

## 5. Damped Harmonic Motion

The foregoing analysis of harmonic oscillator is somewhat idealized in that we have failed to take into account frictional forces.

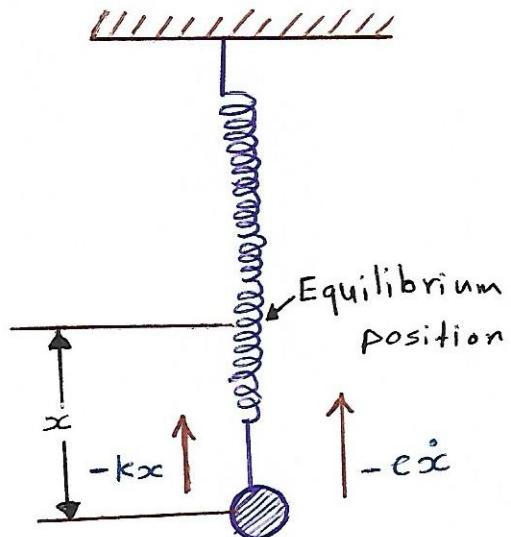
- \* There are always present in a mechanical system to some extent.
- \* There is always a certain amount of resistance in an electrical circuit.

Suppose there is an object of mass "m" that is supported by a light spring of stiffness  $k$ . Assume that there is a linear retarding force to the velocity.

$$m\ddot{x} = -kx - c\dot{x}$$

↑                   ↑  
restoring      retarding  
force            force

Thus,  $m\ddot{x} + kx + c\dot{x} = 0$



or  $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$

If we substitute the damping factor  $\gamma$ , defined as

$$\gamma = \frac{c}{2m}$$

and  $\omega_0^2 (= k/m)$

$$\ddot{x} + 2\delta \dot{x} + \omega_0^2 x = 0$$

The general solution is:

$$x(t) = A_1 e^{-(\delta-\eta)t} + A_2 e^{-(\delta+\eta)t}$$

Where  $\eta = \sqrt{\delta^2 - \omega_0^2}$

There are three possible scenarios:

- I.  $\eta$  real  $> 0$  "overdamping"
- II.  $\eta$  real  $= 0$  "critical damping"
- III.  $\eta$  imaginary  $< 0$  "underdamping"



- $A_1$  and  $A_2$  are constants, determined by the initial conditions.
- The motion is an exponential decay with decay constants:

$$(\delta-\eta) \text{ and } (\delta+\eta)$$

- The mass, given some initial displacement and released from a rest  $\Rightarrow$  returns slowly to equilibrium position and prevented from oscillating by the strong damping force.



- $\zeta = 0 \Rightarrow x(t) = A_1 e^{-\delta t} + A_2 t e^{-\delta t}$

$$\text{let } A = A_1 + A_2$$

$$\therefore x(t) = A e^{-\delta t}$$

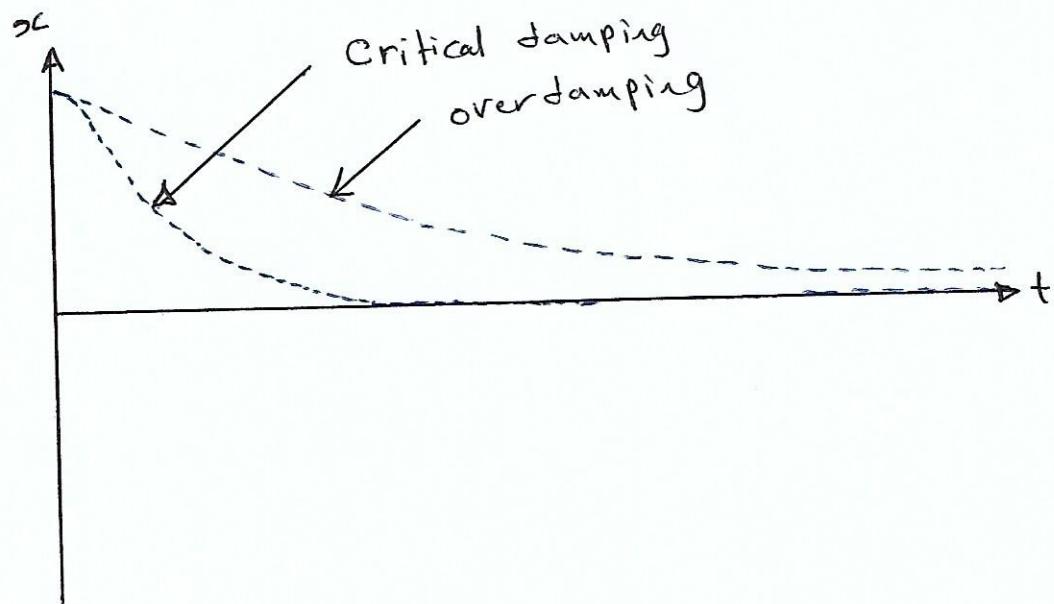
- The solution requires two different functions and independent constants to satisfy the boundary conditions specified by an initial position and velocity.

- The solution is  $x(t) = A t e^{-\delta t} + B e^{-\delta t}$

Note that, we have two different functions ( $t e^{-\delta t}$ ) and ( $e^{-\delta t}$ ). Also, we have two different constants A and B

- If the mass (m) displaced and released from a rest  $\Rightarrow$  the motion is nonoscillatory and returning to equilibrium.

- Critical damping is desirable in many systems such as the mechanical suspension systems of motor vehicles.



### III. Underdamping

If the damping Factor ( $\gamma$ ) is small enough that:

$$\gamma^2 - \omega_0^2 < 0 \Rightarrow q = \sqrt{\gamma^2 - \omega_0^2} = \text{imaginary}$$

$$\text{i.e } q = \sqrt{\gamma^2 - \omega_0^2} = \sqrt{-1} \sqrt{\omega_0^2 - \gamma^2} = i\omega_d$$

$$\text{where } \omega_d = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

$\omega_0$  is the angular frequency of the undamped harmonic oscillator,  $\omega_d$  is the angular frequency of the underdamped harmonic oscillators.

The general solution is

$$X(t) = e^{-\delta t} [A \sin(\omega_d t + \phi_0)]$$

From this equation, we note that the solution of

an underdamped oscillator is nearly identical to that of the undamped oscillator.

There are two differences:

- 1 The presence of the real exponential factor  $[e^{-\delta t}]$  leads to a gradual death of the oscillations.
- 2 The underdamped oscillator's angular frequency is  $\omega_d \neq \omega_0$ , because the presence of the damping force.

