Chapter 7

Hamilton's Principle-Lagrangian and Hamiltonian

7.1 Introduction

Experience has shown that a particle's motion in an internal reference frame is correctly described by the Newtonian equation (see chapter 2)

$$F = \dot{p} \qquad (7\text{-}1)$$

If the particle is not required to move in some complicated manner and if rectangular coordinates are used to describe the motion, then usually the equations of motion are **<u>relatively simple</u>** and we can use Newtonian equation to solve the problem and to find the equation of motion. But if either of these restrictions is removed, the equations can can become quite complex and difficult to manipulate.

In fact, to solve the problem by using the Newtonian procedure (see Chapter 2), we must know *all the forces* because the quantity *F* that appears in the fundamental equation is the total force acting on a body. To circumvent some of the practical difficulties that arise in attempts to apply Newton's equations to particular problems, alternate procedure may be developed. The method is known as **Hamilton's Principle** and the equations of motion resulting from the application of this principle are called the **Lagrangian's equations.**

If **Lagrangian's equations** are to constitute a proper description of dynamics of particles, they must be equivalent to Newton's equations. On the other hand,

Hamilton's Principle can be applied to a wide range of physical phenomena (*particularly those involving fields*) not usually associated with Newton's equations.

7.2 Hamilton's Principle

In terms of calculus of variations, Hamilton's Principle becomes

$$\delta \int_{t_1}^{t_2} (T - U)dt = 0 \qquad (7.2)$$

Where δ is an operation that represents a variation of any particular system parameter. While T is the kinetic energy of the system and U is the potential energy of the system.

The kinetic energy of a particle expressed in fixed, Cartesian (rectangular) coordinated is a function only of the $\dot{x}_i$ and if the particle moves in a conservative force field, the potential energy is a function only of the $\dot{x}_i$;

$$T = T(\dot{x}_i), \qquad U = U(x_i) \qquad (7.3)$$

If we define the differences of these quantities to be

$$L = T - U = L(x_i, \dot{x}_i) \qquad (7.4)$$

L : The differences between the kinetic and potential energies of a system, called the **Lagrangian** of the system.

The Euler-Lagrange equations is given by

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \quad i = 1,2,3 \qquad (7.5)$$

These are the Lagrange equation of motion for the particle.

Example 7.1: Use the Lagrange equation to obtain the equation of motion for one-dimensional harmonic oscillator.

Answer:

With the usual expressions for the kinetic and potential energies, we have

$$L = T - U \qquad (7.6)$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \qquad (7.7)$$

$$\frac{\partial L}{\partial x} = -kx \qquad (7.8)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad (7.9)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad (7.10)$$

Substituting these results into equation 5.5, this leads to

$$m\ddot{x} + kx = 0 \qquad (7.11)$$

Equation 5.11 is identical with the equation of motion obtained using Newtonian mechanics (See equation 3.5, Chapter 3).

Example 7.2: Use the Lagrange equation to obtain the equation of motion of Simple pendulum.

Answer:

The kinetic and potential energies of the system are given by:

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

$$U = mgl(1 - \cos\theta)$$

Thus,

$$L = T - U$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \theta} = -mgl\ sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

Therefore,

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$

$$-mgl\ sin\theta - ml^2\ddot{\theta} = 0$$

$$g\ sin\theta + l\ddot{\theta} = 0$$

$$\ddot{\theta} + \left(\frac{g}{l}\right)sin\theta = 0$$

This result is identical with the Newtonian result (See equation 3.15, Chapter 3).

This is a remarkable result; it has been obtained by calculating the kinetic and potential energies in terms of θ rather than x and then applying a set of operations designed for use with rectangular rather than angular coordinates.

Another important characteristic of the method used in two preceding simple examples is that nowhere in the calculation did there enter any statement **regarding force.**

7.3 Generalized coordinates.

Depending on the problem at hand, it may prove more convenient to choose some of the parameters with dimensions of *energy*, some with dimensions of $(lenght)^2$, some that are *dimmentionless* and so forth. We give the name **generalized coordinates** to any set of quantities that completely specifies the state of a system. The generalized coordinates are customarily written as $q_1, q_2, q_3, \ldots$ or simply as the q_i. A set of independent generalized coordinates whose number equals the number s of degrees of freedom of the system and not restricted by the constraints is called a *proper* set of generalized coordinates.

In addition to the generalized coordinates, we may define a set of quantities consisting of time derivatives of $\dot{q}_1, \dot{q}_2 \ldots \ldots \ldots$ or simply $\dot{q}_j$.

Example 7.3. Use the (x, y) coordinate system in Figure 5.1 to find the kinetic energy T, potential energy U, and the Lagrangian L for a simple pendulum (length l, mass bob m) moving in x,y plane .Determine the transformation equations from the (x, y) rectangular system to the coordinate θ. Find the equation of motion.
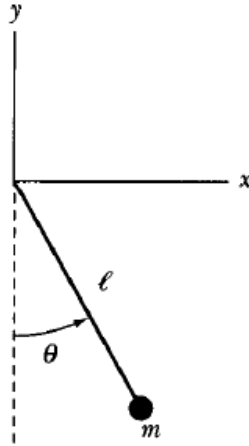

Fig. 7.1 A simple pendulum of length l and bob of mass m.

Answer:

The kinetic and potential energies and the Lagrangian become

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

$$U = mgy$$

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

Inspection of figure 5.1 reveals that the motion can be better described by using θ and $\dot{\theta}$. Let's transform x and y into the coordinate θ and then find L in terms of θ.

$$x = l\sin\theta$$

$$y = -l\cos\theta$$

We now find for $\dot{x}$ and $\dot{y}$

$$\dot{x} = l\dot{\theta}\cos\theta$$

$$\dot{y} = l\dot{\theta}\sin\theta$$

$$L = \frac{m}{2}\left(l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta\right) + mgl\cos\theta$$

$$L = \frac{m}{2}l^2\dot{\theta}^2 + mgl\cos\theta$$

We can follow the same way used in Example 7.2 to get the equation of motion

$$\ddot{\theta} + \left(\frac{g}{l}\right)\sin\theta = 0$$

7.4 Lagrange's Equations of motion in Generalized coordinates.

To set up the variational form of Hamilton's principle in generalized coordinate, we may take advantage of an important property of the Lagrangian we have not so far emphasized. The Lagrangian for a system is defined to be the differences between the kinetic and potential energies. But energy is a scale quantity and so the Lagrangian is a scale function. Hence the Lagrangian must be invariant with respect to coordinate transformations. However, certain transformations that change the Lagrangian but leave the equations of motion unchanged are allowed.

We express the Lagrangian in terms of $x_{a,i}$ and $\dot{x}_{a,i}$ or q_j and $\dot{q}_j$

$$L = T(\dot{x}_{a,i}) - U(x_{a,i}) \qquad (7.12)$$

$$L = T(q_j, \dot{q}_j, t) - U(q_j, t) \qquad (7.13)$$

that is

$$L = L(q_1, q_1, \ldots\ldots\ldots, q_s; \dot{q}_1, \dot{q}_2, \ldots\ldots\ldots, \dot{q}_s, t) \qquad (7.14)$$

$$L = L(q_j, \dot{q}_{j,}t) \qquad (7.15)$$

Thus, Hamilton's Principle becomes

$$\delta \int_{t_1}^{t_2} L(q_j, \dot{q}_{j,}t)\, dt = 0 \qquad (7.16)$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0, \qquad j = 1,2,\ldots\ldots\ldots\ldots, s \qquad (7.17)$$

It is important to realize that the validity of Lagrange's equation requires the following two conditions:

1- The force acting on the system (apart from any forces of constraint) must be derivable from the potential

2- The equations of constraint must be relations that connect the coordinates of the particles and may be functions of the time.

Example 7.4: Consider the case of projectile motion under gravity in two dimensions (as was discussed in Chapter 2). Find the equations of motion in both Cartesian and polar coordinates.

Answer:

In Cartesian coordinate, we use x (horizantoal) and y (vertical). In polar coordinate, we use r (in radial direction) and θ (elevation angle from horizontal)
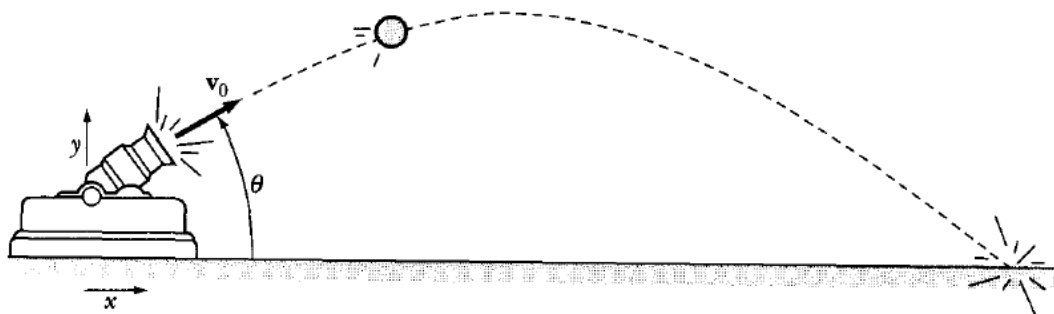
Fig. 7.2 Projectile motion in two-dimensions

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

$$U = mgy$$

Where $U = 0$ at $y = 0$

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

We find the questions of motion by using Equation 7.17

x:
$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \qquad (7.18)$$

$$0 - \frac{d}{dt}m\dot{x} = 0$$

$$\ddot{x} = 0$$

y:
$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0 \qquad (7.19)$$

$$-mg - \frac{d}{dt}(m\dot{y}) = 0$$

$$\ddot{y} = -g$$

In polar coordinates, we have

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2$$

$$U = mgr\sin\theta$$

Where $U = 0$ for $\theta = 0$

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\sin\theta$$

r:
$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \qquad\qquad (7.20)$$

$$mr\dot{\theta}^2 - mg\sin\theta - \frac{\dot{d}}{dt}(m\dot{r}) = 0$$

$$r\dot{\theta}^2 - g\sin\theta - \ddot{r} = 0$$

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θ:
$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$-mgr\cos\theta - \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$-gr\cos\theta - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} = 0$$

Example 7.5: The point of support of a simple pendulum of length b moves on massless rim of radius a rotating with constant angular velocity ω. Obtain the expression for the Cartesian components of the velocity and acceleration of the mass m. Obtain also the angular acceleration for the angle θ shown in the figure below.
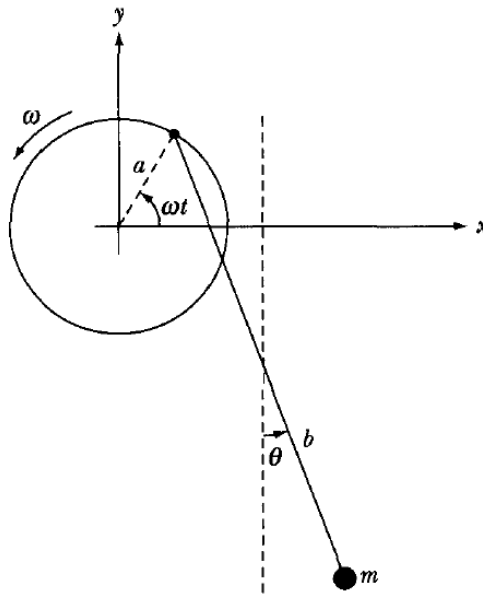
We choose the origin of our coordinate system to be at the center of the rotating rim. The Cartesian components of mass m become

$$x = a \cos \omega t + b \sin \theta$$

$$y = a \sin \omega t - b \cos \theta$$

The velocities are

$$\dot{x} = -a\omega \sin \omega t + b\dot{\theta} \cos \theta$$

$$\dot{y} = a\omega \cos \omega t + b\dot{\theta} \sin \theta$$

Taking the time derivative once again gives the acceleration:

$$\ddot{x} = -a\omega^2 \cos \omega t + b(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta)$$

$$\ddot{y} = -a\,\omega^2 \sin \omega t + b(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta)$$

It should be clear that the single generalized coordinate is θ. The kinetic and potential

energies are

$$T = \frac{1}{2}\,m\,(\dot{x}^2 + \dot{y}^2)$$

$$U = mgy$$

Where $U = 0$ at $y = 0$, The Lagrangian is

$$L = T - U = \frac{m}{2}[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega \sin(\theta - \omega t)] - mg(a \sin \omega t - b \cos \theta)$$

The derivatives for the Lagrange equation equation of motion for θ are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mb^2\ddot{\theta} + mba\omega\left(\dot{\theta} - \omega\right)\cos\left(\theta - \omega t\right)$$

$$\frac{\partial L}{\partial \theta} = mb\dot{\theta}a\omega \cos(\theta - \omega t) - mgb \sin \theta)$$

We can get the equation of motion after solving for $\ddot{\theta}$

$$\ddot{\theta} = \frac{\omega^2 a}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin \theta$$

Notice that this result reduces to well-known equation of motion for a simple pendulum

if $\omega = 0$.