

The Newtonian Mechanics

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Effect of Retarding Force

$$F = m\ddot{x}$$

We should emphasize that the force is not necessarily constant, and indeed, it may consist of several distinct parts.

F is not constant

For example, if a particle falls in a constant gravitational field, the gravitational force is $\mathbf{F}_g = m\mathbf{g}$, where $\mathbf{g}$ is the acceleration of gravity. If, in addition, a retarding force $\mathbf{F}_r$ exists that is some function of the instantaneous speed, then the total force is

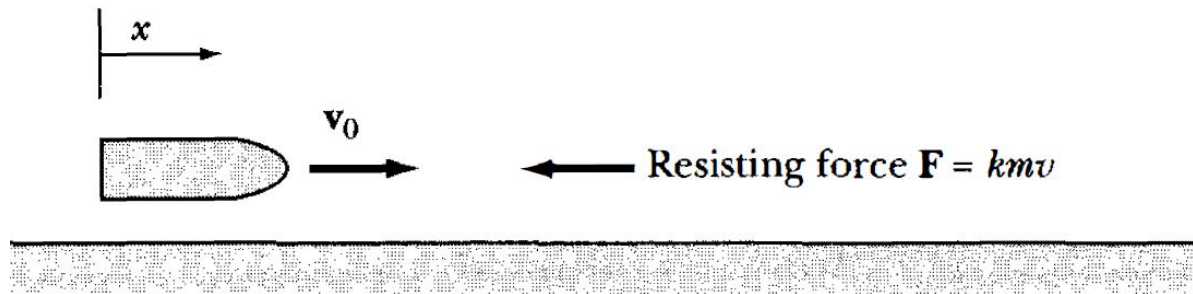
$$\mathbf{F} = \mathbf{F}_g + \mathbf{F}_r(v)$$

$$\mathbf{F} = m\mathbf{g} + \mathbf{F}_r(v)$$

$$F_r \propto v^n \begin{cases} v & \text{at low speed} \\ v^2 & \text{at high speed} \end{cases}$$

Example 4:

As the simplest example of the resisted motion of a particle, find the displacement and velocity of horizontal motion in a medium in which the retarding force is proportional to the velocity.



x-direction

$$ma = m \frac{dv}{dt} = -kmv$$

$$\int \frac{dv}{v} = -k \int dt$$

$$\ln v = -kt + C_1$$

The integration constant in Equation 2.23 can be evaluated if we prescribe the initial condition $v(t = 0) \equiv v_0$. The $C_1 = \ln v_0$, and

$$v = v_0 e^{-kt} \quad (2.24)$$

We can integrate this equation to obtain the displacement x as a function of time:

$$\begin{aligned} v &= \frac{dx}{dt} = v_0 e^{-kt} \\ x &= v_0 \int e^{-kt} dt = -\frac{v_0}{k} e^{-kt} + C_2 \end{aligned} \quad (2.25a)$$

The initial condition $x(t = 0) \equiv 0$ implies $C_2 = v_0/k$. Therefore

$$x = \frac{v_0}{k} (1 - e^{-kt}) \quad (2.25b)$$

We can obtain the velocity as a function of displacement by writing :

$$\frac{dv}{dx} = \frac{dv}{dt} \frac{dt}{dx}$$

[Chain rule]

$$= \frac{dv}{dt} \left(\frac{1}{v} \right)$$

[as $\frac{dx}{dt} = \text{velocity}$]

So that

$$v \frac{dv}{dx} = \frac{dv}{dt} \quad \longrightarrow \quad v \frac{dv}{dx} = -k v_0 e^{-kt}$$

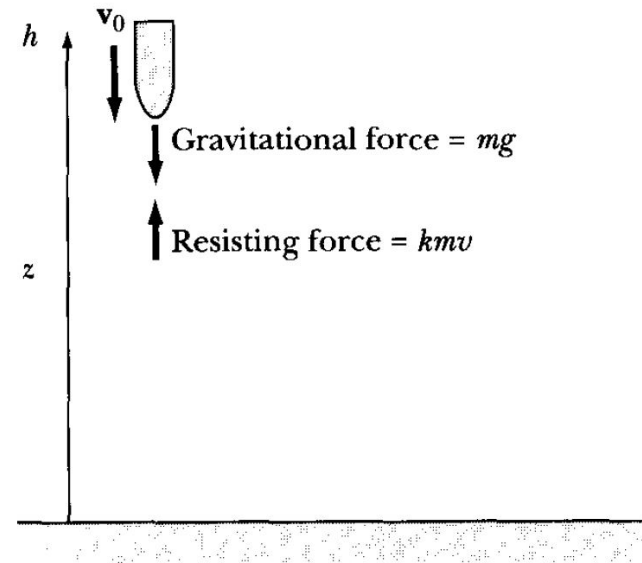
$$v = v_0 e^{-kt}$$

~~$$v \frac{dv}{dx} = -kv$$~~
$$\longrightarrow \int_{v_0}^v dv = -k \int_0^x dx$$

$$v = v_0 - kx$$

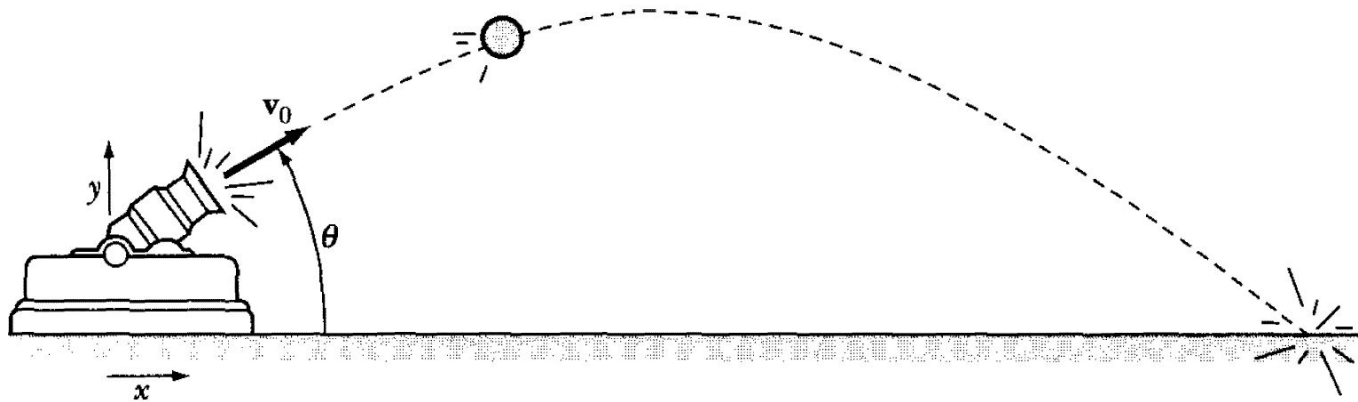
Therefore, the velocity decreases linearly with displacement:

Example 5 & Terminal Velocity see Pdf file:



Example 6:

Consider projectile motion in two dimensions, without considering air resistance. Let the muzzle velocity of the projectile be v_0 and the angle of elevation be (θ) (Figure 2-7). Calculate the projectile's displacement, velocity, and range



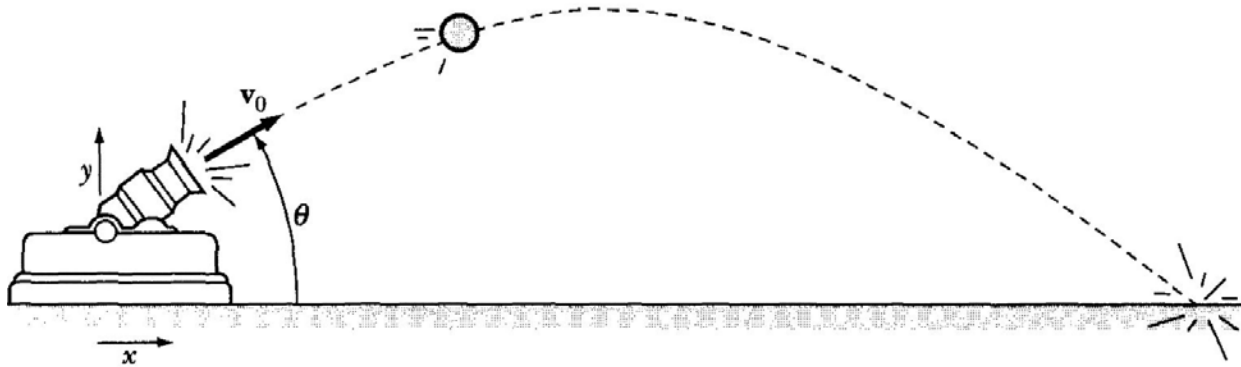
Solution. Using $F = mg$, the force components become

x-direction

$$0 = m\ddot{x}$$

y-direction

$$-mg = m\ddot{y}$$



Neglect the height of the gun, and assume $x = y = 0$ at $t = 0$. Then

$$\ddot{x} = 0$$

$$\dot{x} = v_0 \cos \theta$$

$$x = v_0 t \cos \theta$$

$$\ddot{y} = -g$$

$$\dot{y} = -gt + v_0 \sin \theta$$

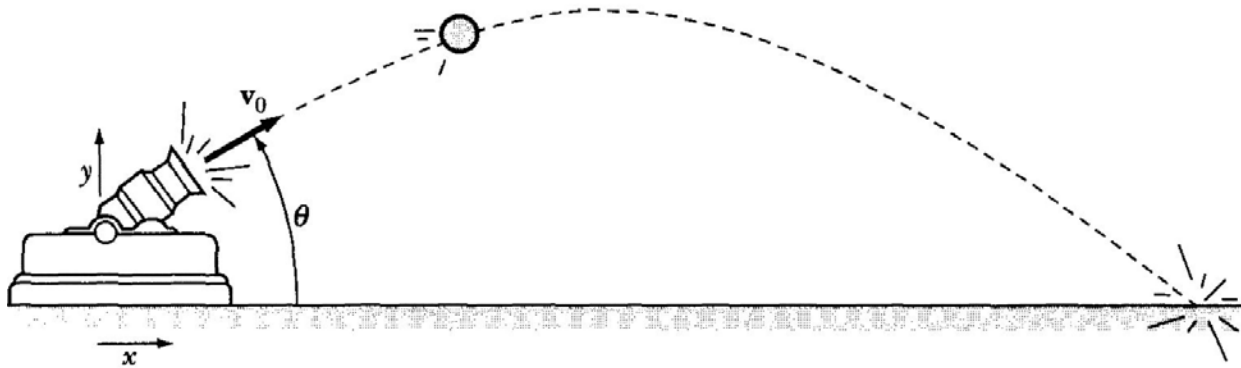
$$y = \frac{-gt^2}{2} + v_0 t \sin \theta$$

The speed and total displacement as functions of time are found to be

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = (v_0^2 + g^2 t^2 - 2v_0 g t \sin \theta)^{1/2}$$

and

$$r = \sqrt{x^2 + y^2} = \left(v_0^2 t^2 + \frac{g^2 t^4}{4} - v_0 g t^3 \sin \theta \right)^{1/2}$$



We can find the range by determining the value of x when the projectile falls back to ground, that is, when $y = 0$.

$$y = t \left(\frac{-gt}{2} + v_0 \sin \theta \right) = 0 \quad (2.36)$$

One value of $y = 0$ occurs for $t = 0$ and the other one for $t = T$.

$$\frac{-gT}{2} + v_0 \sin \theta = 0$$

$$T = \frac{2v_0 \sin \theta}{g} \quad (2.37)$$

The range R is found from

$$x(t = T) = \text{range} = \frac{2v_0^2}{g} \sin \theta \cos \theta \quad (2.38)$$

$$R = \text{range} = \frac{v_0^2}{g} \sin 2\theta \quad (2.39)$$

Notice that the maximum range occurs for $\theta = 45^\circ$.

Let us use some actual numbers in these calculations. The Germans used a long-range gun named Big Bertha in World War I to bombard Paris. Its muzzle velocity was 1,450 m/s. Find its predicted range, maximum projectile height, and projectile time of flight if $\theta = 55^\circ$. We have $v_0 = 1450$ m/s and $\theta = 55^\circ$, so the range (from Equation 2.39) becomes

$$R = \frac{(1450 \text{ m/s})^2}{9.8 \text{ m/s}^2} [\sin(110^\circ)] = 202 \text{ km}$$

Big Bertha's actual range was 120 km. The difference is a result of the real effect of air resistance.

To find the maximum predicted height, we need to calculate y for the time $T/2$ where T is the projectile time of flight:

$$\begin{aligned} T &= \frac{(2)(1450 \text{ m/s})(\sin 55^\circ)}{9.8 \text{ m/s}^2} = 242 \text{ s} \\ y_{\max} \left(t = \frac{T}{2} \right) &= \frac{-gT^2}{8} + \frac{v_0 T}{2} \sin \theta \\ &= \frac{-(9.8 \text{ m/s}^2)(242 \text{ s})^2}{8} + \frac{(1450 \text{ m/s})(242 \text{ s}) \sin(55^\circ)}{2} \\ &= 72 \text{ km} \end{aligned}$$

Example 7:

Next, we add the effect of air resistance to the motion of the projectile in the **previous example**. Calculate the decrease in range under the assumption that the force caused by air resistance is directly proportional to the projectile's velocity.

Solution.  The initial conditions are the same as in the previous example.

$$\left. \begin{aligned} x(t=0) &= 0 = y(t=0) \\ \dot{x}(t=0) &= v_0 \cos \theta \equiv U \\ \dot{y}(t=0) &= v_0 \sin \theta \equiv V \end{aligned} \right\} \quad (2.40)$$

However, the equations of motion, Equation 2.31, become

$$m\ddot{x} = -\boxed{km\dot{x}} \quad (2.41)$$

$$m\ddot{y} = -\boxed{km\dot{y}} - mg \quad (2.42)$$

Air Resistance

Equation 2.41 is exactly that used in Example 2.4. The solution is therefore

$$x = \frac{U}{k}(1 - e^{-kt}) \quad (2.43)$$

Similarly , we can find y by letting $h=0$

H.W

$$y = -\frac{gt}{k} + \frac{kV + g}{k^2}(1 - e^{-kt}) \quad (2.44)$$

The trajectory is shown in Figure 2-8 for several values of the retarding force constant k for a given projectile flight.

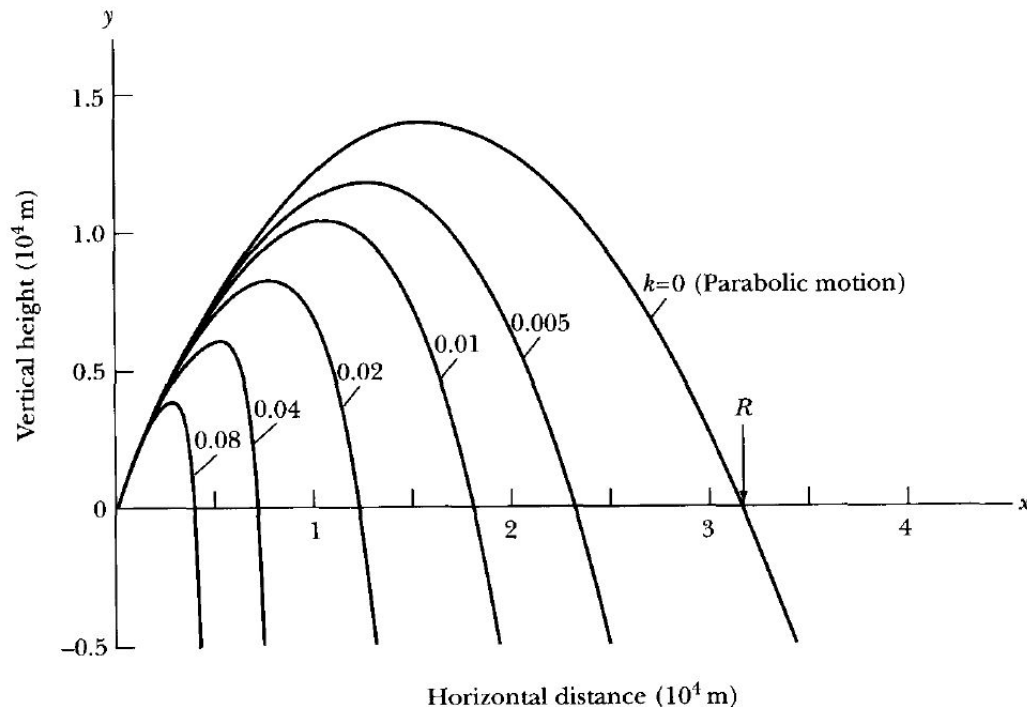


FIGURE 2-8 The calculated trajectories of a particle in air resistance ($F_{\text{res}} = -kmv$) for various values of k (in units of s^{-1}). The calculations were performed for values of $\theta = 60^\circ$ and $v_0 = 600 \text{ m/s}$. The values of y (Equation 2.44) are plotted versus x (Equation 2.43).

The range R' , which is the range including air resistance, can be found as previously by calculating the time T required for the entire trajectory and then substituting this value into Equation for x . The time T is found as previously by finding **$t = T$ when $y = 0$** . From, we find

$$T = \frac{kV + g}{gk} (1 - e^{-kT})$$

$$y = -\frac{gt}{k} + \frac{kV + g}{k^2} (1 - e^{-kt})$$

This is a transcendental equation, and therefore we cannot obtain an **analytic** expression for T . Nonetheless, we still have **powerful methods** to use to solve. To solve this equation, we need to follow **the perturbation method**.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$



Expand the exponential term

and $(1 \pm x)^{-1} = 1 \pm x \pm x^2 \pm x^3 + \dots$



Taylor Series

$$T = \frac{kV + g}{gk} \left(kT - \frac{1}{2}k^2T^2 + \frac{1}{6}k^3T^3 - \dots \right)$$

$$T = \frac{kV + g}{gk} \left(kT - \frac{1}{2}k^2T^2 + \frac{1}{6}k^3T^3 - \dots \right)$$

If we keep only terms in the expansion through k^3 , this equation can be rearranged to yield



[Return to pdf file](#)

$$T = \frac{2V/g}{1 + kV/g} + \frac{1}{3}kT^2$$

We now have the expansion parameter k in the denominator of the first term on the right-hand side of this equation. We need to expand this term in a power series (Taylor series,

$$\frac{1}{1 + kV/g} = 1 - (kV/g) + (kV/g)^2 - \dots$$

where we have kept only terms through k^2

$$T = \frac{2V}{g} + \left(\frac{T^2}{3} - \frac{2V^2}{g^2} \right)k + O(k^2)$$

$$T = \frac{2V}{g} + \left(\frac{T^2}{3} - \frac{2V^2}{g^2} \right) k + O(k^2)$$

where we choose to neglect $O(k^2)$, the terms of order k^2 and higher. In the limit $k \rightarrow 0$ (no air resistance), Equation 2.49 gives us the same result as in the previous example:

$$T(k=0) = T_0 = \frac{2V}{g} = \frac{2v_0 \sin \theta}{g}$$

Therefore, if k is small (but nonvanishing), the flight time will be *approximately* equal to T_0 . If we then use this approximate value for $T = T_0$ in the right-hand side of Equation 2.49, we have

$$T \cong \frac{2V}{g} \left(1 - \frac{kV}{3g} \right)$$

Next, we write the equation for x (Equation 2.43) in expanded form:

$$x = \frac{U}{k} (1 - e^{-kt})$$

$$x = \frac{U}{k} \left(kt - \frac{1}{2} k^2 t^2 + \frac{1}{6} k^3 t^3 - \dots \right)$$

Because $x(t = T) \equiv R'$, we have approximately for the range

$$R' \cong U \left(T - \frac{1}{2} k T^2 \right) \quad x = \frac{U}{k} \left(kt - \frac{1}{2} k^2 t^2 + \frac{1}{6} k^3 t^3 - \dots \right)$$

where again we keep terms only through the first order of k . We can now evaluate this expression by using the value of T from Equation 2.50. If we retain only terms linear in k , we find

$$R' \cong \frac{2UV}{g} \left(1 - \frac{4kV}{3g} \right) \quad (2.53)$$

The quantity $2UV/g$ can now be written (using Equations 2.40) as

$$\frac{2UV}{g} = \frac{2v_0^2}{g} \sin \theta \cos \theta = \frac{v_0^2}{g} \sin 2\theta = R \quad (2.54)$$

which will be recognized as the range R of the projectile when air resistance is neglected. Therefore

$$R' \cong R \left(1 - \frac{4kV}{3g} \right)$$