Sheet 4

Q1:-

(i) Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence. State precisely what it means that the sequence (x_n) converges to $x \in X$ in (X, d).

(ii) $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence that converges to $x \in X$ and $(y_n)_{n\in\mathbb{N}} \subset X$ be another sequence that also converges to x. Prove that

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Q2:- In the Euclidean space $\mathbb{R}^2$, sketch first several elements of the sequences and find their limits (if exist):

(i) $\lim_{n\to\infty} (\frac{1}{n^2}, \frac{1}{n})$.

(ii) $\lim_{n\to\infty} (n, \frac{1}{n^2})$.

(iii) $\lim_{n\to\infty} (cos(\pi/n), sin(\pi/n))$.

(iv) $\lim_{n\to\infty} (sin(\pi n), cos(\pi n))$.

Q3:- In the Euclidean space $\mathbb{R}^4$ find the following limit.

$$\lim_{n\to\infty} \left(\frac{n^2 - 3}{2n^2 + 1}, \frac{1}{n} cos(n), \frac{e^{-n}}{n+1}, \sqrt{\frac{n^3 + 3n + 1}{4n^3 + n + 1}} \right).$$

Q4:- Let $(\mathbb{R}^N, d_1)$ be a metric . A sequence

$$\left(x^{(n)}\right)_{n\in\mathbb{N}} = \left(x_1^{(n)}, x_2^{(n)}, \dots, x_N^{(n)}\right)_{n\in\mathbb{N}}$$

converges in $\mathbb{R}^N$ to the limit $x = (x_1, x_2, \dots, x_N)$.

Q5:- Let $(\mathbb{R}^N, d_\infty)$ be a metric . A sequence

$$\left(x^{(n)}\right)_{n\in\mathbb{N}} = \left(x_1^{(n)}, x_2^{(n)}, \dots, x_N^{(n)}\right)_{n\in\mathbb{N}}$$

converges in $\mathbb{R}^N$ to the limit $x = (x_1, x_2, \dots, x_N)$.

Q6:- In the remark 4.2.2 Show that the statements (a), (b) and (c) are indeed equivalent.

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