Harmonic Oscillator in Three Dimensions

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4.3.2 The Three-Dimensional Isotropic Harmonic Oscillator

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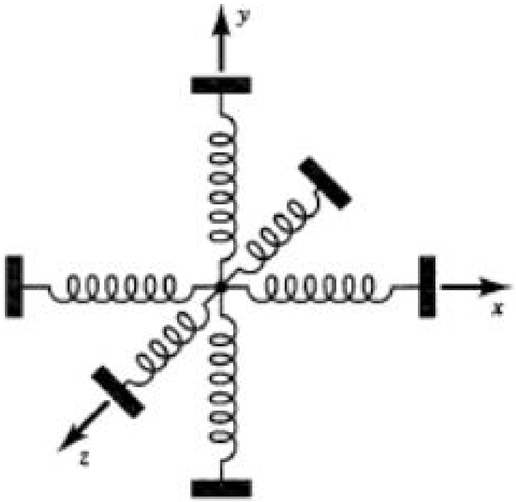
4.3.4 Energy Considerations:

4.3.1 Harmonic Oscillator in Three Dimensions



$$\boxed{F = -kr}$$

Accordingly, the deferential equation of motion is simply expressed as

$$\boxed{m\frac{d^2r}{dt^2} = -kr}$$

In the case of three-dimensional motion, the deferential equation of motion is equivalent to the three equations

$$\begin{aligned} m\ddot{x} &= -kx \\ m\ddot{y} &= -ky \qquad (4.50) \\ m\ddot{z} &= -kz \end{aligned}$$

$$x = A\cos(\omega t + \alpha)$$
$$y = B\cos(\omega t + \beta) \tag{4.51}$$
$$z = C\cos(\omega t + \gamma)$$

As $\boxed{\omega = \sqrt{k/m}}$

4.3.3 Non-isotropic Oscillator in Three-Dimensional

The previous discussion considered the motion of the **isotropic oscillator**, wherein the restoring force is independent on the direction of the displacement. If the magnitudes of the components of the restoring force depend on the direction of the displacement, we have the case of the non-isotropic oscillator

$$m\ddot{x} = -k_1 x$$
$$m\ddot{y} = -k_2 y \tag{4.52}$$
$$m\ddot{z} = -k_3 z$$

Here we have a case of three different frequencies of oscillation, $\omega_1 = \sqrt{k_1/m}$, $\omega_2 = \sqrt{k_2/m}$ and $\omega_3 = \sqrt{k_3/m}$ and the motion is given by the solutions

$$x = A\cos(\omega_1 t + \alpha)$$

$$y = B\cos(\omega_2 t + \beta) \tag{4.53}$$

$$z = C\cos(\omega_3 t + \gamma)$$

Again, the six constants of integration in the above equations are determined from the initial conditions.

4.3.4 Energy Considerations:

In the preceding chapter we showed that the potential energy function of the one dimensional harmonic oscillator is quadratic in the displacement, $V(x) = \frac{1}{2}kx^2$. For the general three-dimensional case, it is easy to verify that

$$V(x, y, z) = \frac{1}{2}kx^2 + \frac{1}{2}ky^2 + \frac{1}{2}kz^2 \tag{4.54}$$

because $F_x = -\frac{\partial V}{\partial x} = -k_1 x$, and similarly for F_y and F_z. If $k_1 = k_2 = k_3 = k$, we have the isotropic case, and

$$V(x, y, z) = \frac{1}{2}k(x^2 + y^2 + z^2) = \frac{1}{2}kr^2 \tag{4.55}$$

Example (5)

A particle of mass m moves in two dimensions under the following potential energy function:

$$V(r) = \frac{1}{2}k(x^2 + 4y^2)$$

Find the resulting motion, given the initial condition at $t = 0,\ x = a\ , y = 0, \dot{x} = 0\ and\ \dot{y} = v_\circ$

Solution: This is an isotropic oscillator potential. The force function is

Thus $F = -\nabla V$

$$F = -2kxi - 4kyj = m\ddot{r}$$

$$m\ddot{x} + kx = 0 \qquad m\ddot{y} + 4ky = 0$$

The x-motion has angular frequency $\omega = (k/m)^{1/2}$, while the y-motion has angular frequency just twice that, namely, $\omega_y = (4k/m)^{1/2} = 2\omega$. We shall write the general solution in the form

$$x = A_1 \cos\omega t + B_1 \sin\omega t$$
$$y = A_2 \cos 2\omega t + B_2 \sin 2\omega$$

To use the initial condition we must first differentiate with respect to t to find the general expression for the velocity components:

$$\dot{x} = -A_1\omega \sin\omega t + B_1\omega \cos\omega t$$
$$\dot{y} = -2A_2\omega \sin 2\omega t + 2B_2\omega \cos 2\omega t$$

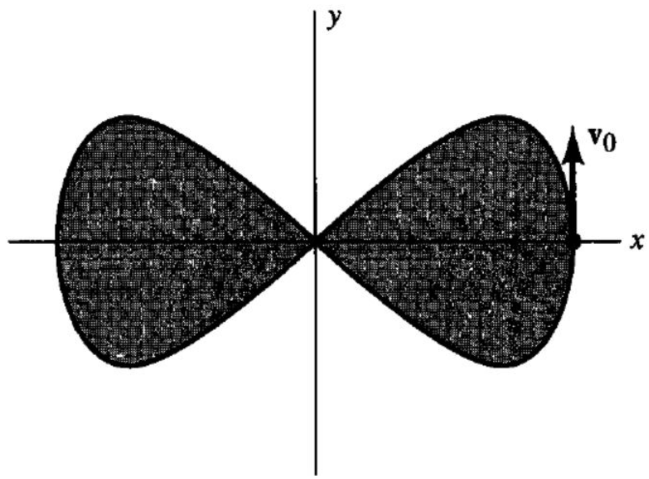
Thus, at t = 0, we see that the above equations for the components of position and velocity reduce to

$$a = A_1 \qquad 0 = A_2 \qquad 0 = B_1\omega \qquad v_0 = 2B_2\omega$$

These equations give directly the values of the amplitude coefficients, $A_1 = a$, $A_2 = B_1 = 0$, and $B_2 = v_0/2\omega$, so the final equations for the motion are

$$x = a \cos\omega t$$
$$y = \frac{v_0}{2\omega} \sin 2\omega t$$

The path is a Lissajous figure having the shape of a figure-eight as shown in Figure 4.4.3.



H.W

H.W 5 A particle of mass m moving in three dimensions under the potential energy function $V(x,y,z) = \alpha x + \beta y^2 + \gamma z^3$ has speed v_0 when it passes through the origin.

(a) what will its speed be if and when it passes through the point $(1,1,1)$?

(b) If the point $(1,1,1)$ is a turning point in the motion $(v = 0)$, what is v_0 ?

(c) what are the component differential equations of motion of the particle?

[Note: you do not have to solve the differential equations of motion in this problem]