

# Chapter 3

## Oscillations

### 1. Introduction

We begin by considering the oscillatory motion of a particle constrained to move in one dimension. We assume that position of stable equilibrium exists for the particle, and we designate this point as the origin. If the particle is displaced from the origin (in either direction), a certain force tends to restore the particle to its original position. An example is an atom in a long molecular chain. The restoring force is, in general, some complicated function of the displacement and perhaps of the particle's velocity or even of some higher time derivative of position coordinate. We consider here only cases in which the restoring force  $F$  is a function only of the displacement:  $F = F(x)$ .

We assume that the function  $F(x)$  that describes the restoring force possesses continuous derivatives of all orders so that the function can be expanded in a Taylor series:

$$F(x) = F_0 + x \left( \frac{dF}{dx} \right)_0 + \frac{1}{2!} x^2 \left( \frac{d^2 F}{dx^2} \right) + \frac{1}{3!} x^3 \left( \frac{d^3 F}{dx^3} \right) + \dots \quad (3.1)$$

where  $F_0$  is the value of  $F(x)$  at the origin position ( $x=0$ ), it is equal to zero when we define

the origin as the equilibrium position. In addition if the displacement ( $x$ ) is sufficiently small  $\Rightarrow x^2$  or higher order  $\Rightarrow$  Zero [can be neglected]

i.e.  $F(x) = -kx$  --- (3.2)

Where  $k = - (dF/dx)_0$ . The restoring force is always directed toward the equilibrium position (the origin), so the derivative  $(dF/dx)_0$  is negative, and  $k$  is positive. The restoring force is a linear force and it obeys Hook's law.

One of the classes of physical process that can be treated by applying Hook's law is that involving elastic deformations. As long as the displacements are small and the elastic limits are not exceeded, a linear restoring force can be used for problems of stretched springs, elastic springs, bending beams, and the like.

## 2. Simple Harmonic Oscillator

The equation of motion in this case can be obtained by substituting the Hook's law force in the Newtonian equation ( $F=ma$ ). Thus

$$-kx = m\ddot{x} \quad (3.3)$$

If we define  $\omega_0^2 = k/m$  (3.4)

Then, eq. 3.3 becomes

$$\ddot{X} + \omega_0^2 X = 0 \quad (3.5)$$

To solve this equation, we can express  $x(t)$  in either of the form:

$$X(t) = A \sin(\omega_0 t - \delta) \quad (3.6 a)$$

$$X(t) = A \cos(\omega_0 t - \phi) \quad (3.6 b)$$

Where the phases  $\delta$  and  $\phi$  differ by  $\frac{\pi}{2}$ . Equations 3.6 a and b exhibit the well-known sinusoidal behavior of the displacement of the simple harmonic oscillator.

We can obtain the relationship between the total energy of the oscillator and the amplitude of its motion as follows. Using eq 3.6 a for  $x(t)$ , we can find for the kinetic energy.

$$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \omega_0^2 A^2 \cos^2(\omega_0 t - \delta)$$

$$\therefore T = \frac{1}{2} k A^2 \cos^2(\omega_0 t - \delta) \quad \dots (3.7)$$

The incremental amount of work  $dW$  necessary to move the particle by an amount  $dx$  against the restoring force  $F$  is:

$$dW = -F dx = kx dx \quad \dots (3.8)$$

Integration from 0 to  $x$  and setting the work done on the particle equal to the potential energy

$$U = \int dW = -k \int x$$

$$\therefore U = -\frac{1}{2} k x^2$$

$$\underline{\text{or}} \quad U = -\frac{1}{2} k A^2 \sin^2(\omega_0 t - \delta) \quad \dots (3.9)$$

by combining the expressions for  $T$  and  $U$  [eqs. (3.7) and (3.9)]

$$E = T + U = \frac{1}{2} k A^2 [\cos^2(\omega_0 t - \delta) + \sin^2(\omega_0 t - \delta)]$$

$$\therefore E = T + U = \frac{1}{2} k A^2 \quad \dots (3.10)$$

Notice that the total energy is proportional to the square of the amplitude; this is a general result for linear system.

Notice also that  $E$  is independent of the time. Energy is conserved because we have been considering a system without frictional losses or other external forces.

The period  $T_0$  of the motion is defined:

$$\omega_0 T_0 = 2\pi \quad \dots (3.11)$$

$$\therefore T_0 = 2\pi \sqrt{\frac{m}{k}} \quad \dots (3.12)$$

It is clear that  $\omega_0$  represents the angular frequency of the motion, which is related to the frequency  $\nu_0$  by

$$\boxed{\omega_0 = 2\pi \nu_0 = \sqrt{\frac{k}{m}}} \quad \dots (3.13)$$

$$T_0 = \frac{1}{\nu_0} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \dots (3.14)$$

Notice that, the period of the simple harmonic oscillator is independent of the amplitude (or total energy).

### 3. The Simple pendulum

The small mass "m" swinging at the end of a tight, inextensible string of length  $l$ .

The motion is along circular arc. The restoring force along the path of motion is:

$$F_s = -mg \sin \theta$$

$$m \ddot{s} = -mg \sin \theta$$

$$\therefore m \ddot{s} + mg \sin \theta = 0$$

$$s = l\theta \Rightarrow \ddot{s} = l \ddot{\theta}$$

Also, for small  $\theta$  [ $\sin \theta \approx \theta$ ]

$$\text{Thus, } \ddot{\theta} + \frac{g}{l} \theta = 0 \quad [\text{Simple pendulum}] \quad \dots (3.15)$$

The differential equation is mathematically identical to that of the linear harmonic oscillator [ $\ddot{x} + \frac{k}{m} x = 0$ ]. We will replace  $(\frac{k}{m})$  by  $(\frac{g}{l})$  in the solution,

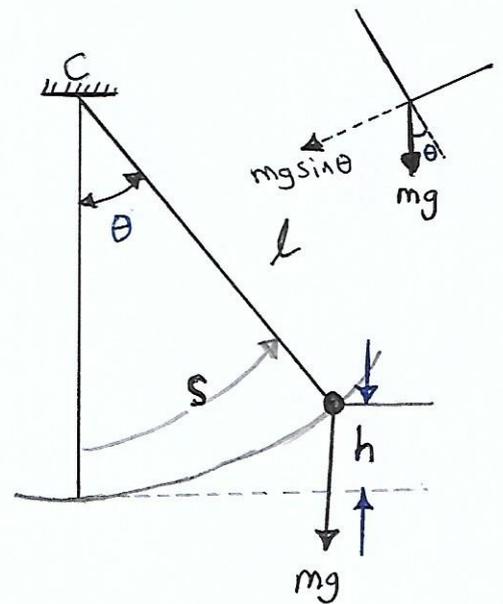


Fig. 1

$$\omega_0 = \sqrt{\frac{g}{l}}$$

--- (3.16)

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{l}{g}}$$

--- (3.17)

#### 4. Energy Considerations in Harmonic Motion

Consider a particle under the action of a linear restoring force  $F = -kx$ . Let calculate the work done by an external force  $F_{ext}$  in moving the particle from equilibrium position ( $x=0$ ) to some position  $x$ .

- Consider
- ① The particle moves slowly [not gain any kinetic energy]
  - ② Applied external force is greater in magnitude than the restoring force

$$\text{Hence, } F_{ext} = -F_x = kx$$

$$W = \int_0^x F_{ext} dx = \int_0^x kx dx$$

$$\therefore W = \frac{1}{2} kx^2 \quad \text{--- (3.18)}$$

If the spring obeys Hooke's law, the work is stored as a potential energy  $W = V(x)$

$$W = V(x) = \frac{1}{2} k x^2$$

The "total energy" when the particle is undergoing harmonic motion, is given by the sum of the kinetic and potential energies.

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \quad \dots (3.19)$$

\* The kinetic energy is quadratic in the velocity variable

\* The potential energy is quadratic in the displacement variable.

\* The total energy is constant [no other force]

Solving for the velocity:

$$\dot{x} = \pm \left( \frac{2E}{m} - \frac{kx^2}{m} \right)^{1/2} \quad \dots (3.20)$$

We can integrate to give  $t$  as a function of  $x$ :

$$\frac{dx}{dt} = \pm \left( \frac{2E}{m} - \frac{kx^2}{m} \right)^{1/2}$$

$$\therefore dt = \frac{\pm dx}{\left( \frac{2E}{m} - \frac{kx^2}{m} \right)^{1/2}}$$

$$dt = \frac{\pm dx}{\left( \frac{2E}{m} \right)^{1/2} \left[ 1 - \frac{kx^2}{2E} \right]^{1/2}}$$

$$\text{let } x = \sqrt{\frac{2E}{K}} \cos \theta$$

$$\text{Thus, } x^2 = \frac{2E}{K} \cos^2 \theta \quad ; \quad dx = -\sqrt{\frac{2E}{K}} \sin \theta d\theta$$

$$\therefore dt = \frac{\pm (-\sqrt{\frac{2E}{K}} \sin \theta) d\theta}{\left(\frac{2E}{m}\right)^2 [1 - \cos^2 \theta]^{1/2}}$$

$$\therefore dt = \frac{\mp (\sqrt{\frac{2E}{K}} \sin \theta) d\theta}{\left(\frac{2E}{m}\right)^2 \cancel{\sin \theta}}$$

$$\therefore \int dt = \int \frac{\mp (\sqrt{\frac{2E}{K}}) d\theta}{\left(\frac{2E}{m}\right)^2}$$

$$t = \mp \frac{\sqrt{\frac{2E}{K}}}{\left(\frac{2E}{m}\right)^2} \theta + C$$

$$t = \mp \sqrt{\frac{2E}{K}} \left[\frac{m}{2E}\right]^2 \theta + C$$

$$= \mp \sqrt{\frac{2E}{K}} \left[\frac{m}{2E}\right]^2 \cos^{-1} \left[ \frac{x}{\sqrt{\frac{2E}{K}}} \right] + C$$

$$= \mp \sqrt{\frac{2E \cdot m}{K \cdot 2E}} \cos^{-1} \left[ \frac{x}{A} \right] + C$$

$$\therefore \boxed{t = \mp \sqrt{\frac{m}{K}} \cos^{-1} \left[ \frac{x}{A} \right] + C} \quad \dots (3-21)$$

Where  $\boxed{A = \sqrt{\frac{2E}{K}}}$  A is the amplitude  $\dots (3-22)$

We can obtain the relation of A from the energy equation (3-19).

□ The value of  $x$  must lie between  $\pm A$  in order

For  $\dot{x}$  to be real (see the figure below)  $[E = \frac{1}{2} k A^2]$

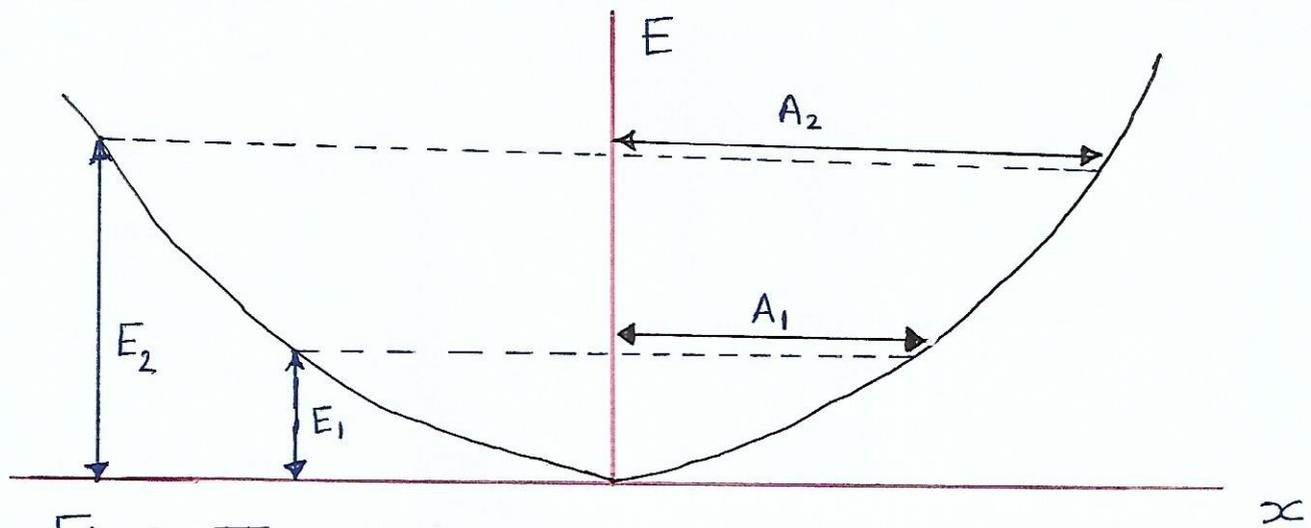


Fig. 2: The parabolic potential energy as a function of the harmonic oscillator.

[2] The maximum value of the speed  $[v_{max}]$  occurs at  $x=0$

$$E = \frac{1}{2} m v_{max}^2 = \frac{1}{2} k A^2$$

**Example 1** Find the total energy of the simple pendulum?

The potential energy of the simple pendulum (Fig 1) is given by the expression

$$V = mgh$$

$h$  is the vertical distance

From Fig. 1, the displacement through an angle  $\theta$ , we see that  $h = l - l \cos \theta$

$$\therefore V(\theta) = mg(l - l \cos \theta)$$

Using  $\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$

Taking the first two terms, ... Yield

$$V(\theta) = \frac{1}{2} mgl \theta^2$$

$$s = l\theta$$

$$V(s) = \frac{1}{2} \frac{mg}{l} s^2$$

The potential energy function is quadratic in the displacement variable; the total energy is given by:

$$E = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} \frac{mg}{l} s^2$$

### Example 2

Calculate the average kinetic, potential and total energies of the harmonic oscillator.

$$\langle \text{Energy} \rangle = \frac{1}{T_0} \int_0^{T_0} \text{energy}(t) dt$$

We use  $K$  for the kinetic energy and  $T_0$  for the period of the motion.

$$\begin{aligned} \langle K \rangle &= \frac{1}{T_0} \int_0^{T_0} K(t) dt \\ &= \frac{1}{T_0} \int_0^{T_0} \frac{1}{2} m \dot{x}^2 dt \end{aligned}$$

$$x = A \sin(\omega t - \delta)$$

[See 3-6a]

$$\dot{x} = \omega_0 A \cos(\omega_0 t - \delta)$$

let  $\delta = 0$  and  $u = \omega_0 t = \left(\frac{2\pi}{T_0}\right)t \Rightarrow dt = \frac{T_0}{2\pi} du$

$$\langle K \rangle = \frac{1}{T_0} \left[ \frac{1}{2} m \omega_0^2 A^2 \int_0^{T_0} \cos^2(\omega_0 t) dt \right]$$

$$\therefore \langle K \rangle = \frac{1}{T_0} \left[ \frac{1}{2} m \omega_0^2 A^2 \frac{T_0}{2\pi} \int_0^{2\pi} \cos^2 u du \right]$$

$$\langle K \rangle = \frac{1}{2\pi} \left[ \frac{1}{2} m \omega_0^2 A^2 \int_0^{2\pi} \cos^2 u du \right]$$

We can make use of the fact that:

$$\frac{1}{2\pi} \int_0^{2\pi} (\sin^2 u + \cos^2 u) du = \frac{1}{2\pi} \int_0^{2\pi} du = 1$$

Thus,  $\frac{1}{2\pi} \int_0^{2\pi} \cos^2 u du = \frac{1}{2}$

because the areas under the  $\cos^2$  and  $\sin^2$  terms throughout one cycle are identical.

$$\langle K \rangle = \frac{1}{4} m \omega_0^2 A^2$$

Average kinetic energy

The average potential energy proceeds along similar lines

$$V = \frac{1}{2} k x^2$$

$$= \frac{1}{2} k A^2 \sin^2 \omega_0 t$$

$$\langle V \rangle = \frac{1}{T_0} \int_0^{T_0} V(t) dt$$

$$= \frac{1}{T_0} \frac{1}{2} k A^2 \int_0^{T_0} \sin^2 \omega_0 t dt$$

$$= \frac{1}{2} k A^2 \frac{1}{2\pi} \int_0^{2\pi} \sin^2 u du$$

$$= \frac{1}{4} k A^2$$



$$\frac{k}{m} = \omega_0^2 \quad \text{or} \quad k = m\omega_0^2$$

$$\text{Thus, } \langle V \rangle = \frac{1}{4} k A^2 = \frac{1}{4} m \omega_0^2 A^2 = \langle K \rangle$$

$$\therefore \langle E \rangle = \langle K \rangle + \langle V \rangle$$

$$\langle E \rangle = \frac{1}{2} m \omega_0^2 A^2$$

$$\langle E \rangle = \frac{1}{2} k A^2 = E$$

The average kinetic energies and potential energies are equal. Therefore, the average energy of the oscillator is equal to its total instantaneous energy.