Chapter 2

Normed Spaces

2.1 Definition of a normed space

Recall that by a vector space over an algebraic field $\mathbb{F}$ we understand a nonempty set X equipped with two operations:

1. $(x, y) \longrightarrow x + y$ from $X \times X$ into X, (addition).
2. $(\lambda, x) \longrightarrow \lambda x$ from $\mathbb{F} \times X$ into X, (scalar multiplication).

Elements of the vector space X are called vectors, elements of the field $\mathbb{F}$ are called scalars (or sometimes numbers). Here we consider only real vector spaces with $\mathbb{F} = \mathbb{R}$. A norm on a vector space is function which roughly speaking has a meaning of the length of a vector. More precisely, we have the following definition.

Definition 2.1.1. (Normed space). Let X be a vector space. A norm on X is a function $\|\cdot\| : X \longrightarrow \mathbb{R}$, which satisfy the following properties:

(N1) $\|x\| = 0$ if and only if $x = 0$.

(N2) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$. (scalar multiplication)

(N3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. (Triangle inequality)

A normed space is a pair $(X, \|\cdot\|)$, where X is a vector space and $\|\cdot\|$ is a normed on X.

Example 2.1.2. (Positivity of the norm). Let $(X, \|\cdot\|)$ be a normed space. Prove that

$$(N1') \qquad \|x\| \geq 0 \qquad \forall x \in X. \tag{2.1}$$

Example

$(\mathbb{R}, |\cdot|)$ is normed space.

Theorem 2.1.3. *Let $(X, \|\cdot\|)$ be a normed space. Then*

$$d(x, y) := \|x - y\| \tag{2.2}$$

defines a metric on X, i.e. (X, d) is a metric space.

Remark 2.1.4. Theorem 2.1.3 states that every normed space is also a metric space with the induced metric. However, not every metric space can be made into a normed space. For example, the metric space of all positive rational numbers Q_+ with the metric $d(x, y) = |log(\frac{x}{y})|$ from Example 1.1.4 is not a normed space, simply because Q_+ is not a vector space.

2.2 Vector space $\mathbb{R}^N$

The space $\mathbb{R}^N$ of N-vectors of real numbers, defined in Example 1.1.6 is a vector space. The number N is called the dimension of the space $\mathbb{R}^N$. For vectors $x, y \in \mathbb{R}^N$ and scalar $\lambda \in \mathbb{R}$, the addition and scalar multiplication are defined as follows:

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_N + y_N)$$

$$\lambda x = (\lambda x_1, \lambda x_2, \cdots, \lambda x_N)$$

Proposition 2.2.1. *(The Cauchy-Schwarz inequality). For any* $x, y \in \mathbb{R}^N$,

$$|x \cdot y| \leq \|x\|\|y\|. \tag{2.3}$$

Example 2.2.2. (Euclidean norm on $\mathbb{R}^N$). The function

$$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_N|^2} = \sqrt{\sum_{i=1}^{N} |x_i|^2}$$

defines the Euclidean norm on $\mathbb{R}^N$.

Corollary 2.2.3. $(\mathbb{R}^N, d_2)$ *is metric space*

Example 2.2.4. (Taxi-cab and ∞ norm on R^N). Let $\mathbb{R}^N$ be the N-dimensional vector space as before. We define the taxi-cab norm $\|\cdot\|_1 : \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$\|x\|_1 = |x_1| + \cdots + |x_N|$$

and infinity norm $\|\cdot\|_\infty : \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$\|x\|_\infty = \max\{|x_1|, \cdots, |x_N|\}$$

Example

Explain why $\|x\| = \min\{x_1, x_2\}$ does not define a norm on $\mathbb{R}^2$.

2.3 Vector space of continuous functions $C([a, b])$

Let $C([a, b])$ be the vector space of all continuous functions on the closed interval $[a, b]$. For functions $f, g \in C([a, b])$ and scalar $\lambda \in \mathbb{R}$, the addition and scalar multiplication are defined in a natural pointwise way:

$$(f + g)(x) = f(x) + g(x)$$

$$(\lambda f)(x) = \lambda f(x)$$

Example 2.3.1. (Uniform convergence norm on $C([a, b])$). The function

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

is a norm on $C([a, b])$, known as a uniform convergence norm, or ∞-norm on $C([a, b])$.