Sheet 5

Q1:Let (X, d) be a metric space and E $\subset$ X be a subset of X.

i. What it means to say that the set E is compact ?
ii. Identify which of the following subsets of $\mathbb{R}^2$ are compact and which are not. If E is not compact find the smallest compact set K (if there is one) such that E $\subset$ K.

a) $A = \{f(x, y) \in \mathbb{R}^2 : a \leq x^2 + y^2 \leq b\}$ for some real numbers 0 < a < b.
b) $B = \left\{(\frac{1}{n}, 0) \in \mathbb{R}^2, n \in \mathbb{N}\right\}$.
c) $C = \left\{f(x, y) \in \mathbb{R}^2 : y = \sin\left(\frac{1}{x}\right) ; x \in (0,1)\right\}$.
d) $\mathrm{D} = \{\mathrm{f}(\mathrm{x}, \mathrm{y}) \in \mathbb{R}^2 : |xy| \leq 1\}$.

Q2:- The distance between sets A and B in a metric space (X, d) is denoted by

$$dist(A, B) = \{\inf d(a, b), a \in A, b \in B\}$$

Consider the following sets in $\mathbb{R}^2$:

$A = \{(x, y) \in \mathbb{R}^2 | \, x = 0\}, B = \{(x, y) \in \mathbb{R}^2 | \, xy = 1\}$.

Prove that $dist(A, B) = 0$.

Hint. You may wish to sketch the sets A and B in order to prove this result.

Q3:- Suppose that (X, d) is a metric space, x $\in$ X and $(x_n)_{n \in \mathbb{N}}$ is a sequence in X converging to x. Show that for every $y \in X$, $d(x_n, y) \longrightarrow$ d(x, y) as $n \longrightarrow \infty$.

Q4:- In $\mathbb{R}^N$ we define

$$d_1(x, y) = \sum_{i=1}^{N} |x_i - y_i|,$$

$$d_2(x, y) = \left(\sum_{i=1}^{N} |x_i - y_i|^2\right)^{\frac{1}{2}},$$

$$d_\infty(x, y) = \max_{1 \leq i \leq N} |x_i - y_i|.$$

Show that

a) $d_\infty(x, y) \leq d_1(x, y) \leq N d_\infty(x, y)$.
b) $d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{N}\, d_\infty(x, y)$.