Chapter4 -General Motion of Particle in Three Dimensions

By
Dr. Mohammed F. Al-Mudhaffer
Department of Physics-College of Education for Pure Science

Outlines:

4.1 Introduction: General Principles

4.1.1 The work Principle

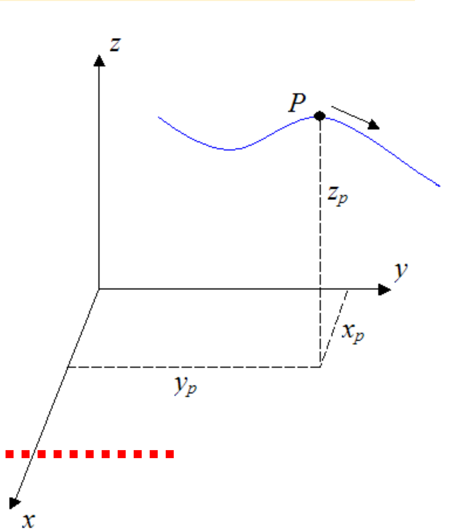
4.1.2 Conservative Farce and Force Fields

4.2 The Potential Energy Function in Three Dimensions Motion: The Del Operator

4.1 Introduction: General Principles:

- We now study the general case of motion of particle in three dimensions. The vector form of the equation of motion (Newton's Second Law)for such particle is :

$$F = \frac{dp}{dt} \qquad \textbf{OR} \quad F = ma \qquad (4.1)$$



- as $p = mv$, is the linear momentum of the particle. In Cartesian coordinates , we can write:

$$
\begin{aligned}
F_x &= m\ddot{x} \\
F_y &= m\ddot{y} \qquad (4.2) \\
F_z &= m\ddot{z}
\end{aligned}
$$

The simplest solution is assuming
a function (F) of spatial coordinates only(i.e. F=F(r)).

4.1.1 The work Principle:

- Work done on a particle causes it to gain or lose kinetic energy. Now, we will use equation (1) with taking the dot product with velocity (v), so, we have

$$\vec{F}.\vec{v} = \frac{dp}{dt}.\vec{v} = \frac{d(mv)}{dt}.\vec{v} \qquad (4.4)$$
$$= m[\dot{v}.v + v.\dot{v}]$$
$$= 2m[\dot{v}.v]$$

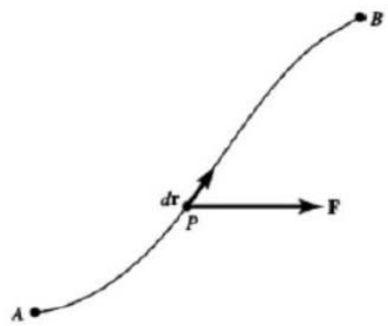


Figure 4.1: The work done by a force F is the line integral

So that ,

$$\vec{F}.\vec{v} = \frac{d}{dt}(\frac{1}{2}mv^2) = \frac{dT}{dt} \qquad (4.5)$$

Where **T is the Kinetic Energy**. However, $v = dr/dt$, so equation (4.5) becomes

$$\vec{F}.\frac{d\vec{r}}{dt} = \frac{dT}{dt} \qquad (4.6)$$

$$\int \vec{F}.d\vec{r} = \int dT \qquad (4.7)$$

$$W = T_f - T_i \qquad (4.8)$$

4.1.2 Conservative Force and Force Fields:

- If the forces acting on a particle were conservative , it could be drive from the derivative of a scalar potential function as following:

$$F_x = -\frac{dV(x)}{dx}$$
$$W = \int F_x dx = -\int dV(x) \qquad (4.9)$$

$$W = \int F_x dx = -\triangle V(x) = V(A) - V(B) \qquad (4.10)$$

- This mean , we need to know the potential energy at started point (A) and the ended point (B). From equation (8)and (10), we can find a general form of the conservation law for the total energy of the particle:

$$\int \vec{F}.d\vec{r} = \int dT \qquad (4.7)$$

$$W = T_f - T_i \qquad (4.8)$$

$$W = \int F_x dx = -\triangle V(x) = V(A) - V(B) \qquad (4.10)$$

$$E_{total} = V(A) + T(A) = V(B) + T(B) = \text{Constant at the particle motion} \qquad (4.11)$$

4.2 The Potential Energy Function in Three Dimensions Motion: The Del Operator

- Assume we have F and V in three dimensions so we can write:

As F(r)=iFx + jFy + kFz

, V(r)=iVx + jVy + kVz

$$F(r) = -\frac{dV(r)}{dr} \tag{4.12}$$

$$F_x = -\frac{\partial V(x)}{\partial x}$$
$$F_y = -\frac{\partial V(y)}{\partial y} \tag{4.13}$$
$$F_z = -\frac{\partial V(z)}{\partial z}$$

$$F = -i\frac{\partial V(x)}{\partial x} - j\frac{\partial V(y)}{\partial y} - k\frac{\partial V(z)}{\partial z} \tag{4.14}$$

$$F = -\bigtriangledown V$$

Del Operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \qquad (4.15)$$

Del Operator Properties

$1 - \nabla F =$ **is a numerical quantity**

$2 - \nabla \times F =$ **Curl F** ;lead to a new Vector

$3 - \nabla \times (\nabla F) = 0$; Curl of any grediat =0

(4.16)

Del Operator

Now , we express ***Curl F*** using Del operator

$$\text{Curl } \mathbf{F} = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \qquad (4.17)$$

$$\nabla \times F = i(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) + j(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}) + k(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) = 0 \qquad (4.18)$$

- Now, we can generalize the conservation of energy principle to three dimensions. The work done by a conservative force in moving a particle from point A to point B can be written as :

$$\int_A^B \vec{F}.dr = -\int_A^B \nabla V(\vec{r}).dr \quad \text{as} \quad F = -\nabla V$$
$$= \int_A^B (i\frac{\partial V}{\partial x} + j\frac{\partial V}{\partial y} + k\frac{\partial V}{\partial z}) \cdot (idx + jdy + kdz) \qquad (4.19)$$

.

- H.W

$$\int_A^B dV(\vec{r}) = \Delta V]_A^B = V(B) - V(A) \qquad (4.20)$$

- The last step illustrates the fact that $\nabla V . dr$ is an exact deferential equal to dV . The work done by any net force is always equal to the change in kinetic energy

$$\text{i.e.} \int_A^B F.dr = \Delta T = -\Delta V \qquad (4.21)$$

$$\Delta(T+V) = 0$$

$$T(A) + V(A) = T(B) + V(B) = E = \text{Constant} \qquad (4.22)$$

- We have arrived at our desired law of conservation of total energy. NOTE: If F is nonconservative force , it can not be equal to ($-\nabla V$) this lead to the work ($F.dr$) is not an exact deferential and can not be equaled to (dV). So, the general form of work -energy theorem becomes:

$$\int_A^B F.dr = \Delta(T+V) = -\Delta E \qquad (4.23)$$

Example (1)

Given the two-dimensional potential energy function

$$V(\mathbf{r}) = V_0 - \frac{1}{2}k\delta^2 e^{-r^2/\delta^2}$$

where $\mathbf{r} = \mathbf{i}x + \mathbf{j}y$ and V_0, k, and δ are constants, find the force function.

Solution:

We first write the potential energy function as a function of x and y,

$$V(x,y) = V_0 - \frac{1}{2}k\delta^2 e^{-(x^2+y^2)/\delta^2}$$

and then apply the gradient operator:

$$F = -\nabla V$$

$$F = -\left(i\frac{\partial V}{\partial x} + j\frac{\partial V}{\partial y}\right)$$

$$V(x,y) = V_0 - \frac{1}{2}k\delta^2 e^{-(x^2+y^2)/\delta^2}$$

$$F = -\left(i\frac{\partial V}{\partial x} + j\frac{\partial V}{\partial y}\right)$$

$$\frac{\partial V}{\partial x} = \frac{1}{2}k\delta^2\left(\frac{2x}{\delta^2}\right)e^{-(x^2+y^2)/\delta^2} \quad\Rightarrow\quad \frac{\partial V}{\partial x} = kx\,e^{-(x^2+y^2)/\delta^2}$$

$$\frac{\partial V}{\partial y} = \frac{1}{2}k\delta^2\left(\frac{2y}{\delta^2}\right)e^{-(x^2+y^2)/\delta^2} \quad\Rightarrow\quad \frac{\partial V}{\partial y} = ky\,e^{-(x^2+y^2)/\delta^2}$$

$$F = -\left(ikx\,e^{-(x^2+y^2)/\delta^2} + jky\,e^{-(x^2+y^2)/\delta^2}\right)$$

$$F = -k(ix + jy)e^{-(x^2+y^2)/\delta^2}$$

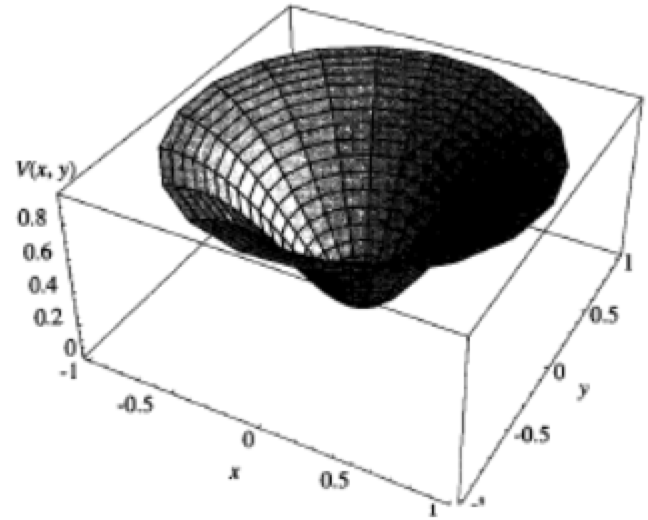


Figure 4.2.2a The potential energy function $V(x, y) = V_0 - \frac{1}{2}k\delta^2 e^{-(x^2+y^2)/\delta^2}$.

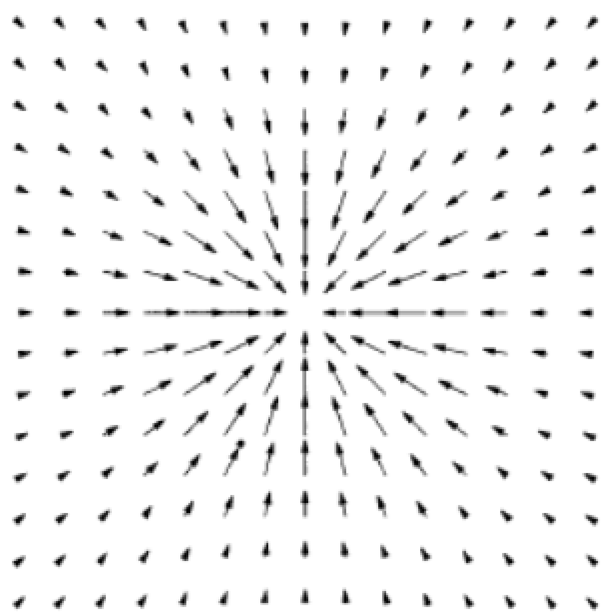


Figure 4.2.2b Force field gradient of potential energy function in Figure 4.2.2(a); $\mathbf{F} = -\Delta V - k(\mathbf{i}x + \mathbf{j}y)e^{-(x^2+y^2)/\delta^2}$.