Sheet 2

Q1:- Define a metric d_1 on $C([a,b])$ by

$$d_1(f,g) = \int_a^b |f(x) - g(x)|dx\,, f,g \in C([a,b]).$$

(i) Show that d_1 is indeed metric on $C([a,b])$.

(ii) Dose the function $\|f\|_1 = \int_a^b |f(x)|dx$ defines a norm on $C([a,b])$.

Q2:- Consider linear vector space

$$C^1\,([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ and } \acute{f} \text{ are continous on } [a,b]\}$$

a) Verify that

$$\|f\|_{\infty,1} := \max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |\acute{f}(x)|$$

defines a norm on $C^1\,([a,b])$.

b) Let $f(x) = \sin(\pi x)$. Find $\|f\|_{\infty,1}$ in $C^1([0,1])$.

Q3:- Let (X_1, d_1) and (X_2, d_2) be metric spaces. Show that $X_1 \times X_2$ can be made into a metric space by the following definition of a metric *d*:

$$d((x_1,x_2),(y_1,y_2)) = d_1(x_1,y_1) + d_2(x_2,y_2).$$

Q4:- In a set X is given a function $d': X \times X \to \mathbb{R}$ that satisfies

$$d'(x,y) = 0 \Leftrightarrow x = y \quad \forall\, x,y \in X,$$

$$d'(x,y) \le d'(z,x) + d'(z,y) \quad \forall\, x,y,z \in X.$$

Show that (X, d') is a metric space.

Q5:- Let (X,d) be a metric space. The diameter of a non-empty subset *M* of *X* is defined as

$$\delta(M) = \sup_{x,y\in M} d(x,y) \qquad (\le \infty).$$

Show that $\delta(M) = 0$ if and only if *M* contains only one point.