The Fundamental Concepts of Vectors

By
Dr. Mohammed F. Al-Mudhaffer
Department of Physics-College of Education for Pure Science
2020-2021

Outlines:

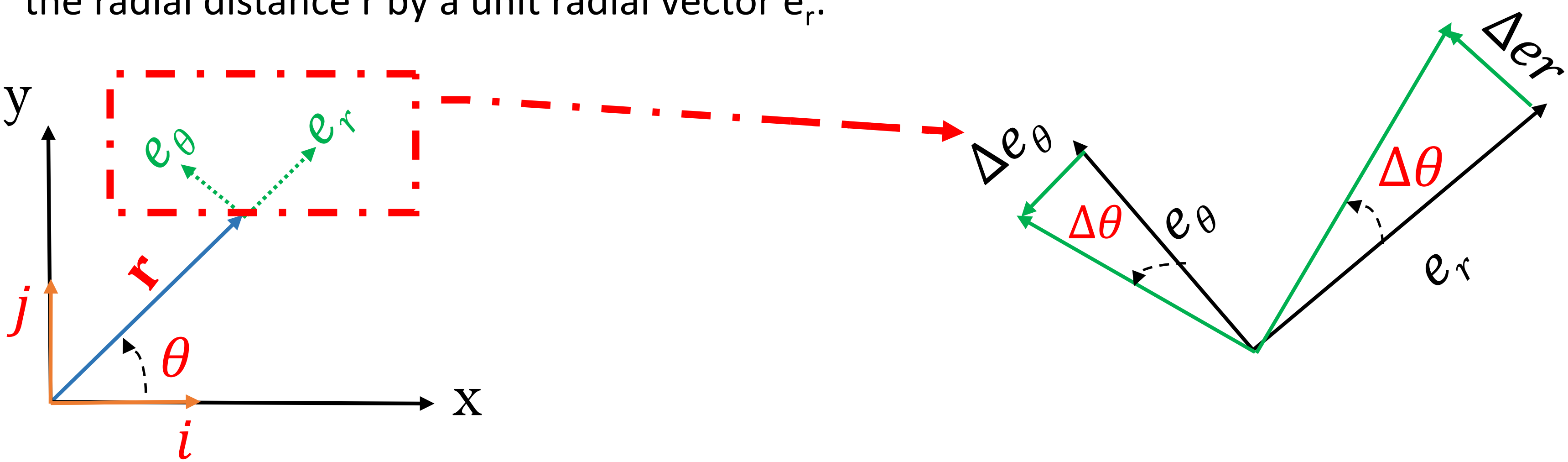
1.9 Velocity and Acceleration in Plane Polar Coordinates:

1.10.a Velocity and Acceleration in Cylinder Coordinates:

1.10.b Velocity and Acceleration in Spherical Coordinates:

1.9 Velocity and Acceleration in Plane Polar Coordinates:

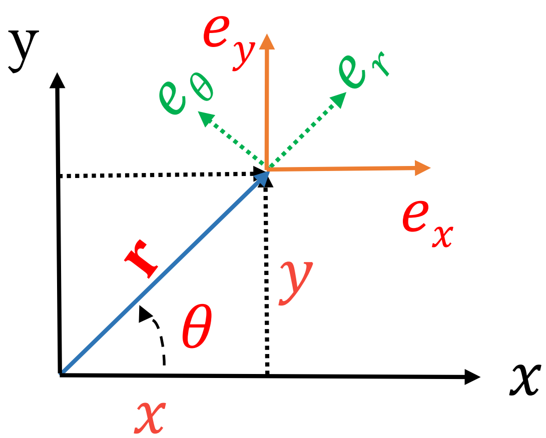
It is often convenient to employ polar coordinates r, θ to express the position of a particle moving in a plane. Vertically, the position of the particle can be written as the product of the radial distance r by a unit radial vector e_r:



$$\mathbf{r} = r e_r$$ → **Position vector**

As Particle Moving → $r \,\&\, e_r$ **both are vary with (t)**

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r} e_r + r \frac{de_r}{dt}$$



$$\mathbf{v} = \frac{dr}{dt} = \dot{r}e_r + r\boxed{\frac{de_r}{dt}}$$

$$x = r\,cos\theta \;and\; y = r\,sin\theta$$

$$e_r = cos\theta\;\; e_x + sin\theta\; e_y$$

$$e_\theta = -sin\theta e_x + cos\theta e_y$$

$$\frac{de_r}{dt} = \frac{d(cos\theta)}{dt}e_x + cos\theta\cancel{\frac{de_x}{dt}} + \frac{d(sin\theta)}{dt}e_y + sin\theta\cancel{\frac{de_y}{dt}}$$

$$\frac{de_r}{dt} = \frac{d(cos\theta)}{dt}e_x + \frac{d(sin\theta)}{dt}e_y$$

$$\frac{de_r}{dt} = -sin\theta\frac{d\theta}{dt}e_x + cos\theta\frac{d\theta}{dt}e_y = \frac{d\theta}{dt}e_\theta$$

$$\boxed{\frac{de_r}{dt} = \dot{\theta}e_\theta}$$

In similar procedure we can find

$$\boxed{\frac{de_\theta}{dt} = -\dot{\theta}e_r}$$

$$\mathbf{v} = \frac{dr}{dt} = \dot{r}e_r + r\dot{\theta}e_\theta$$

Velocity vector

$$\mathbf{v} = \frac{dr}{dt} = \boxed{\dot{r}e_r} + \boxed{r\dot{\theta}e_\theta}$$

$$\mathbf{v}_r = \dot{r}e_r$$

$$\mathbf{v}_\theta = r\dot{\theta}e_\theta$$

$$\mathbf{v} = \frac{dr}{dt} = \dot{r}e_r + r\dot{\theta}e_\theta$$

$$\mathbf{a} = \frac{dv}{dt} = \ddot{\boldsymbol{r}}e_r + \dot{r}\frac{de_r}{dt} + \left(r\ddot{\theta} + \dot{\boldsymbol{r}}\dot{\theta}\right)e_\theta + r\dot{\theta}\,\frac{de_\theta}{dt}$$

$\frac{de_r}{dt}$ and $\frac{de_\theta}{dt}$ are given

$$\boxed{\frac{de_r}{dt} = \dot{\theta}e_\theta}$$

$$\boxed{\frac{de_\theta}{dt} = -\dot{\theta}e_r}$$

$H.W$

$$\mathbf{a} = \boxed{(\ddot{\boldsymbol{r}} - r\dot{\theta}^2)e_r} + \boxed{\left(r\ddot{\theta} + 2\dot{\boldsymbol{r}}\dot{\theta}\right)e_\theta} \rightarrow \textbf{Acceleration vector}$$

$$\mathbf{a}_r = \ddot{\boldsymbol{r}} - r\dot{\theta}^2$$

$$\mathbf{a}_\theta = \left(r\ddot{\theta} + \mathbf{2}\dot{\boldsymbol{r}}\dot{\theta}\right)$$

Example

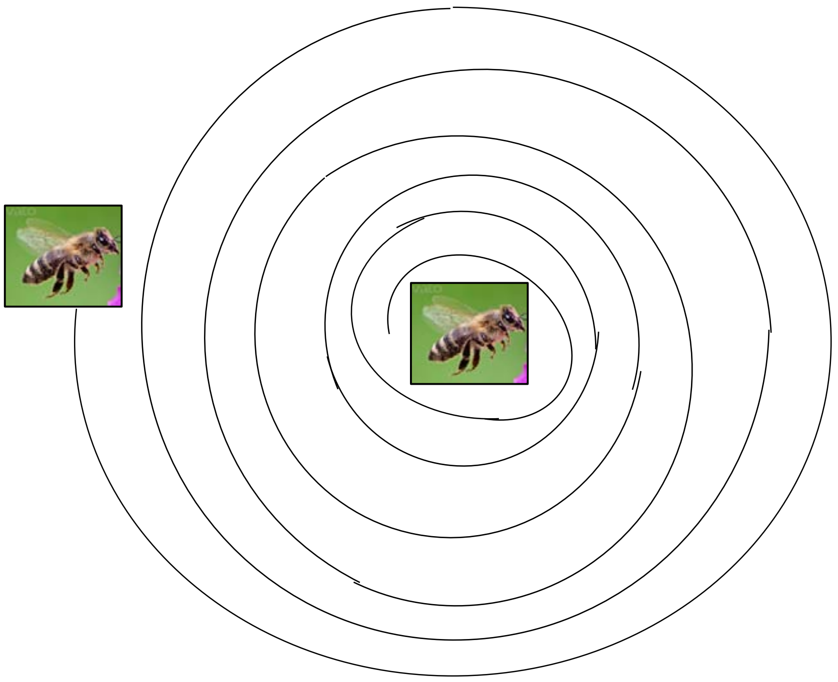
A honeybee hones in on its hive in a spiral path in such a way that the radial distance decreases at constant rate, $r = b - ct$, while the angular speed increases at constant rate, $\dot{\theta} = \mathrm{kt}$. Find the speed as a function of time

$$\mathbf{v} = \dot{r}e_r + r\dot{\theta}e_\theta$$

We have $\dot{r} = -c \; and \; \dot{\theta} = kt$

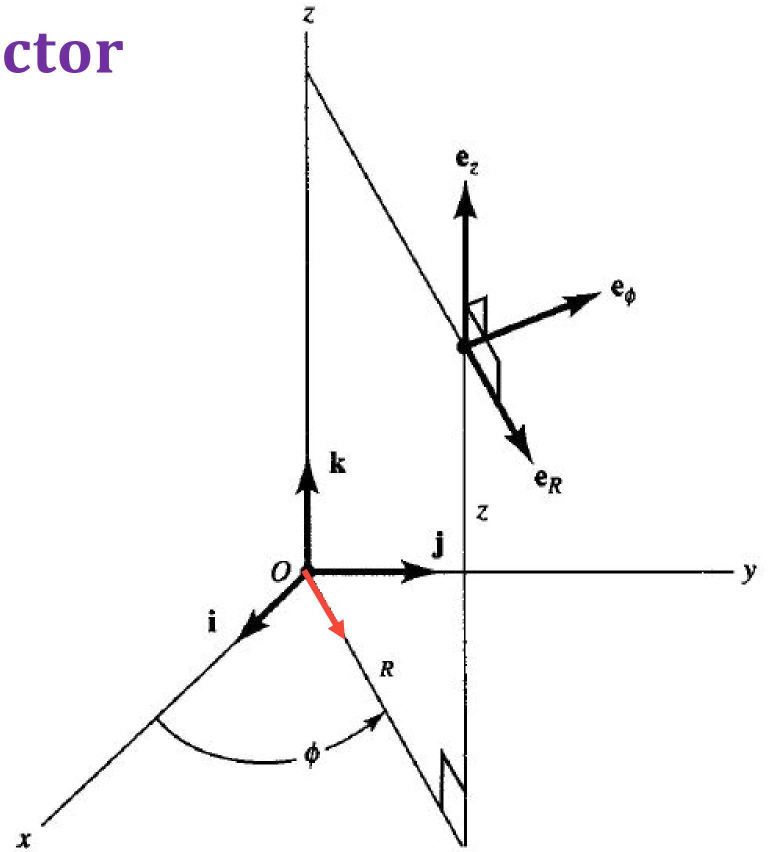
$$\mathbf{v} = -\boldsymbol{c}\; e_r + (b - ct)\; kt\; e_\theta$$

$$\mathbf{v} = \sqrt{\boldsymbol{c}^{\mathbf{2}} \; + (b - ct)^2\, k^2 t^2}$$



1.10.a Velocity and Acceleration in Cylinder Coordinates:

$\mathbf{r} = R\, e_R + z\, ez$ → **Position vector**



$e_R = i\, cos\phi + j\, sin\phi$

$e_\phi = -i\, sin\phi + j\, cos\phi$

$e_z = k$

$\frac{de_R}{dt} = e_\phi\, \dot{\phi}\,, \quad \frac{de_\phi}{dt} = er\, \dot{\phi}\ and\ \frac{de_z}{dt} = 0$

$$v = R\frac{de_R}{dt} + \dot{R}e_R + \dot{z}e_z$$

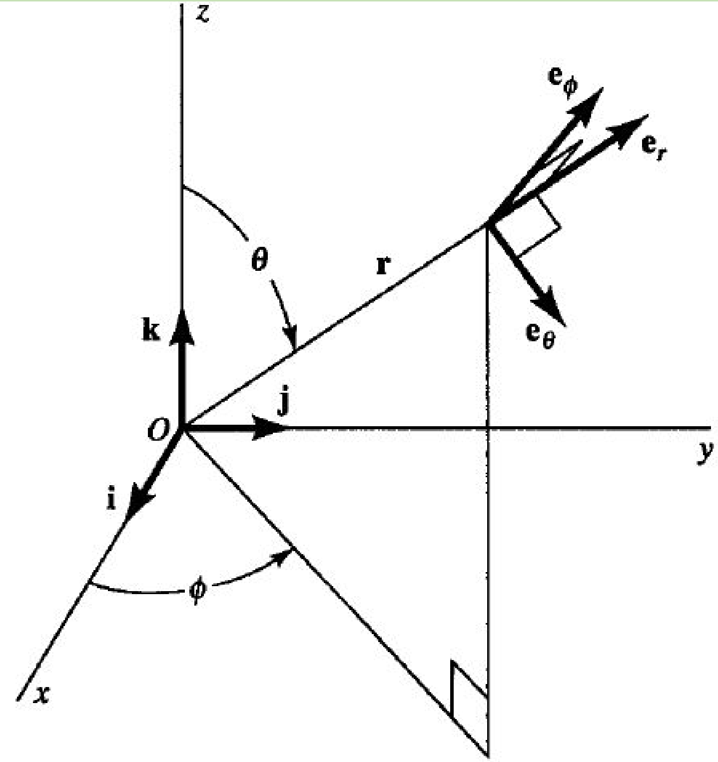
$$v = \dot{R}e_R + R\,\dot{\phi}e_\phi + \dot{z}e_z$$ → **Velocity vector**

H.W.

$$a = (\ddot{R} - R\,\dot{\phi}^2)e_R + (2\dot{R}\,\dot{\phi} + R\ddot{\phi})e_\phi + \ddot{z}e_z$$

1.10.b. Velocity and Acceleration in Spherical Coordinates:

$\mathbf{r} = r\, e_R$ → **Position vector**



$$e_r = i\; sin\theta cos\phi + j \sin\theta\; sin\phi + k \cos\theta$$

$$e_\theta = i\; cos\theta sin\phi + j \cos\theta\; sin\phi - k \sin\theta$$

$$e_\phi = -i\; sin\phi + j cos\phi$$

$$\boxed{\frac{de_r}{dt} = \dot{\theta} e_\theta + \; \dot{\phi}\; sin\theta\; e_\phi}$$

$$\frac{de_\theta}{dt} = -\dot{\theta} e_r + \phi\; cos\theta\; e_\phi$$

$$\frac{de_\phi}{dt} = -\; \dot{\phi}\; sin\theta\; e_r - \; \dot{\phi}\; cos\theta\; e_\phi$$

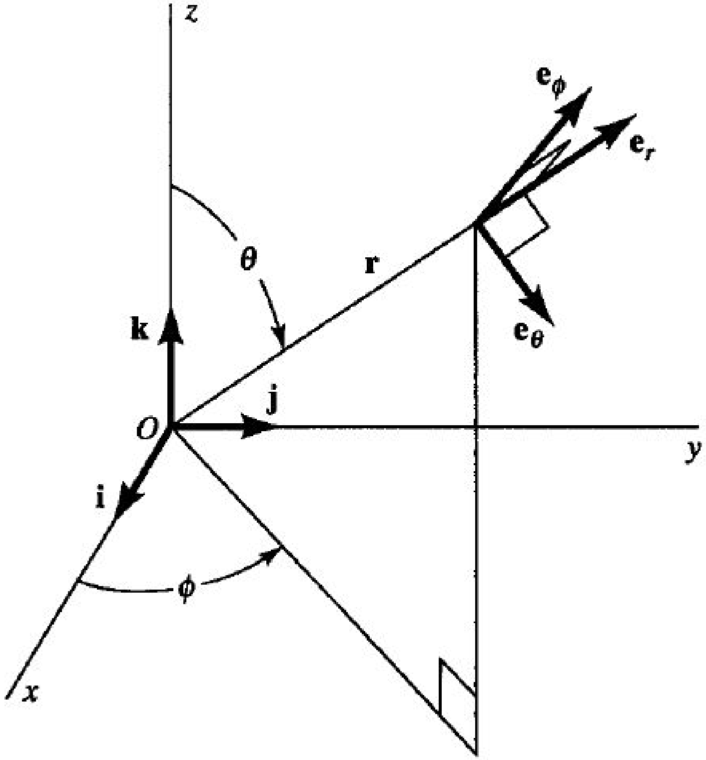
$$v = \dot{r} e_r + r \frac{de_r}{dt}$$

→ **Velocity is the derivative of Position vector**

$$v = \dot{r}e_r + r\frac{de_r}{dt}$$

$$v = \dot{r}e_R + r\,\dot{\phi}\,sin\theta\;e_\phi + r\dot{\theta}e_\theta$$

$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\mathbf{e}_\theta$$
$$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\mathbf{e}_\phi$$

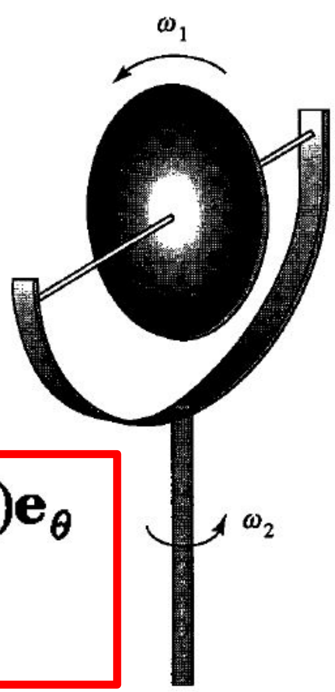


Example:

A wheel of radius b is placed in a gimbal mount and is made to rotate as follows. The wheel spins with constant angular speed ω_1 about its own axis, which in turn rotates with constant angular speed ω_2 about a vertical axis in such a way that the axis of the wheel stays in a horizontal plane and the center of the wheel is motionless. Use spherical coordinates to find the acceleration of any point on the rim of the wheel. In particular, find the acceleration of the highest point on the wheel.

$$r = \mathrm{b}, \theta = \omega_1 t \; and \; \phi = \omega_2 t$$

$$\dot{r} = 0, \dot{\theta} = \omega_1 , \ddot{\theta} = 0 , \dot{\phi} = \omega_2 \text{ and } \ddot{\phi} = 0$$

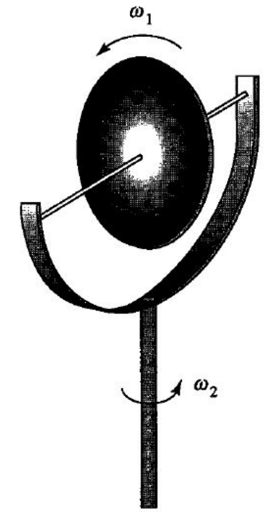


In general the acceleration can be defined as following :

$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2 \sin^2\theta - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta)\mathbf{e}_\theta + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\mathbf{e}_\phi$$

$$r = \mathrm{b}, \theta = \omega_1 t \ and \ \phi = \omega_2 t$$

$$\dot{r} = 0, \dot{\theta} = \omega_1 , \ddot{\theta} = 0 , \dot{\phi} = \omega_2 \text{ and } \ddot{\phi} = 0$$



In general the acceleration can be defined as following :

$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2 \sin^2\theta - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta)\mathbf{e}_\theta + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\mathbf{e}_\phi$$

$$a = (0 - \mathrm{b}\omega_2{}^2 \sin^2\theta - b\,\omega_1{}^2)\mathrm{er} + (0 + 0 - \mathrm{b}\omega_2{}^2 \sin\theta \cos\theta)e_\theta + (0 + 0 + 2b\,\omega_1\,\omega_2 \cos\theta)\,\mathrm{e}_\phi$$

$$a = (\mathrm{b}\omega_2{}^2 \sin^2\theta - b\,\omega_1{}^2)\mathrm{er} - (\mathrm{b}\omega_2{}^2 \sin\theta \cos\theta)e_\theta + (2b\,\omega_1\,\omega_2 \cos\theta)\,\mathrm{e}_\phi$$

The point at the top has coordinate $\theta = 0$, so at that point

$$a = -b\,\omega_1{}^2\,\mathrm{e_r} + 2b\,\omega_1\,\omega_2\,e_\phi$$

Thanks for your attention