Chapter 7

Ordinary Differential Equations Theory

7.1 Ordinary Differential Equations Theory

In this section we will use some abstract results of metric spaces presented in the previous sections. Particularly, we want to study the existence and uniqueness of solutions to ordinary differential equations (ODE). For $[a, b] \subset \mathbb{R}$, let $y : [a, b] \longrightarrow \mathbb{R}^N$ be a function, the form

$$y'(t) = F(t, y(t)) \tag{7.1}$$

represent an ODE, where $F : [a, b] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is some continuous function. So we might think of y as the trajectory of a particle in $N-$dimensional space, the ODE then specifies the velocity of the particle at a given time, which is allowed to depend on either or both of the current time and the current position of the particle. In the same manner one can define higher-order ODE by

$$y^{(m)}(t) = F(t, y(t), \cdots, y^{(m-1)}(t)), \tag{7.2}$$

for $y : [a, b] \longrightarrow \mathbb{R}^N$, which can be reduced to system of first order ODE.

Lemma 7.1.1. *Given a continuous function $F : [a, b] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$,*

$y_0 \in \mathbb{R}^N$ and $y : [a,b] \longrightarrow \mathbb{R}^N$, the following are equivalent:

1. *y is continuously differentiable (C^1) and is a solution to the initial value problem*

$$y'(t) = F(t, y(t)), t \in [a,b]; y(a) = y_0. \tag{7.3}$$

2. *y is continuous, and for each $t \in [a,b]$ we have*

$$y(t) = y_0 + \int_a^t F(s, y(s))ds. \tag{7.4}$$

Proof. Assuming (1) we have (by the fundamental theorem of calculus)

$$y(t) - y(a) = y(t) - y_0 = \int_a^t y'(s)ds = \int_a^t F(s, y(s))ds. \tag{7.5}$$

Conversely, assuming (2), we obviously have $y(a) = y_0$ and for $h \neq 0$ we have

$$\frac{y(t+h) - y(t)}{h} = \frac{1}{h}\int_t^{t+h} F(s, y(s))ds \tag{7.6}$$

the right hand side tends to $F(t, y(t))$ as $h \longrightarrow 0$. So we deduce that the derivative $y'(t)$ exists and it is equal to $F(t, y(t))$. Since $F(t, y(t))$ is continuous function of t thus y is continuously differentiable. □

Remark 7.1.2. Recall that $(C([a,b];\mathbb{R}), d_\infty)$ and $(C([a,b];\mathbb{R}^N), d_\infty)$ are complete metric spaces. Indeed from the definition of complete metric space, so we have that $\mathbb{R}$ and $\mathbb{R}^N$ are complete (with the usual distance) in addition $(C([a,b];\mathbb{R}), d_\infty)$ and $(C([a,b];\mathbb{R}^N), d_\infty)$ are complete metric spaces.

Theorem 7.1.3 (Contractive Mapping Principle)**.** *Let (X, d) be a complete metric space and let $\phi : X \longrightarrow X$ be a function satisfying the following property: For some $r < 1$, we have, for each $x, y \in X$*

$$d(\phi(x), \phi(y)) \leq rd(x, y). \tag{7.7}$$

Then there is one and only one point $x_0 \in X$ such that

$$\phi(x_0) = x_0 \tag{7.8}$$

Proof. We know that there is no more than one fixed point of ϕ, since if $\phi(x_0) = x_0$ and $\phi(x_1) = x_1$ then our assumption shows

$$d(\phi(x_1), \phi(x_0)) = d(x_1, x_0) \leq rd(x_1, x_0). \tag{7.9}$$

Since $r < 1$ then $x_1 = x_0$ (Otherwise $r \geq 1$).
So, we only have to prove the existence of x_0. Let $x_1 \in X$ be any point, for $n \geq 1$ define $x_{n+1} = \phi(x_n)$. Then the sequence $x_n \longrightarrow x_0$ as $n \longrightarrow \infty$. Indeed, Let $A = d(x_1, x_2)$, so for any n we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(\phi(x_n), \phi(x_{n+1})) \\ &\leq rd(x_n, x_{n+1}) \\ &= rd(\phi(x_{n-1}), \phi(x_n)) \\ &\leq r^2 d(x_{n-1}, x_n) \\ &\quad \vdots \\ &\leq r^n A. \end{aligned}$$

So by the triangle inequality, we find

$$d(x_n, x_{n+k}) \leq \sum_{j=0}^{k-1} d(x_{n+j}, x_{n+j+1}) \leq \sum_{j=0}^{k-1} Ar^{n-1+j} = Ar^{n-1} \sum_{j=0}^{k-1} r^j \leq \frac{Ar^{n-1}}{1-r}$$

Hence, if $n, m > N$, we get

$$d(x_n, x_m) \leq \frac{Ar^N}{1-r} \longrightarrow 0 \qquad \text{as } N \longrightarrow \infty. \tag{7.10}$$

this shows that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence and it is converges to some $x_0 \in X$ i.e. $x_n \longrightarrow x_0$ (since (X, d) is complete). The assumption on ϕ

implies that ϕ is continuous (give $\epsilon > 0$ and take $\delta = aeps$). So,

$$\phi(x_0) = \lim_{n \longrightarrow \infty} \phi(x_n) = \lim_{n \longrightarrow \infty} (x_{n+1}) = x_0$$

□

Definition 7.1.4. $F : [a, b] \times R^N \longrightarrow R$ is called uniformly Lipschitz with Lipschitz constant M if, for every $t \in R$ and $x, y \in R^N$ we have

$$|F(t, x) - F(t, y)| \leq M|x - y|.$$

Proposition 7.1.5. *Suppose that $F : [a, b] \times R^N \longrightarrow R^N$ is continuous and uniformly Lipschitz with Lipschitz constant M and $y_0 \in R^N$, and define $\Phi : C([a, b], R^N) \longrightarrow C([a, b], R^N)$ by*

$$(\Phi(y))(t) = y_0 + \int_a^t F(s, y(s))ds \tag{7.11}$$

Then, for each $y, z \in C([a, b], R^N)$ we have

$$d(\Phi(y), \Phi(z)) \leq M(b - a)d(y, z). \tag{7.12}$$

Proof. By definition, we see that

$$d(\Phi(y), \Phi(z)) = \max_{t \in [a,b]} \left| \int_a^t (F(s, y(s)) - F(s, z(s)))ds \right|$$

So by the uniform Lipschitz assumption, we get

$$d(\Phi(y), \Phi(z)) \leq \max_{t \in [a,b]} \int_a^t M|y(s) - z(s)|ds \leq \max_{t \in [a,b]} \int_a^t Md(f, g)ds \leq M(b-a)d(f, g) \tag{7.13}$$

□

Corollary 7.1.6. *Let $F : [a, b] \times R^N \longrightarrow R^N$ be continuous and uniformly Lipschitz with Lipschitz constant M satisfying $b - a < 1/M$. Then for any $y_0 \in R^N$, the initial value problem*

$$y_0(t) = F(t, y(t)) \qquad y(a) = y_0 \tag{7.14}$$

has exactly one solution.

Proof. Proposition 7.1.5 gives $d(\Phi(y), \Phi(z)) \le M(b - a)d(y, z)$, and our assumption is that $M(b - a) < 1$. Hence we can apply the contractive mapping principle to Φ, which shows that there is one and only one $y \in C([a, b], R^N)$ such that $\Phi(y) = y$. But we observed earlier that $y \in C([a, b], R^N)$ satisfies $\Phi(y) = y$ precisely when y is a solution to the initial value problem. □

Theorem 7.1.7. *Suppose that $F : [a, b] \times R^N \longrightarrow R^N$ is continuous and uniformly Lipschitz. Then there is one and only one solution $y : [a, b] \longrightarrow R^N$ to the initial value problem*

$$y_0(t) = F(t, y(t)), \qquad y(a) = y_0. \tag{7.15}$$