convenient on occasion to help with perspective and so we'll often do this with 3D graphs and sketches.



Note that at this point we can now write down the equations for each of the coordinate planes as well using this idea.

$$\begin{aligned} z &= 0 \qquad xy-\text{plane} \\ y &= 0 \qquad xz-\text{plane} \\ x &= 0 \qquad yz-\text{plane} \end{aligned}$$

Let's take a look at a slightly more general example.

Example 18

Graph $y = 2x - 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

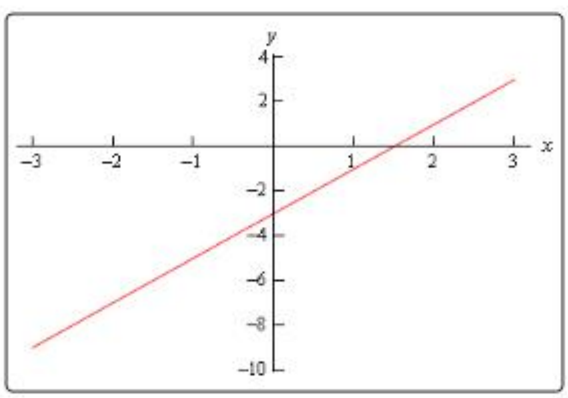
Solution

Note we had to throw out $\mathbb{R}$ for this example since there are two variables which means that we can't be in a 1-D space (1-D space has only one variable!).

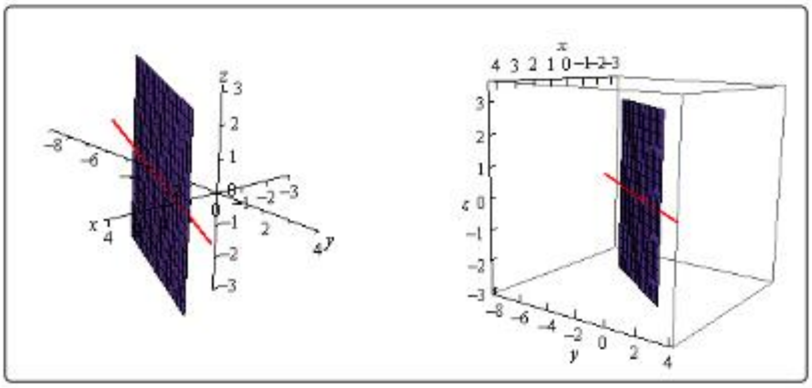
In $\mathbb{R}^2$ this is a line with slope 2 and a y intercept of -3.

However, in $\mathbb{R}^3$ this is not necessarily a line. Because we have not specified a value of z we are forced to let z take any value. This means that at any particular value of z we will get a copy of this line. So, the graph is then a vertical plane that lies over the line given by $y = 2x - 3$ in the xy-plane.

Here is the graph in $\mathbb{R}^2$.



here is the graph in $\mathbb{R}^3$.



Notice that if we look to where the plane intersects the xy-plane we will get the graph of the line in $\mathbb{R}^2$ as noted in the above graph by the red line through the plane.

Let's take a look at one more example of the difference between graphs in the different coordinate systems.

Example 19

Graph $x^2 + y^2 = 4$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

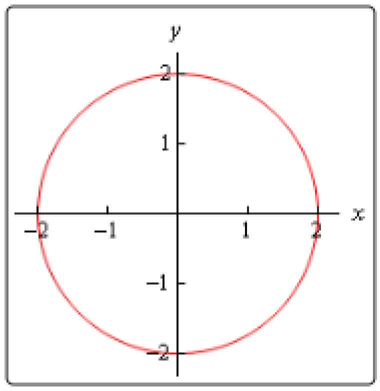
Solution

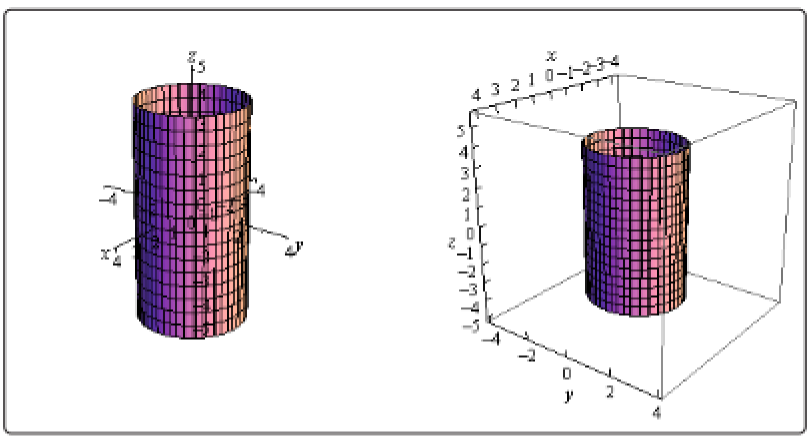
As with the previous example this won't have a 1-D graph since there are two variables.

In $\mathbb{R}^2$ this is a circle centered at the origin with radius 2.

In $\mathbb{R}^3$ however, as with the previous example, this may or may not be a circle. Since we have not specified z in any way we must assume that z can take on any value. In other words, at any value of z this equation must be satisfied and so at any value z we have a circle of radius 2 centered on the z-axis. This means that we have a cylinder of radius 2 centered on the z-axis.

Here are the graphs for this example.





Notice that again, if we look to where the cylinder intersects the xy-plane we will again get the circle from $\mathbb{R}^2$.

We need to be careful with the last two examples. It would be tempting to take the results of these and say that we can't graph lines or circles in $\mathbb{R}^3$ and yet that doesn't really make sense. There is no reason for there to not be graphs of lines or circles in $\mathbb{R}^3$. Let's think about the example of the circle. To graph a circle in $\mathbb{R}^3$ we would need to do something like $x^2 + y^2 = 4$ at $z = 5$. This would be a circle of radius 2 centered on the z-axis at the level of $z = 5$. So, as long as we specify a z we will get a circle and not a cylinder. We will see an easier way to specify circles in a later section.

We could do the same thing with the line from the second example. However, we will be looking at lines in more generality in the next section and so we'll see a better way to deal with lines in $\mathbb{R}^3$ there.

The point of the examples in this section is to make sure that we are being careful with graphing equations and making sure that we always remember which coordinate system that we are in.

Another quick point to make here is that, as we've seen in the above examples, many graphs of equations in $\mathbb{R}^3$ are surfaces. That doesn't mean that we can't graph curves in $\mathbb{R}^3$. We can and will graph curves in $\mathbb{R}^3$ as well as we'll see later in this chapter.

1.6 Equations of Lines

In this section we need to take a look at the equation of a line in $\mathbb{R}^3$. As we saw in the previous section the equation $y = mx + b$ does not describe a line in $\mathbb{R}^3$, instead it describes a plane. This doesn't mean however that we can't write down an equation for a line in 3-D space. We're just going to need a new way of writing down the equation of a curve.

So, before we get into the equations of lines we first need to briefly look at vector functions. We're going to take a more in depth look at vector functions later. At this point all that we need to worry about is notational issues and how they can be used to give the equation of a curve.

The best way to get an idea of what a vector function is and what its graph looks like is to look at an example. So, consider the following vector function.

$$\vec{r}(t) = \langle t, 1 \rangle$$

A vector function is a function that takes one or more variables, one in this case, and returns a vector. Note as well that a vector function can be a function of two or more variables. However, in those cases the graph may no longer be a curve in space.

The vector that the function gives can be a vector in whatever dimension we need it to be. In the example above it returns a vector in $\mathbb{R}^2$. When we get to the real subject of this section, equations of lines, we'll be using a vector function that returns a vector in $\mathbb{R}^3$

Now, we want to determine the graph of the vector function above. In order to find the graph of our function we'll think of the vector that the vector function returns as a position vector for points on the graph. Recall that a position vector, say $\vec{v} = \langle a, b \rangle$, is a vector that starts at the origin and ends at the point (a, b).

So, to get the graph of a vector function all we need to do is plug in some values of the variable and then plot the point that corresponds to each position vector we get out of the function and play connect the dots. Here are some evaluations for our example.

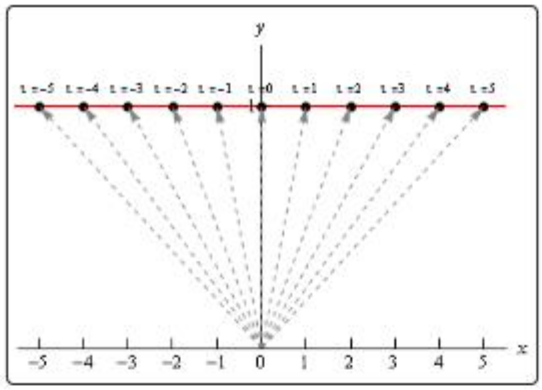
$$\vec{r}(-3) = \langle -3, 1 \rangle \qquad \vec{r}(-1) = \langle -1, 1 \rangle \qquad \vec{r}(2) = \langle 2, 1 \rangle \qquad \vec{r}(5) = \langle 5, 1 \rangle$$

So, each of these are position vectors representing points on the graph of our vector function. The points,

$$(-3, 1) \qquad (-1, 1) \qquad (2, 1) \qquad (5, 1)$$

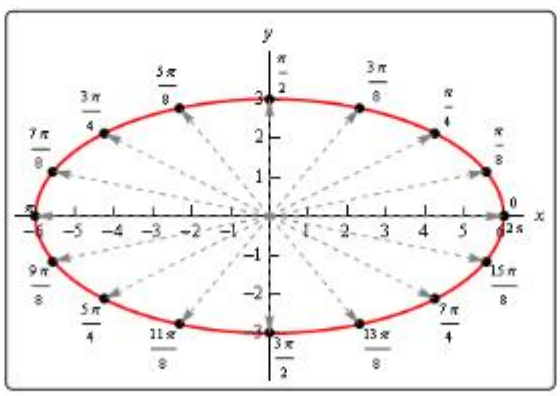
are all points that lie on the graph of our vector function.

If we do some more evaluations and plot all the points we get the following sketch.



In this sketch we've included the position vector (in gray and dashed) for several evaluations as well as the t (above each point) we used for each evaluation. It looks like, in this case the graph of the vector equation is in fact the line $y = 1$.

Here's another quick example. Here is the graph of $\vec{r}(t) = \langle 6\cos(t), 3\sin(t)\rangle$.



In this case we get an ellipse. It is important to not come away from this section with the idea that vector functions only graph out lines. We'll be looking at lines in this section, but the graphs of vector functions do not have to be lines as the example above shows.

We'll leave this brief discussion of vector functions with another way to think of the graph of a vector function. Imagine that a pencil/pen is attached to the end of the position vector and as we increase the variable the resulting position vector moves and as it moves the pencil/pen on the end sketches out the curve for the vector function.

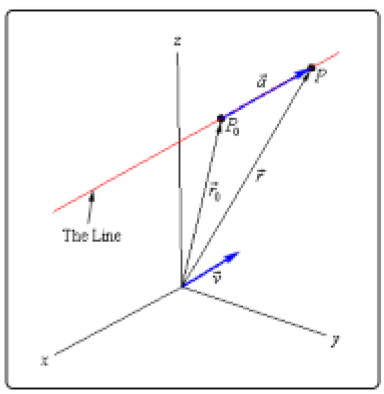
Okay, we now need to move into the actual topic of this section. We want to write down the equation of a line in $\mathbb{R}^3$ and as suggested by the work above we will need a vector function to do this. To see how we're going to do this let's think about what we need to write down the equation of a line in $\mathbb{R}^2$. In two dimensions we need the slope (m) and a point that was on the line in order to write down the equation.

In $\mathbb{R}^3$ that is still all that we need except in this case the "slope" won't be a simple number as it was in two dimensions. In this case we will need to acknowledge that a line can have a three dimensional slope. So, we need something that will allow us to describe a direction that is potentially in three dimensions. We already have a quantity that will do this for us. Vectors give directions and can be three dimensional objects.

So, let's start with the following information. Suppose that we know a point that is on the line, $P_0 = (x_0, y_0, z_0)$, and that $\vec{v} = \langle a, b, c \rangle$ is some vector that is parallel to the line. Note, in all likelihood, $\vec{v}$ will not be on the line itself. We only need $\vec{v}$ to be parallel to the line. Finally, let $P = (x, y, z)$ be any point on the line.

Now, since our "slope" is a vector let's also represent the two points on the line as vectors. We'll do this with position vectors. So, let $\vec{r_0}$ and $\vec{r}$ be the position vectors for P_0 and P respectively. Also, for no apparent reason, let's define $\vec{a}$ to be the vector with representation $\overrightarrow{P_0P}$.

We now have the following sketch with all these points and vectors on it.



Now, we've shown the parallel vector, $\vec{v}$, as a position vector but it doesn't need to be a position vector. It can be anywhere, a position vector, on the line or off the line, it just needs to be parallel to the line.

Next, notice that we can write $\vec{r}$ as follows,

$$\vec{r} = \vec{r_0} + \vec{a}$$

If you're not sure about this go back and check out the sketch for vector addition in the vector arithmetic section. Now, notice that the vectors $\vec{a}$ and $\vec{v}$ are parallel. Therefore there is a number, t, such that

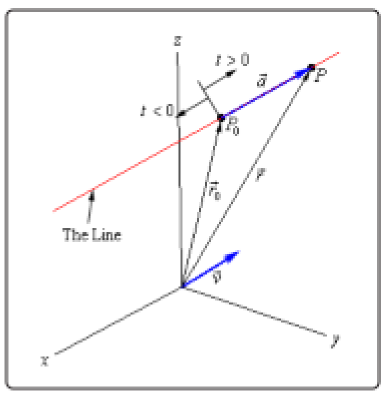
$$\vec{a} = t\,\vec{v}$$

We now have,

Vector Form of a Line

$$\vec{r} = \vec{r_0} + t\,\vec{v} = \langle x_0, y_0, z_0 \rangle + t\,\langle a, b, c \rangle$$

The only part of this equation that is not known is the t. Notice that $t\,\vec{v}$ will be a vector that lies along the line and it tells us how far from the original point that we should move. If t is positive we move away from the original point in the direction of $\vec{v}$ (right in our sketch) and if t is negative we move away from the original point in the opposite direction of $\vec{v}$ (left in our sketch). As t varies over all possible values we will completely cover the line. The following sketch shows this dependence on t of our sketch.



There are several other forms of the equation of a line. To get the first alternate form let's start with the vector form and do a slight rewrite.

$$\vec{r} = \langle x_0, y_0, z_0 \rangle + t\,\langle a, b, c \rangle$$
$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

The only way for two vectors to be equal is for the components to be equal. In other words,

Parametric Form of a Line

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

Notice that this is really nothing more than an extension of the parametric equations we've seen previously. The only difference is that we are now working in three dimensions instead of two dimensions.

To get a point on the line all we do is pick a t and plug into either form of the line. In the vector form of the line we get a position vector for the point and in the parametric form we get the actual coordinates of the point.

There is one more form of the line that we want to look at. If we assume that a, b, and c are all non-zero numbers we can solve each of the equations in the parametric form of the line for t. We can then set all of them equal to each other since t will be the same number in each. Doing this gives the following,

Symmetric Equations of a Line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

If one of a, b, or c does happen to be zero we can still write down the symmetric equations. To see this let's suppose that $b = 0$. In this case t will not exist in the parametric equation for y and so we will only solve the parametric equations for x and z for t. We then set those equal and acknowledge the parametric equation for y as follows,

$$\frac{x - x_0}{a} = \frac{z - z_0}{c} \qquad y = y_0$$

Let's take a look at an example.

Example 20

Write down the equation of the line that passes through the points $(2, -1, 3)$ and $(1, 4, -3)$. Write down all three forms of the equation of the line.

Solution

To do this we need the vector $\vec{v}$ that will be parallel to the line. This can be any vector as long as it's parallel to the line. In general, $\vec{v}$ won't lie on the line itself. However, in this case it will. All we need to do is let $\vec{v}$ be the vector that starts at the second point and ends at the first point. Since these two points are on the line the vector between them will also lie on the line and will hence be parallel to the line. So,

$$\vec{v} = \langle 1, -5, 6 \rangle$$

Note that the order of the points was chosen to reduce the number of minus signs in the vector. We could just have easily gone the other way.

Once we've got $\vec{v}$ there really isn't anything else to do. To use the vector form we'll need a point on the line. We've got two and so we can use either one. We'll use the first point. Here is the vector form of the line.

$$\vec{r} = \langle 2, -1, 3 \rangle + t \langle 1, -5, 6 \rangle = \langle 2 + t, -1 - 5t, 3 + 6t \rangle$$

Once we have this equation the other two forms follow. Here are the parametric equations of the line.

$$x = 2 + t$$
$$y = -1 - 5t$$
$$z = 3 + 6t$$

Here is the symmetric form.

$$\frac{x-2}{1} = \frac{y+1}{-5} = \frac{z-3}{6}$$

Example 21

Determine if the line that passes through the point $(0, -3, 8)$ and is parallel to the line given by $x = 10 + 3t$, $y = 12t$ and $z = -3 - t$ passes through the xz-plane. If it does give the coordinates of that point.

Solution

To answer this we will first need to write down the equation of the line. We know a point on the line and just need a parallel vector. We know that the new line must be parallel to the line given by the parametric equations in the problem statement. That means that any vector that is parallel to the given line must also be parallel to the new line.

Now recall that in the parametric form of the line the numbers multiplied by t are the components of the vector that is parallel to the line. Therefore, the vector,

$$\vec{v} = \langle 3, 12, -1 \rangle$$

is parallel to the given line and so must also be parallel to the new line.

The equation of new line is then,

$$\vec{r} = \langle 0, -3, 8 \rangle + t \langle 3, 12, -1 \rangle = \langle 3t, -3 + 12t, 8 - t \rangle$$

If this line passes through the xz-plane then we know that the y-coordinate of that point must be zero. So, let's set the y component of the equation equal to zero and see if we can solve for t. If we can, this will give the value of t for which the point will pass through the xz-plane.

$$-3 + 12t = 0 \qquad \Rightarrow \qquad t = \frac{1}{4}$$

So, the line does pass through the xz-plane. To get the complete coordinates of the point all we need to do is plug $t = \frac{1}{4}$ into any of the equations. We'll use the vector form.

$$\vec{r} = \left\langle 3\left(\frac{1}{4}\right), -3 + 12\left(\frac{1}{4}\right), 8 - \frac{1}{4}\right\rangle = \left\langle \frac{3}{4}, 0, \frac{31}{4}\right\rangle$$

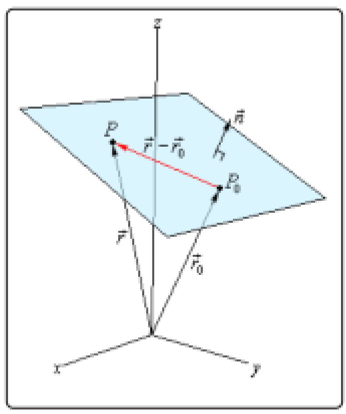
Recall that this vector is the position vector for the point on the line and so the coordinates of the point where the line will pass through the xz-plane are $\left(\frac{3}{4}, 0, \frac{31}{4}\right)$.

1.7 Equations of Planes

In the first section of this chapter we saw a couple of equations of planes. However, none of those equations had three variables in them and were really extensions of graphs that we could look at in two dimensions. We would like a more general equation for planes.

So, let's start by assuming that we know a point that is on the plane, $P_0 = (x_0, y_0, z_0)$. Let's also suppose that we have a vector that is orthogonal (perpendicular) to the plane, $\vec{n} = \langle a, b, c \rangle$. This vector is called the **normal vector**. Now, assume that $P = (x, y, z)$ is any point in the plane. Finally, since we are going to be working with vectors initially we'll let $\vec{r_0}$ and $\vec{r}$ be the position vectors for P_0 and P respectively.

Here is a sketch of all these vectors.



Notice that we added in the vector $\vec{r} - \vec{r_0}$ which will lie completely in the plane. Also notice that we put the normal vector on the plane, but there is actually no reason to expect this to be the case. We put it here to illustrate the point. It is completely possible that the normal vector does not touch the plane in any way.

Now, because $\vec{n}$ is orthogonal to the plane, it's also orthogonal to any vector that lies in the plane. In particular it's orthogonal to $\vec{r} - \vec{r_0}$. Recall from the Dot Product section that two orthogonal vectors will have a dot product of zero. In other words,

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \qquad \Rightarrow \qquad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

This is called the **vector equation of the plane**.

A slightly more useful form of the equations is as follows. Start with the first form of the vector equation and write down a vector for the difference.

$$\langle a, b, c \rangle \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0$$
$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Now, actually compute the dot product to get,

Scalar equation of the plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Often this will be written as,

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

This second form is often how we are given equations of planes. Notice that if we are given the equation of a plane in this form we can quickly get a normal vector for the plane. A normal vector is,

$$\vec{n} = \langle a, b, c \rangle$$

Let's work a couple of examples.

Example 22

Determine the equation of the plane that contains the points $P = (1, -2, 0)$, $Q = (3, 1, 4)$ and $R = (0, -1, 2)$.

Solution

In order to write down the equation of plane we need a point (we've got three so we're cool there) and a normal vector. We need to find a normal vector. Recall however, that we saw how to do this in the Cross Product section.

We can form the following two vectors from the given points.

$$\overrightarrow{PQ} = \langle 2, 3, 4 \rangle \qquad \overrightarrow{PR} = \langle -1, 1, 2 \rangle$$

These two vectors will lie completely in the plane since we formed them from points that were in the plane. Notice as well that there are many possible vectors to use here, we just chose two of the possibilities.

Now, we know that the cross product of two vectors will be orthogonal to both of these vectors. Since both of these are in the plane any vector that is orthogonal to both of these will also be orthogonal to the plane. Therefore, we can use the cross product as the normal vector.

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 2 & 3 \\ -1 & 1 \end{matrix} = 2\vec{i} - 8\vec{j} + 5\vec{k}$$

The equation of the plane is then,

$$\begin{aligned} 2(x-1) - 8(y+2) + 5(z-0) &= 0 \\ 2x - 8y + 5z &= 18 \end{aligned}$$

We used P for the point but could have used any of the three points.

Example 23

Determine if the plane given by $-x + 2z = 10$ and the line given by $\vec{r} = \langle 5, 2-t, 10+4t \rangle$ are orthogonal, parallel or neither.

Solution

This is not as difficult a problem as it may at first appear to be. We can pick off a vector that is normal to the plane. This is $\vec{n} = \langle -1, 0, 2 \rangle$. We can also get a vector that is parallel to the line. This is $v = \langle 0, -1, 4 \rangle$.

Now, if these two vectors are parallel then the line and the plane will be orthogonal. If you think about it this makes some sense. If $\vec{n}$ and $\vec{v}$ are parallel, then $\vec{v}$ is orthogonal to the plane, but $\vec{v}$ is also parallel to the line. So, if the two vectors are parallel the line and plane will be orthogonal.

Let's check this.

$$\vec{n} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 2 \\ 0 & -1 & 4 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ -1 & 0 \\ 0 & -1 \end{matrix} = 2\vec{i} + 4\vec{j} + \vec{k} \neq \vec{0}$$

So, the vectors aren't parallel and so the plane and the line are not orthogonal.

Now, let's check to see if the plane and line are parallel. If the line is parallel to the plane then any vector parallel to the line will be orthogonal to the normal vector of the plane. In other words, if $\vec{n}$ and $\vec{v}$ are orthogonal then the line and the plane will be parallel.

Let's check this.

$$\vec{n} \cdot \vec{v} = 0 + 0 + 8 = 8 \neq 0$$

The two vectors aren't orthogonal and so the line and plane aren't parallel.

So, the line and the plane are neither orthogonal nor parallel.

1.8 Quadric Surfaces

In the previous two sections we've looked at lines and planes in three dimensions (or $\mathbb{R}^3$) and while these are used quite heavily at times in a Calculus class there are many other surfaces that are also used fairly regularly and so we need to take a look at those.

In this section we are going to be looking at quadric surfaces. Quadric surfaces are the graphs of any equation that can be put into the general form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where $A, \ldots, J$ are constants.

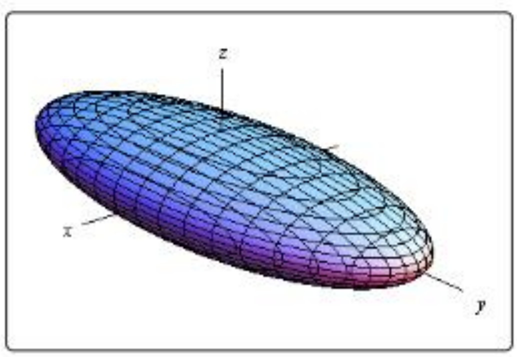
There is no way that we can possibly list all of them, but there are some standard equations so here is a list of some of the more common quadric surfaces.

Ellipsoid

Here is the general equation of an ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical ellipsoid.



If $a = b = c$ then we will have a sphere.

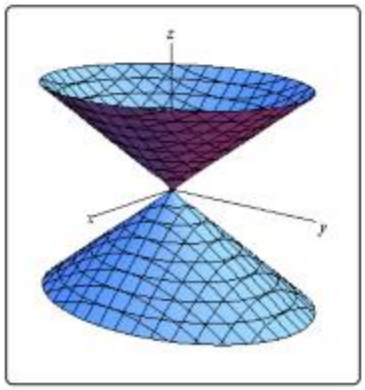
Notice that we only gave the equation for the ellipsoid that has been centered on the origin. Clearly ellipsoids don't have to be centered on the origin. However, in order to make the discussion in this section a little easier we have chosen to concentrate on surfaces that are "centered" on the origin in one way or another.

Cone

Here is the general equation of a cone.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Here is a sketch of a typical cone.



Now, note that while we called this a cone it is more of an hour glass shape rather than what most would call a cone. Of course, the upper and the lower portion of the hour glass really are cones as we would normally think of them.

That brings up the question of what if we really did just want the upper or lower portion (*i.e.* a cone in the traditional sense)? That is easy enough to answer. All we need to do is solve the given equation for z as follows,

$$z^2 = c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = \frac{c^2}{a^2}x^2 + \frac{c^2}{b^2}y^2 = A^2x^2 + B^2y^2 \quad \rightarrow \quad z = \pm\sqrt{A^2x^2+B^2y^2}$$

We simplified the coefficients a little to make it the equation(s) easier to deal with. Now, we know that square roots always return positive numbers and so we can then see that $z = \sqrt{A^2x^2+B^2y^2}$ will always be positive and so be the equation for just the upper portion of the "cone" above. Likewise, $z = -\sqrt{A^2x^2+B^2y^2}$ will always be negative and so be the equation of just the lower portion of the "cone" above.

Also, note that this is the equation of a cone that will open along the z-axis. To get the equation of a cone that opens along one of the other axes all we need to do is make a slight modification of the equation. This will be the case for the rest of the surfaces that we'll be looking at in this section as well.

In the case of a cone the variable that sits by itself on one side of the equal sign will determine the axis that the cone opens up along. For instance, a cone that opens up along the x-axis will have the equation,

$$\frac{y^2}{b^2}+\frac{z^2}{c^2} = \frac{x^2}{a^2}$$

For most of the following surfaces we will not give the other possible formulas. We will however acknowledge how each formula needs to be changed to get a change of orientation for the surface.

Cylinder

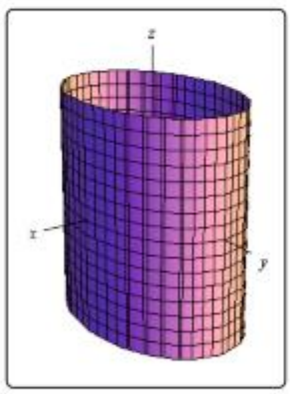
Here is the general equation of a cylinder.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is a cylinder whose cross section is an ellipse. If $a = b$ we have a cylinder whose cross section is a circle. We'll be dealing with those kinds of cylinders more than the general form so the equation of a cylinder with a circular cross section is,

$$x^2 + y^2 = r^2$$

Here is a sketch of typical cylinder with an ellipse cross section.



The cylinder will be centered on the axis corresponding to the variable that does not appear in the equation.

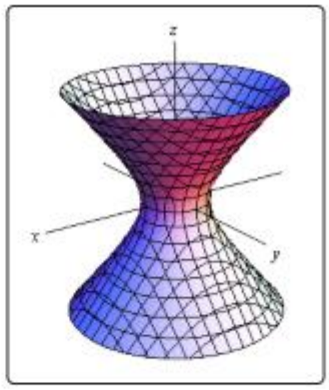
Be careful to not confuse this with a circle. In two dimensions it is a circle, but in three dimensions it is a cylinder.

Hyperboloid of One Sheet

Here is the equation of a hyperboloid of one sheet.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical hyperboloid of one sheet.



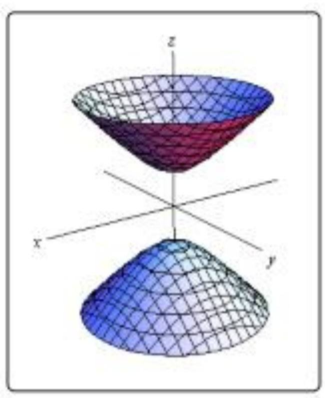
The variable with the negative in front of it will give the axis along which the graph is centered.

Hyperboloid of Two Sheets

Here is the equation of a hyperboloid of two sheets.

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical hyperboloid of two sheets.



The variable with the positive in front of it will give the axis along which the graph is centered.

Notice that the only difference between the hyperboloid of one sheet and the hyperboloid of two sheets is the signs in front of the variables. They are exactly the opposite signs.

Also note that just as we could do with cones, if we solve the equation for z the positive portion will give the equation for the upper part of this while the negative portion will give the equation for the lower part of this.

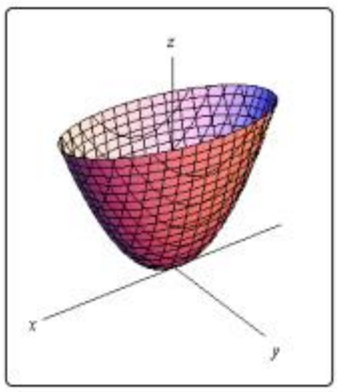
Elliptic Paraboloid

Here is the equation of an elliptic paraboloid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

As with cylinders this has a cross section of an ellipse and if $a = b$ it will have a cross section of a circle. When we deal with these we'll generally be dealing with the kind that have a circle for a cross section.

Here is a sketch of a typical elliptic paraboloid.



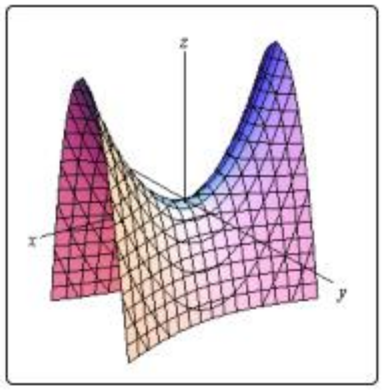
In this case the variable that isn't squared determines the axis upon which the paraboloid opens up. Also, the sign of c will determine the direction that the paraboloid opens. If c is positive then it opens up and if c is negative then it opens down.

Hyperbolic Paraboloid

Here is the equation of a hyperbolic paraboloid.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

Here is a sketch of a typical hyperbolic paraboloid.



These graphs are vaguely saddle shaped and as with the elliptic paraboloid the sign of c will determine the direction in which the surface "opens up". The graph above is shown for c positive.

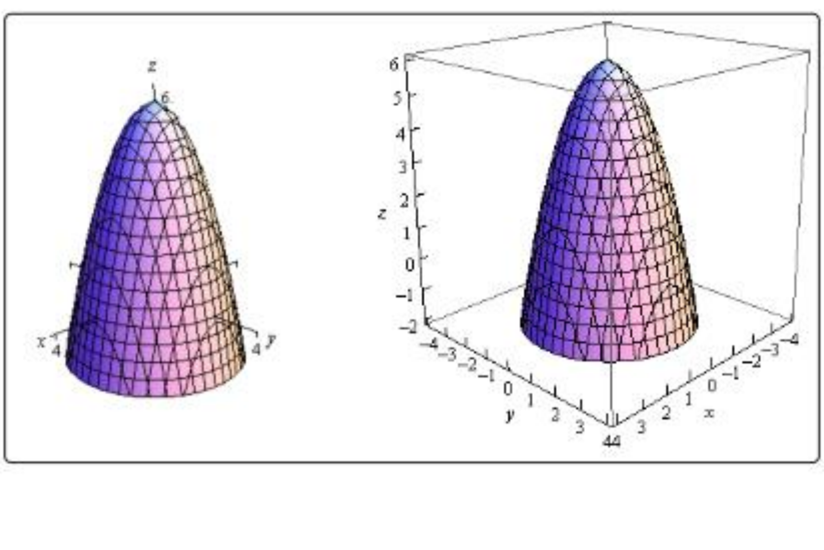
With both of the types of paraboloids discussed above note that the surface can be easily moved up or down by adding/subtracting a constant from the left side.

For instance

$$z = -x^2 - y^2 + 6$$

is an elliptic paraboloid that opens downward (be careful, the "$-$" is on the x and y instead of the z) and starts at $z = 6$ instead of $z = 0$.

Here are a couple of quick sketches of this surface.

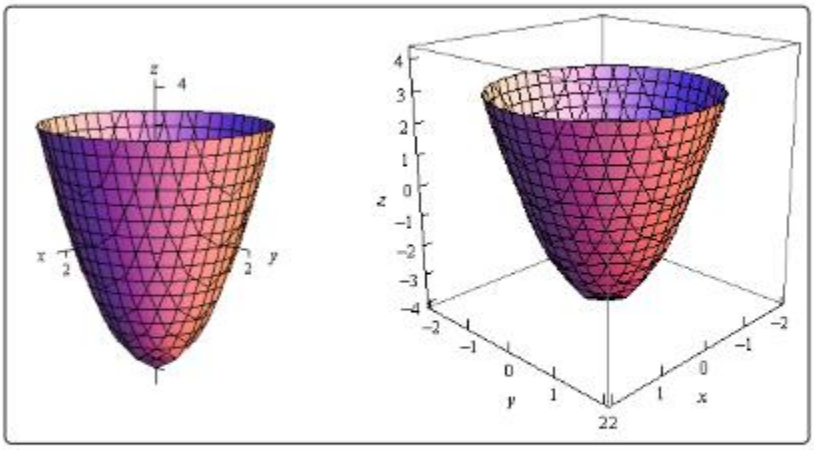


Note that we've given two forms of the sketch here. The sketch on the left has the standard set of axes but it is difficult to see the numbers on the axis. The sketch on the right has been "boxed" and this makes it easier to see the numbers to give a sense of perspective to the sketch. In most sketches that actually involve numbers on the axis system we will give both sketches to help get a feel for what the sketch looks like.

1.9 Functions of Several Variables

In this section we want to go over some of the basic ideas about functions of more than one variable.

First, remember that graphs of functions of two variables, $z = f(x, y)$ are surfaces in three dimensional space. For example, here is the graph of $z = 2x^2 + 2y^2 - 4$.



This is an elliptic paraboloid and is an example of a quadric surface. We saw several of these in the previous section. We will be seeing quadric surfaces fairly regularly later on when we start discussion Multi-variable Calculus.

Another common graph that we'll be seeing quite a bit in this course is the graph of a plane. We have a convention for graphing planes that will make them a little easier to graph and hopefully visualize.

Recall that the equation of a plane is given by

$$ax + by + cz = d$$

or if we solve this for z we can write it in terms of function notation. This gives,

$$f(x, y) = Ax + By + D$$

To graph a plane we will generally find the intersection points with the three axes and then graph the triangle that connects those three points. This triangle will be a portion of the plane and it will give us a fairly decent idea on what the plane itself should look like. For example, let's graph the plane given by,

$$f(x, y) = 12 - 3x - 4y$$

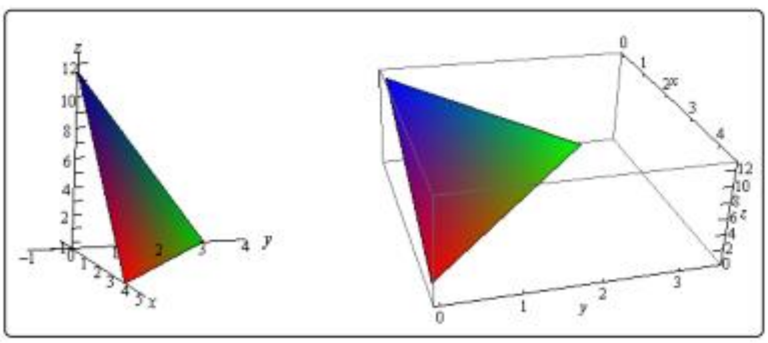
For purposes of graphing this it would probably be easier to write this as,

$$z = 12 - 3x - 4y \quad \Rightarrow \quad 3x + 4y + z = 12$$

Now, each of the intersection points with the three main coordinate axes is defined by the fact that two of the coordinates are zero. For instance, the intersection with the z-axis is defined by $x = y = 0$. So, the three intersection points are,

$$\begin{aligned} x - \text{axis}: & \ (4, 0, 0) \\ y - \text{axis}: & \ (0, 3, 0) \\ z - \text{axis}: & \ (0, 0, 12) \end{aligned}$$

Here is the graph of the plane.



Now, to extend this out, graphs of functions of the form $w = f(x, y, z)$ would be four dimensional surfaces. Of course, we can't graph them, but it doesn't hurt to point this out.

We next want to talk about the domains of functions of more than one variable. Recall that domains of functions of a single variable, $y = f(x)$, consisted of all the values of x that we could plug into the function and get back a real number. Now, if we think about it, this means that the domain of a function of a single variable is an interval (or intervals) of values from the number line, or one dimensional space.

The domain of functions of two variables, $z = f(x, y)$, are regions from two dimensional space and consist of all the coordinate pairs, (x, y), that we could plug into the function and get back a real number.

Example 24

Determine the domain of each of the following.

(a) $f(x, y) = \sqrt{x + y}$

(b) $f(x, y) = \sqrt{x} + \sqrt{y}$

(c) $f(x, y) = \ln\left(9 - x^2 - 9y^2\right)$

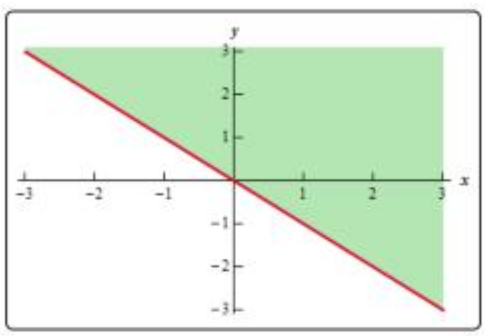
Solution

(a) $f(x, y) = \sqrt{x + y}$

In this case we know that we can't take the square root of a negative number so this means that we must require,

$$x + y \geq 0$$

Here is a sketch of the graph of this region.

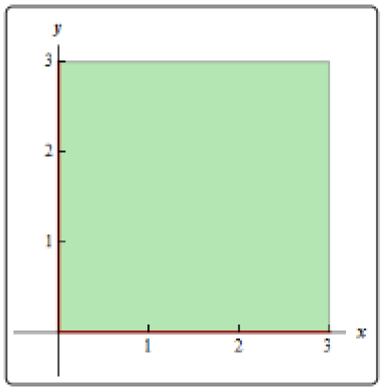


(b) $f(x, y) = \sqrt{x} + \sqrt{y}$

This function is different from the function in the previous part. Here we must require that,

$$x \geq 0 \quad \text{and} \quad y \geq 0$$

and they really do need to be separate inequalities. There is one for each square root in the function. Here is the sketch of this region.

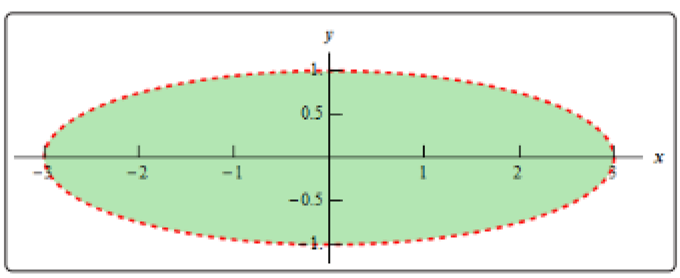


(c) $f(x, y) = \ln(9 - x^2 - 9y^2)$

In this final part we know that we can't take the logarithm of a negative number or zero. Therefore, we need to require that,

$$9 - x^2 - 9y^2 > 0 \qquad \Rightarrow \qquad \frac{x^2}{9} + y^2 < 1$$

and upon rearranging we see that we need to stay interior to an ellipse for this function. Here is a sketch of this region.



Note that domains of functions of three variables, $w = f(x, y, z)$, will be regions in three dimensional space.

Example 25

Determine the domain of the following function,

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2-16}}$$

Solution

In this case we have to deal with the square root and division by zero issues. These will require,

$$x^2+y^2+z^2-16>0 \qquad \Rightarrow \qquad x^2+y^2+z^2>16$$

So, the domain for this function is the set of points that lies completely outside a sphere of radius 4 centered at the origin.

The next topic that we should look at is that of **level curves** or **contour curves**. The level curves of the function $z = f(x,y)$ are two dimensional curves we get by setting $z = k$, where k is any number. So the equations of the level curves are $f(x,y) = k$. Note that sometimes the equation will be in the form $f(x,y,z) = 0$ and in these cases the equations of the level curves are $f(x,y,k) = 0$.

You've probably seen level curves (or contour curves, whatever you want to call them) before. If you've ever seen the elevation map for a piece of land, this is nothing more than the contour curves for the function that gives the elevation of the land in that area. Of course, we probably don't have the function that gives the elevation, but we can at least graph the contour curves.

Let's do a quick example of this.

Example 26

Identify the level curves of $f(x,y) = \sqrt{x^2+y^2}$. Sketch a few of them.

Solution

First, for the sake of practice, let's identify what this surface given by $f(x,y)$ is. To do this let's rewrite it as,

$$z = \sqrt{x^2+y^2}$$

Recall from the Quadric Surfaces section that this the upper portion of the "cone" (or hour glass shaped surface).

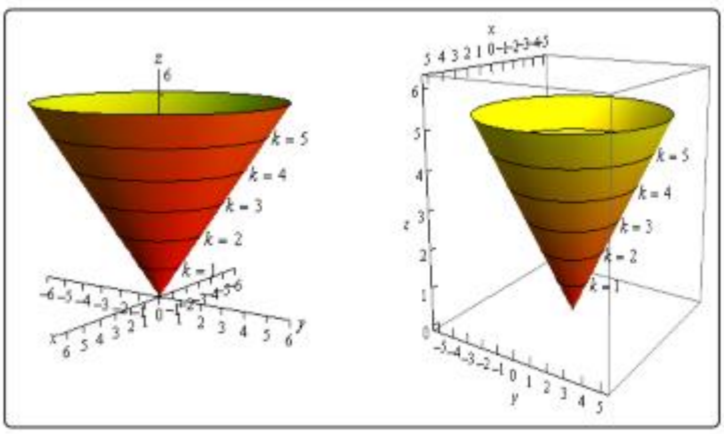
Note that this was not required for this problem. It was done for the practice of identifying the surface and this may come in handy down the road.

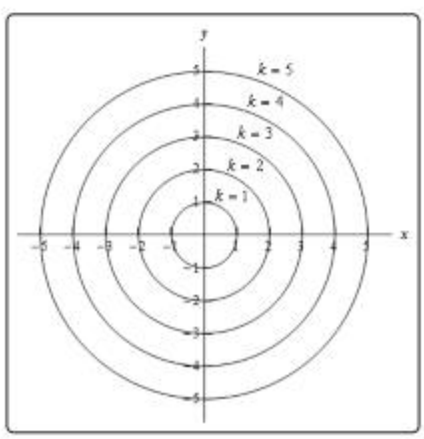
Now on to the real problem. The level curves (or contour curves) for this surface are given by the equation are found by substituting $z = k$. In the case of our example this is,

$$k = \sqrt{x^2 + y^2} \qquad \Rightarrow \qquad x^2 + y^2 = k^2$$

where k is any number. So, in this case, the level curves are circles of radius k with center at the origin.

We can graph these in one of two ways. We can either graph them on the surface itself or we can graph them in a two dimensional axis system. Here is each graph for some values of k.





Note that we can think of contours in terms of the intersection of the surface that is given by $z = f(x, y)$ and the plane $z = k$. The contour will represent the intersection of the surface and the plane.

For functions of the form $f(x, y, z)$ we will occasionally look at **level surfaces**. The equations of level surfaces are given by $f(x, y, z) = k$ where k is any number.

The final topic in this section is that of **traces**. In some ways these are similar to contours. As noted above we can think of contours as the intersection of the surface given by $z = f(x, y)$ and the plane $z = k$. Traces of surfaces are curves that represent the intersection of the surface and the plane given by $x = a$ or $y = b$.

Let's take a quick look at an example of traces.

Example 27

Sketch the traces of $f(x, y) = 10 - 4x^2 - y^2$ for the plane $x = 1$ and $y = 2$.

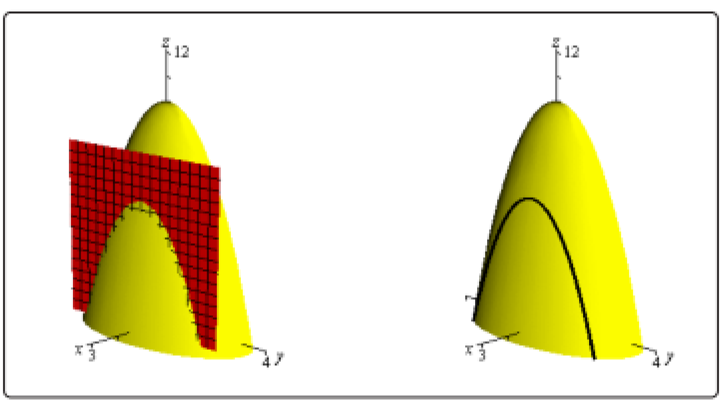
Solution

We'll start with $x = 1$. We can get an equation for the trace by plugging $x = 1$ into the equation. Doing this gives,

$$z = f(1, y) = 10 - 4(1)^2 - y^2 \qquad \Rightarrow \qquad z = 6 - y^2$$

and this will be graphed in the plane given by $x = 1$.

Below are two graphs. The graph on the left is a graph showing the intersection of the surface and the plane given by $x = 1$. On the right is a graph of the surface and the trace that we are after in this part.



For $y = 2$ we will do pretty much the same thing that we did with the first part. Here is the equation of the trace,

$$z = f(x, 2) = 10 - 4x^2 - (2)^2 \qquad \Rightarrow \qquad z = 6 - 4x^2$$

and here are the sketches for this case.