

The period of the underdamped oscillator is given by

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}} \quad \text{--- (3.31)}$$

6. Quality Factor Q

The rate of energy loss of a weakly damped harmonic oscillator is called the quality factor of the oscillator.

$$\frac{\Delta E}{E} = \frac{T_d}{\tau} \quad [\text{energy loss in one cycle}]$$

T_d : period of underdamped oscillator

τ : the time of motion

τ : exponential time constant

$$Q = \frac{2\pi}{T_d/\tau} = \frac{2\pi\tau}{(2\pi/\omega_d)} = \omega_d\tau = \frac{\omega_d}{2\gamma} \quad \text{--- (3.32)}$$

where,

$$\tau = \frac{1}{2\gamma} \quad \text{(3.33)}$$

For weak damping, the period of oscillation T_d is much less than the time constant, τ

Table I: Some values of Q for different kind of oscillators

Kind of oscillators	Q	
Earthquake	250 - 1400	→ Strong damping
Piano string	3000	
Crystal in digital watch	10^4	
Microwave cavity	10^4	
Excited atom	10^7	→ Weak damping
Neutron star	10^{12}	
Excited Fe^{57} nucleus	3×10^{12}	→ Very weak damping

Example 3

An automobile suspension system is critically damped, and its period of free oscillation with no damping is 1 s. If the system is initially displaced by an amount x_0 and released with zero initial velocity, find the displacement at $t = 1$ s.

Answer: Critical damping $\Leftrightarrow \eta = 0$

$$\text{at } t=0 \rightarrow x=x_0 \quad \dot{x}(0)=0$$

$$\text{at } t=1 \rightarrow x(t) = ??$$

$$x(t) = A t e^{-\delta t} + B e^{-\delta t}$$

For critical damping
[see 3.29)

$$\gamma = \sqrt{\delta^2 - \omega_0^2} \Rightarrow \omega = \sqrt{\delta^2 - \omega_0^2} \Rightarrow \boxed{\delta = \omega_0}$$

From 3.13 $\Rightarrow \omega_0 = \frac{2\pi}{T_0}$

$\therefore T_0 = 1 \text{ s} \Rightarrow \omega_0 = 2\pi$

$$\delta = 2\pi$$

Now,

$$x(t) = A t e^{-2\pi t} + B e^{-2\pi t}$$

At $t=0$ and $x = x_0$

$$x(0) = A(0) e^{-2\pi(0)} + B e^{-2\pi(0)}$$

$\cancel{A(0)}$

$$x_0 = B e^0 \Rightarrow \boxed{x_0 = B}$$

Zero initial velocity $[\dot{x}(0) = 0]$

$$\therefore \dot{x}(t) = -2\pi A t e^{-2\pi t} + A e^{-2\pi t} - 2\pi B e^{-2\pi t}$$

$$\dot{x}(0) = 0 + A e^0 - 2\pi B e^0$$

$$0 = A - 2\pi B \Rightarrow A = 2\pi B$$

\downarrow

$$\boxed{A = 2\pi x_0}$$

Now, the expression of displacement is:

$$x(t) = 2\pi \omega_0 t e^{-2\pi t} + x_0 e^{-2\pi t}$$

or

$$x(t) = (2\pi t + 1) x_0 e^{-2\pi t}$$

At $t=1$ see $x(1)$ will be

$$x(1) = (2\pi(1) + 1) x_0 e^{-2\pi(1)}$$

$$\omega(1) = (2\pi + 1) x_0 e^{-2\pi}$$

$$= (7.28) x_0 e^{-6.28}$$

$$\therefore x(1) = 0.0136 x_0$$

The system has practically returned to equilibrium.

Example 4

The frequency of a damped harmonic

oscillator is one-half the frequency of the same oscillator with no damping - Find the ratio of the maxima of successive oscillations.

$$\omega_d = \frac{1}{2} \omega_0 \Rightarrow \sqrt{\omega_0^2 - \gamma^2} = \frac{1}{2} \omega_0$$

$$\therefore \omega_0^2 - \gamma^2 = \frac{\omega_0^2}{4}$$

$$\gamma^2 = \omega_0^2 - \frac{\omega_0^2}{4} \Rightarrow \gamma^2 = \frac{3\omega_0^2}{4}$$

$\therefore \gamma = \frac{\sqrt{3}\omega_0}{2}$

at maxima oscillations $t = T_d$

$$\therefore \gamma t = \gamma T_d = \boxed{\frac{\sqrt{3}\omega_0}{2}} \cdot \boxed{\frac{2\pi}{\omega_0}}$$

$$\therefore \gamma T_d = \sqrt{3}\pi \frac{\omega_0}{\omega_0} = \sqrt{3}\pi \frac{\omega_0}{\omega_0/2} \Rightarrow \boxed{\gamma T_d = 2\sqrt{3}\pi}$$

$$\gamma T_d = 10.88$$

The amplitude ratio is $e^{-\gamma T_d} = e^{-10.88} = 0.00002$

7. Driven oscillations with Damping

The motion of a damped harmonic oscillator that is subjected to a periodic driving force by an external agent is:

$$m\ddot{x} + c\dot{x} + kx = F \quad \text{driving force}$$

$$\text{let } F = F_0 e^{i\omega t}$$

$$\therefore m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t} \quad \dots (3-34)$$

The solution is

$$x(t) = A e^{i(\omega t - \phi)} \quad \dots (3-35)$$

A : Amplitude

ϕ : phase difference

$$m \frac{d^2}{dt^2} A e^{i(\omega t - \phi)} + c \frac{d}{dt} A e^{i(\omega t - \phi)} + k A e^{i(\omega t - \phi)} = F_0 e^{i\omega t}$$

$$-m\omega^2 A e^{i(\omega t - \phi)} + i c \omega A e^{i(\omega t - \phi)} + k A e^{i(\omega t - \phi)} = F_0 e^{i\omega t}$$

$$\text{Note that } e^{i(\omega t - \phi)} = e^{i\omega t} e^{-i\phi}$$

$$\therefore -m\omega^2 A e^{i\omega t} e^{-i\phi} + i c \omega A e^{i\omega t} e^{-i\phi} + k A e^{i\omega t} e^{-i\phi} = F_0 e^{i\omega t}$$

$$[-m\omega^2 A + i c \omega A + k A] e^{-i\phi} = F_0$$

$$-m\omega^2 A + i c \omega A + kA = F_0 e^{i\phi}$$

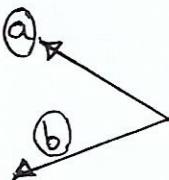
$$e^{i\phi} = \cos\phi + i \sin\phi$$

$$\underbrace{-m\omega^2 A + kA}_{\text{Real}} + \underbrace{i c \omega A}_{\text{Imaginary}} = \underbrace{F_0 \cos\phi}_{\text{Real}} + \underbrace{i F_0 \sin\phi}_{\text{Imaginary}}$$

Equating the real and imaginary parts yields the two equations:

$$\boxed{A(k - m\omega^2) = F_0 \cos\phi} \quad \text{--- (3.36)}$$

$$c \omega A = F_0 \sin\phi$$



$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{c \omega A}{A(k - m\omega^2)} \quad \text{divide } \frac{b}{a}$$

$$\therefore \boxed{\tan\phi = \frac{c \omega}{k - m\omega^2}} \quad \text{--- (3.37)}$$

By squaring both sides of equations 3.36 a and b and adding and employing the identity $\sin^2\phi + \cos^2\phi = 1$ we find after solving for A =

$$\boxed{A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2 \omega^2}}} \quad \text{H.W. (3.38)}$$

We know that $\omega_0^2 = \frac{k}{m}$ and $\delta = \frac{C}{2m}$

$$\downarrow$$

$$k = m\omega_0^2$$

$$\downarrow$$

$$C = 2m\delta$$

Now, we can rewrite (3-37) and (3-38)

$$\tan \phi = \frac{2m\gamma\omega}{m\omega_0^2 - m\omega^2} \Rightarrow \boxed{\tan \phi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}} \quad (3-39)$$

$$A = \frac{F_0}{\sqrt{(m\omega_0^2 - m\omega^2)^2 + 4m^2\gamma^2\omega^2}}$$

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + 4m^2\gamma^2\omega^2}}$$

$$A = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

$$\approx A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \quad (3-40)$$

Amplitude resonance occurs at other ω_r which is given as:

$$\boxed{\omega_r^2 = \omega_0^2 - 2\gamma^2} \quad (3-41)$$

ω_r approaches ω_0 as γ goes to zero

$$\text{Also, } \omega_d = \sqrt{\omega_0^2 - \gamma^2} \Rightarrow \omega_0^2 = \omega_d^2 + \gamma^2$$

$$\therefore \omega_r^2 = \omega_d^2 + \gamma^2 - 2\gamma^2$$

$$\Rightarrow \boxed{\omega_r^2 = \omega_d^2 - \gamma^2} \quad (3.42)$$

$$\text{if } \gamma = 0 \Rightarrow \omega_r \approx \omega_d$$

Thus if the damping is weak, the resonant frequency ω_r , the damped oscillator frequency ω_d , and the natural frequency ω_0 are identical.

At the extreme of strong damping, no amplitude resonance occurs if $\gamma > \frac{\omega_0}{\sqrt{2}}$ because A monotonically decreasing function of ω . To see this, consider the limiting case $\gamma^2 = \frac{\omega_0^2}{2}$, so eq (3.42) $\Rightarrow \omega_r = 0$

Now, substitute in (3.40) \Rightarrow

$$\boxed{A(\omega) = \frac{F_0/m}{\sqrt{\omega_0^4 + \omega^4}}} \quad (3.43)$$

Amplitude of oscillation at resonance peak:

The steady-state amplitude at the resonant frequency which we call A_{\max} is obtained from eqs (3.40) and (3.41)

$$\omega = \omega_r$$

$$\therefore A_{\max} = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_r^2)^2 + 4\gamma^2\omega_r^2}}$$

$$\omega_r^2 = \omega_0^2 - 2\gamma^2$$

$$A_{\max} = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_0^2 + 2\gamma^2)^2 + 4\gamma^2\omega_0^2 - 8\gamma^4}}$$

$$A_{\max} = \frac{F_0/m}{\sqrt{4\gamma^4 + 4\gamma^2\omega_0^2 - 8\gamma^4}}$$

$$A_{\max} = \frac{F_0/m}{\sqrt{4\gamma^2\omega_0^2 - 4\gamma^4}}$$

$$A_{\max} = \frac{F_0/m}{\sqrt{4\gamma^2(\omega_0^2 - \gamma^2)}}$$

$$A_{\max} = \frac{F_0/m}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$$

(3.44)

For weak damping $\Leftrightarrow \delta \approx \text{very small}$

$$A_{\max} = \frac{F_0/m}{2\delta \sqrt{\omega_0^2 - \delta^2}}$$

neglect

$$\therefore A_{\max} = \frac{F_0/m}{2\delta\omega_0} \quad (3-45)$$

Since $C = 2m\delta$

$$\therefore A_{\max} = \frac{F_0}{C\omega_0} \quad (3-46)$$

Sharpness of the resonance : Quality factor

In the case of weak damping $\delta \ll \omega_0$

$$\text{eq (3-40)} \Rightarrow \omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \\ \cong 2\omega_0(\omega_0 - \omega)$$

$$4\delta^2\omega^2 \cong 4\delta^2\omega_0^2$$

$$\therefore A(\omega) = \frac{F_0/m}{\sqrt{(2\omega_0(\omega_0 - \omega))^2 + 4\delta^2\omega_0^2}}$$

$$= \frac{F_0/m}{\sqrt{4\omega_0^2(\omega_0 - \omega)^2 + 4\delta^2\omega_0^2}} = \frac{F_0/m}{2\omega_0 \sqrt{(\omega_0 - \omega)^2 + \delta^2}}$$

From (3-45), we have $\frac{F_0/m}{2\omega_0} = \gamma A_{\max}$

$$\therefore A(\omega) = \frac{A_{\max} \gamma}{\sqrt{(\omega_0 - \omega)^2 + \gamma^2}} \quad \dots \dots (3-47)$$

IF $|\omega_0 - \omega| = \gamma$ or $\omega = \omega_0 \pm \gamma$



$$A(\omega) = \frac{1}{\sqrt{2}} A_{\max}$$

$$A^2(\omega) = \frac{1}{2} A_{\max}^2$$

.....(3-48)

γ is measuring the width of the resonance curve.

The quality factor is defined as $\frac{\omega_d}{2\gamma}$ [see eq 3-32]

In the case of weak damping ($\omega_d \approx \omega_0$), one can write

$$Q = \frac{\omega_0}{2\gamma} \Rightarrow 2\gamma = \frac{\omega_0}{Q}$$

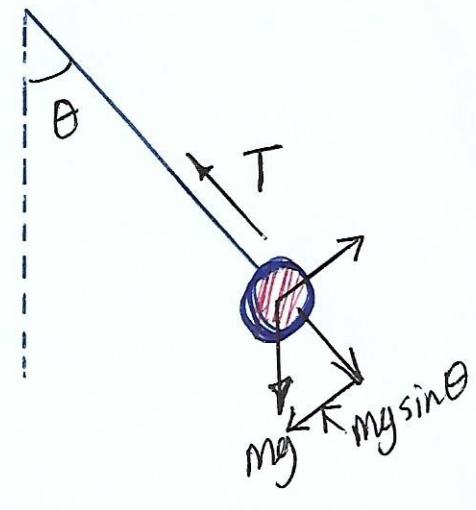
so, the total width at the half-energy point is

$$\Delta\omega = 2\gamma \approx \frac{\omega_0}{Q} \quad \dots \dots (3-49)$$

and $\omega = 2\pi f \Rightarrow \frac{\Delta\omega}{\omega_0} = \frac{\Delta f}{f_0} = \frac{1}{Q}$...3-50

Example 5 Consider a pendulum of length l and a bob of mass m at its end moving through oil with θ decreasing. The massive bob undergoes small oscillations, but the oil retards the bob's motion with a resistive force proportional to speed with $F_{\text{res}} = 2m\sqrt{\frac{g}{l}}(l\dot{\theta})$. The bob is initially pulled back at $t=0$ with $\theta=\alpha$ and $\dot{\theta}=0$. Find the angular displacement θ and velocity $\dot{\theta}$ as a function of time.

$$\text{Force} = m(l\ddot{\theta}) = \text{Restoring Force} + \text{Resistive force}$$



$$\therefore ml\ddot{\theta} = -mg \sin \theta - 2m\sqrt{\frac{g}{l}}l\dot{\theta}$$

$$\ddot{\theta} + 2\sqrt{\frac{g}{l}}\dot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$\omega_0^2 = \frac{g}{l}, \quad \beta^2 = \frac{g}{l} \quad \Rightarrow \quad \omega_0^2 = \beta^2$$

The pendulum is critically damped

$$\theta(t) = (A + Bt)e^{-\beta t}$$

$$\theta(t=0) = \alpha = A$$

$$\ddot{\theta}(t) = B e^{-\beta t} - \beta(A + Bt)e^{-\beta t}$$

$$\dot{\theta}(t=0) = 0 = B - \beta A$$

$$B = \beta A = \beta \alpha$$

$$\theta(t) = \alpha \left(1 + \sqrt{\frac{g}{l}} t\right) e^{-\sqrt{g/l} t}$$

$$\dot{\theta}(t) = -\frac{\alpha g}{l} t e^{-\sqrt{g/l} t}$$

Example 6

A simple harmonic oscillator consists of 100g mass attached to a spring whose force constant is 10^4 dyne/cm. The mass is displaced 3 cm and released from rest. Calculate

(a) The natural frequency ν_0 .

(b) The total energy.

(c) The maximum speed.

$$(a) \nu_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{10^4 \frac{g}{\text{sec}^2} \cdot \frac{\text{cm}}{\text{cm}}}{10^2 g}}$$

$$\nu_0 = \frac{10}{2\pi} \text{ sec}^{-1}$$

$$\nu_0 \approx 1.6 \text{ Hz}$$

$$(b) E = \frac{1}{2} kA^2 = \frac{1}{2} 10^4 \cdot 3^2$$
$$\Rightarrow E = 4.5 \times 10^4 \text{ erg}$$

(c) The maximum Velocity is attained when the total energy of oscillator is equal to the kinetic energy. Therefore

$$\frac{1}{2} m V_{\max}^2 = 4.5 \times 10^4 \text{ erg}$$

$$V_{\max} = \sqrt{\frac{2 \times 4.5 \times 10^4}{100}}$$

$$\therefore V_{\max} = 30 \text{ cm/see}$$

Example 7 A damped harmonic oscillator with $M = 10 \text{ kg}$, $k = 250 \text{ N/m}$ and $C = 60 \text{ kg/s}$ is subject to a driving force given by $F_0 \cos \omega t$ where $F_0 = 48 \text{ N}$.

(a) what value of ω results in steady-state oscillation with maximum amplitude under this condition.

(b) what is the maximum amplitude?

(c) what is the phase shift?

Solution

$$\gamma = \frac{C}{2M} = \frac{60}{2 \times 10} = 3 \text{ sec}^{-1} \Rightarrow \gamma^2 = 9 \text{ sec}^{-2}$$

$$\omega_0^2 = \frac{k}{m} = \frac{250}{10} = 25 \text{ sec}^{-2} \Rightarrow \omega_0 = 5 \text{ sec}^{-1}$$

(a) $\omega_d^2 = \omega_0^2 - \gamma^2$

$$\omega_r^2 = \omega_0^2 - 2\gamma^2$$

$$\therefore \omega_d^2 = 25 - 9 = 16 \text{ sec}^{-2} \Rightarrow \omega_d = 4 \text{ sec}^{-1}$$

$$\omega_r^2 = 25 - 18 = 7 \text{ sec}^{-2} \Rightarrow \omega_r = \sqrt{7} \text{ sec}^{-1}$$

(b) $A_{\max} = \frac{F_0}{2M\gamma\omega_d} = \frac{F_0}{C\omega_d} = \frac{48}{(60)(4)} = 0.2 \text{ m}$

(c) $\tan \phi = \frac{2\gamma\omega_r}{\omega_0^2 - \omega_r^2} = \frac{2\gamma\omega_r}{2\gamma^2} = \frac{\omega_r}{\gamma} = \frac{\sqrt{7}}{3}$

Example 8 The frequency f_d of a damped harmonic oscillator is 100Hz and the ratio of the amplitude of two successive maxima is one half

- (a) what is the undamped frequency of this oscillator?
- (b) what is the resonant frequency?

The ratio of the amplitude is $e^{-\delta T_d} = \frac{1}{2}$

$$\Rightarrow -\delta T_d = \ln \frac{1}{2} \Rightarrow \delta T_d = \ln 2 \Rightarrow \delta = \frac{1}{T_d} \ln 2$$

$$\delta = f_d \ln 2 = 69.315 \text{ sec}^{-1}$$

$$\delta^2 = 4804.57 \text{ sec}^{-2}$$

$$@ \omega_d = 2\pi f_d = 2(3.14)(100) = 628 \text{ sec}^{-1}$$

$$\omega_d^2 = 394384 \text{ sec}^{-2}$$

$$\omega_d^2 = \omega_0^2 - \delta^2 \Rightarrow \omega_0^2 = \omega_d^2 + \delta^2$$

$$\Rightarrow \omega_0 = \sqrt{\omega_d^2 + \delta^2}$$

$$\therefore \omega_0 = 631.81 \text{ sec}^{-1}$$

$$\text{But } f_0 = \frac{\omega_0}{2\pi} = 100.6 \text{ sec}^{-1}$$

(b) $\omega_r^2 = \omega_d^2 - \gamma^2 \Rightarrow \omega_r = \sqrt{\omega_d^2 - \gamma^2}$

$\approx \omega_r = 624.16 \text{ sec}^{-1}$

$$\Rightarrow f_r = \frac{\omega_r}{2\pi} = 99.4 \text{ sec}^{-1}$$