Harmonic Oscillator in Two Dimensions

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Outlines:

Example 2 and 3

4.3.1 Harmonic Oscillator in Two Dimensions

Example (2)

Suppose a particle of mass m is moving in the above force field, and at time t = 0 the particle passes through the origin with speed $v_\circ$. What will the speed of the particle be at some small distance away from the origin given by $r = \Delta\, e_r$, where $\Delta \ll \delta$?

Solution:

The force is conservative because there is a potential energy exist .

$$\mathbf{V(r) = V_0 - \tfrac{1}{2}k\delta^2 e^{-r^2/\delta^2}}$$

$$F = -\nabla V$$

$$F = -k(ix + jy)e^{-(x^2+y^2)/\delta^2}$$

Thus , the total energy given by:

$$E = T + V = E = \frac{1}{2}mv^2 + V(r) = \frac{1}{2}mv_\circ^2 + V(0) \qquad (4.27)$$

By solving equation 4.27 we obtain

$$v^2 - v_\circ^2 = \frac{2}{m}[V(0) - V(r)] \qquad (4.28)$$

From example (1) , we have V (r) defines as following :

$$V(r) = V_\circ - \frac{1}{2}k\delta^2 e^{\frac{-\Delta^2}{\delta^2}}$$
$$V(0) = V_\circ - \frac{1}{2}k\delta^2 \tag{4.29}$$

$as\ r\ = \Delta\ e_r$

Now, we can substitute equation (4.29) in (4.28) to get:

$$v^2 = v_\circ^2 + \frac{2}{m}[(\cancel{V_\circ} - \frac{1}{2}k\delta^2) - (\cancel{V_\circ} - \frac{1}{2}k\delta^2 e^{\frac{-\Delta^2}{\delta^2}})]$$
$$v^2 = v_\circ^2 - \frac{k\delta^2}{m}[1 - e^{\frac{-\Delta^2}{\delta^2}}] \tag{4.30}$$

$$\text{But } \Delta \leq< \delta \Rightarrow e^{\frac{-\Delta^2}{\delta^2}} = (1 - \frac{\Delta^2}{\delta^2})$$

$$v^2 = v_\circ^2 - \frac{k\delta^2}{m}[1 - (1 - \frac{\Delta^2}{\delta^2})] \tag{4.31}$$

$$v^2 = v_\circ^2 - \frac{k\Delta^2}{m} \tag{4.32}$$

Example (3)

Is the force field $F = ixy + jxz + kyz$ conservative ?

Solution:

If $F = -\nabla V$

$$\text{Curl } \mathbf{F} = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0$$

This is a condition of conservative force

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xz & yz \end{vmatrix} = \mathbf{i}(z-x) + \mathbf{j}0 + \mathbf{k}(z-x)$$

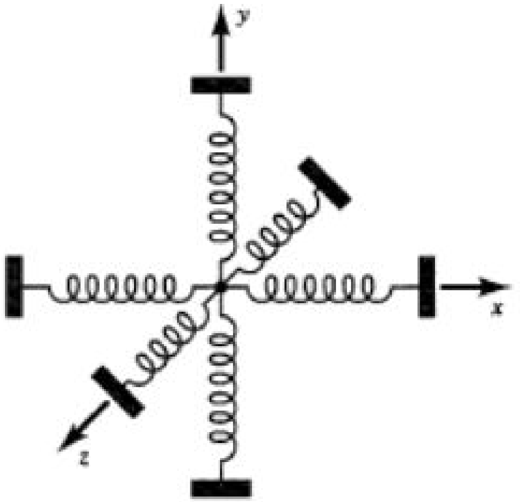
hence, the field is not conservative.

Example (4) For what values of the constants a, b, and c is the force $F = i(ax + by2) + jcxy$ conservative?

Solution: **H.W**

4.3.1 Harmonic Oscillator in Two Dimensions

$$\boxed{F = -kr}$$



Accordingly, the deferential equation of motion is simply expressed as

$$\boxed{m\,\frac{d^2r}{dt^2} = -kr}$$

In the case of motion in a 2D plane, equation of motion is equivalent to the two component equations

$$\begin{aligned} m\ddot{x} &= -kx \\ m\ddot{y} &= -ky \end{aligned} \qquad (4.35)$$

$$m\ddot{x} = -kx$$
$$m\ddot{y} = -ky \tag{4.35}$$

$$x = A\cos(\omega t + \alpha)$$
$$y = B\cos(\omega t + \beta) \tag{4.36}$$

As $\omega = \sqrt{k/m}$

The constants of integration A, B, α and β are determined from the initial conditions in any given case. To find the equation of the path, we eliminate the time t between the two equations. To do this, let us write the second equation in the form

$$y = B\cos(wt + \alpha + \Delta) \quad \text{as} \quad \beta = \alpha + \Delta \tag{4.38}$$

$$y = B[\cos(wt + \alpha)\cos(\Delta) - \sin(wt + \alpha)\sin(\Delta)]$$
$$\frac{y}{B} = \cos(wt + \alpha)\cos(\Delta) - \sin(wt + \alpha)\sin(\Delta) \tag{4.39}$$

But from equation (4.36) we have

$$x = A\cos(wt + \alpha) \Rightarrow \frac{x}{A} = \cos(wt + \alpha) \quad \text{and} \quad \sin\theta = [1 - \cos^2\theta]^{\frac{1}{2}} \tag{4.40}$$

so that , equation (4.39) can be written as following:

$$\frac{y}{B} = \frac{x}{A} cos(\Delta) - [1 - \frac{x^2}{A^2}]^{\frac{1}{2}} sin(\Delta)] \qquad (4.41)$$

$$\frac{x}{A} = \boxed{cos(wt + \alpha)} \text{ and } sin\theta = \boxed{[1 - cos^2\theta]^{\frac{1}{2}}}$$

$$\frac{x}{A} cos(\Delta) - \frac{y}{B} = [1 - \frac{x^2}{A^2}]^{\frac{1}{2}} sin(\Delta) \qquad (4.42)$$

By squaring the two sides and re-range the terms so equation (4.42) becomes:

$$\frac{x^2}{A^2} cos^2(\Delta) + \frac{x^2}{A^2} sin^2(\Delta) + \frac{y^2}{B^2} - \frac{2xy}{AB} cos\Delta = sin^2(\Delta) \qquad (4.43)$$

eliminate $(sin^2 + cos^2) = 1$ from the above equation so we will get

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{xy}{AB} cos\Delta = sin^2(\Delta) \qquad (4.44)$$

which is a quadratic equation in x and y, from equation (4.44) if the phase $\Delta = \frac{\pi}{2}$. So

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \qquad (4.45)$$

Equation (4.45) is the equation of ellipse whose axis coincide with the coordinates axes.

If $\Delta = 0$ then, equation (4.44) becomes:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB} = 0 \tag{4.46}$$

$$y = \frac{B}{A}x \tag{4.47}$$

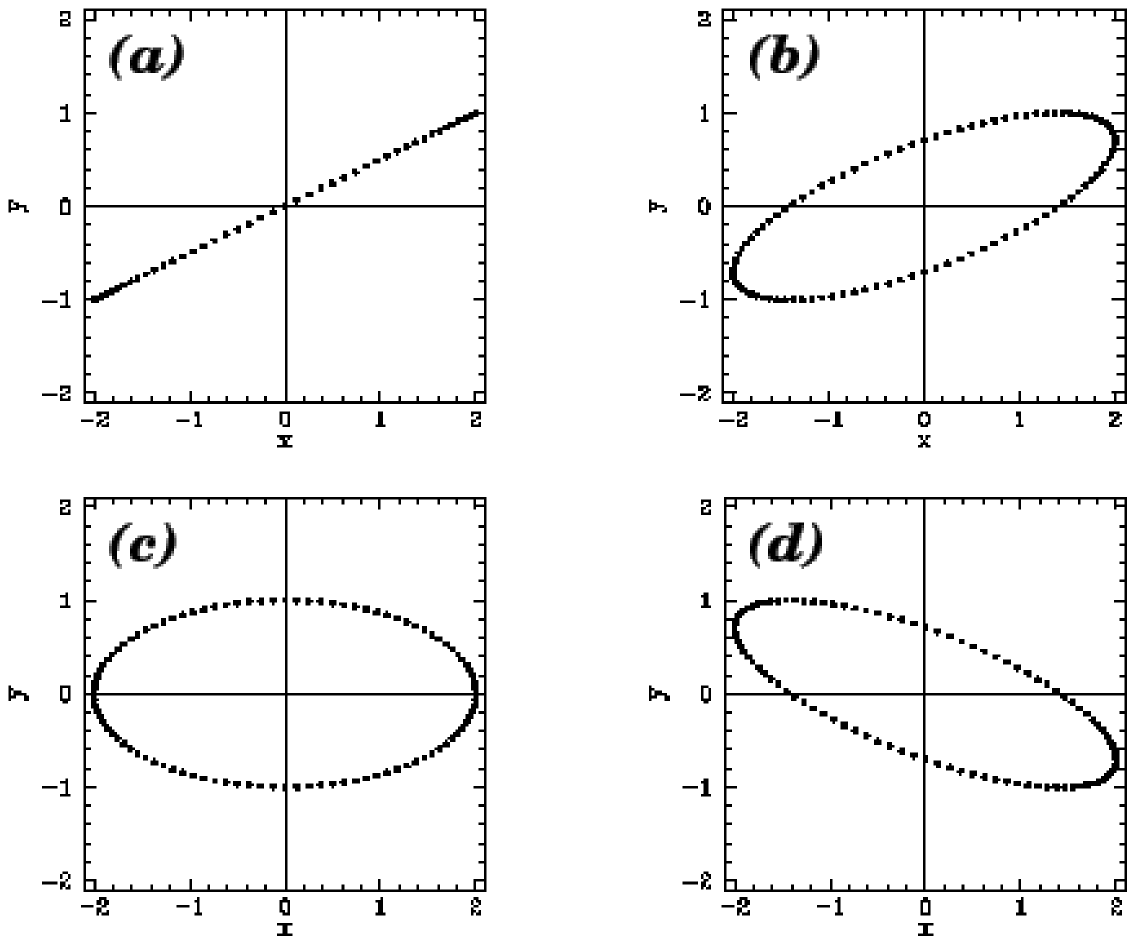
OR if $\Delta = \pi$

$$y = -\frac{B}{A}x \tag{4.48}$$

If the phase difference is 0 or π then the equation of the path reduces to that of a straight line

In the general case it is possible to show that the axis of the elliptical path is inclined to the x-axis by the angle Ψ where :

$$tan2\psi = \frac{2ABcos(\Delta)}{A^2 - B^2} \tag{4.49}$$


(a)
(b)
(c)
(d)
x
y
−2
−1
0
1
2

Q1=find the force if the force field given as

1-$V= cxyz+c$

2-$V=ax^2+By^2+cz^2$