Chapter 6

Continuity

6.1 Uniform convergence

In this section we study in more details the metric of uniform convergence, introduced in Example 1.5. Recall that if $C([a,b])$ denotes the set of continuous functions $f:[a,b]\longrightarrow R$, then

$$d_\infty(f,g)=\max_{x\in[a,b]}|f(x)-g(x)|$$

is a metric on $C([a,b])$, known as the metric of uniform convergence (or ∞-metric).

6.2 Uniform convergence of sequence of functions

Let $(f_n)_{n\in\mathbb{N}}$ denotes a sequence of functions $f_n:[a,b]\longrightarrow\mathbb{R}$,

$$(f_1(x),f_2(x),f_3(x),\cdots,f_n(x),\cdots)$$

In this section we discuss two different types of convergences of sequences of functions. The first one is the pointwise convergence. Another one, the most important, is the uniform convergence.

Definition 6.2.1. (**Pointwise convergence**). A sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise on $[a,b]$ to a limit function $f:[a,b]\longrightarrow\mathbb{R}$, if

$$\lim_{n\longrightarrow\infty}|f_n(x)-f(x)|=0 \qquad \forall x\in[a,b]$$

In this case we often write $f_n\longrightarrow f$ pointwise on the interval $[a,b]$.

Remark 6.2.2. In other words, $f_n\longrightarrow f$ pointwise on the interval $[a,b]$ if $f_n(x)\longrightarrow f(x)$ for every $x\in[a,b]$, in the sense of the standard convergence in $\mathbb{R}$.

Definition 6.2.3. (**Uniform convergence**). A sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on the interval $[a,b]$ to a limit function $f:[a,b]\longrightarrow R$, if

$$\lim_{n\longrightarrow\infty}\max_{x\in[a,b]}|f_n(x)-f(x)|=0$$

In this case we often write $f_n\longrightarrow f$ uniformly.

Remark 6.2.4. In other words, $f_n\longrightarrow f$ uniformly on the interval $[a,b]$ if

$$d_\infty(f_n,f)\longrightarrow 0,$$

that is $f_n\longrightarrow f$ with respect to the metric d_∞.

It is clear that if $f_n\longrightarrow f$ uniformly on the interval $[a,b]$ then $f_n\longrightarrow f$ pointwise on $[a,b]$. The converse is not true in general, as it shown in the following example.

Example 6.2.5. The sequence of continuous functions

$$f_n(x)=\frac{2nx}{1+n^2x^2}$$

converges pointwise to the limit function $f(x)=0$ on the interval $[0,2]$, but (f_n) does not converge uniformly to f.

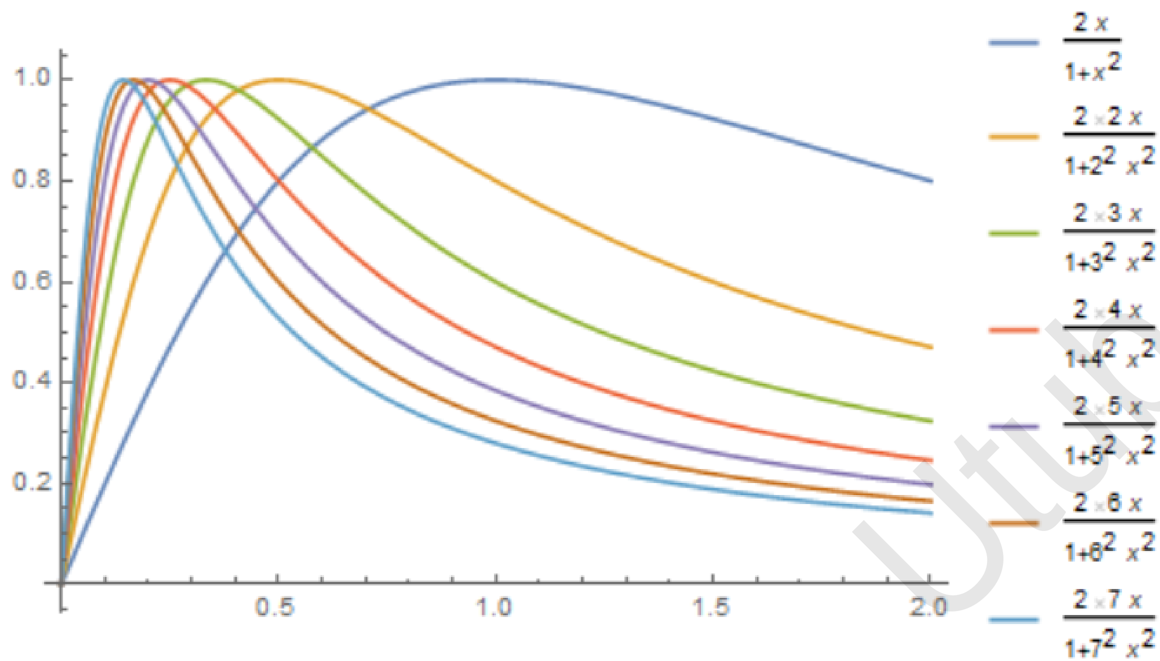

Figure 6.1: The sequence $f_n(x) = \frac{2nx}{1+n^2x^2}$

Example 6.2.6. The sequence of continuous functions

$$f_n(x) = x^n$$

converges pointwise on the interval $[0, 1]$ to the discontinuous limit function

$$f(x) = \begin{cases} 0, & \text{if} \quad x \in [0, 1), \\ 1, & \text{if} \quad x = 1. \end{cases} \tag{6.1}$$

The sequence (f_n) converges uniformly to f on any interval $[0, b]$ with $b < 1$. However, the sequence (f_n) does not converge uniformly to f on the interval $[0, 1]$.

The next theorem shows that the uniform limit of a sequence of continuous functions is a continuous function.

Theorem 6.2.7. *(Weierstrass Uniform Convergence theorem) Let* f_n :

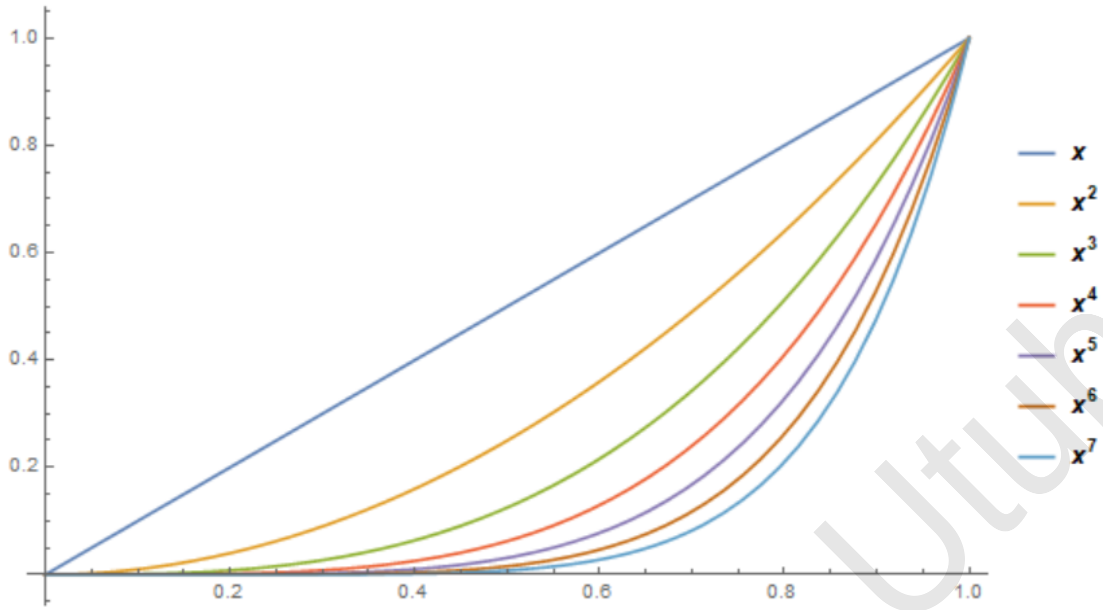

Figure 6.2: The sequence $f_n(x) = x^n$

$[a,b] \longrightarrow \mathbb{R}$ be a sequence of continuous functions. If $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$ to a limit function $f : [a,b] \longrightarrow \mathbb{R}$, then f is continuous on $[a,b]$.

Remark 6.2.8. Theorem 6.2.7 can be used to prove that the sequence $f_n(x) = x^n$ does not converge uniformly on $[0,1]$. Indeed, all the functions f_n are continuous on $[0,1]$. Assume that $f_n \longrightarrow f$ uniformly on $[0,1]$. Then by Theorem 6.2.7 the function f must be continuous on $[0,1]$. But from Example 6.2.6 we know that the pointwise limit of the sequence (f_n) is the discontinuous function f defined in (6.1), which is a contradiction.

Example 6.2.9. Find a pointwise limit of the sequence

$$f_n(x) = \frac{x^n - 1}{x^n + 1} \quad \text{on } [0,2]$$

and the sequence $g_n(x) = (1-x^2)^n$ on $[-1,1]$. Are these convergences uniform ?

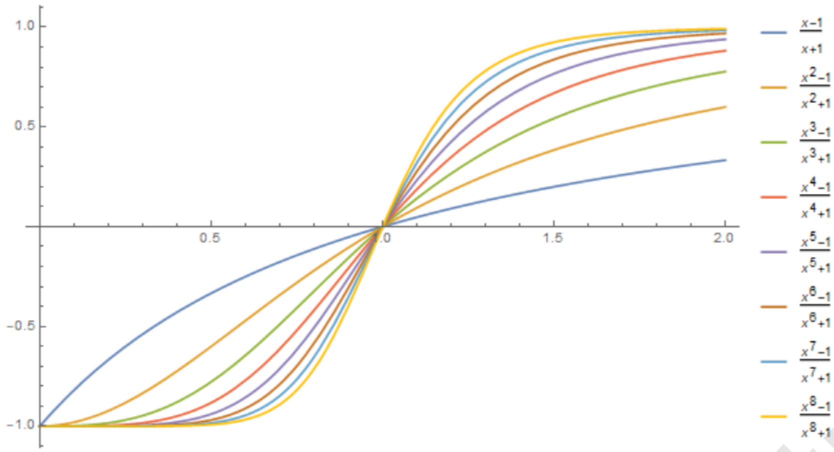

Figure 6.3: The sequence $f_n(x) = \frac{x^n-1}{x^n+1}$

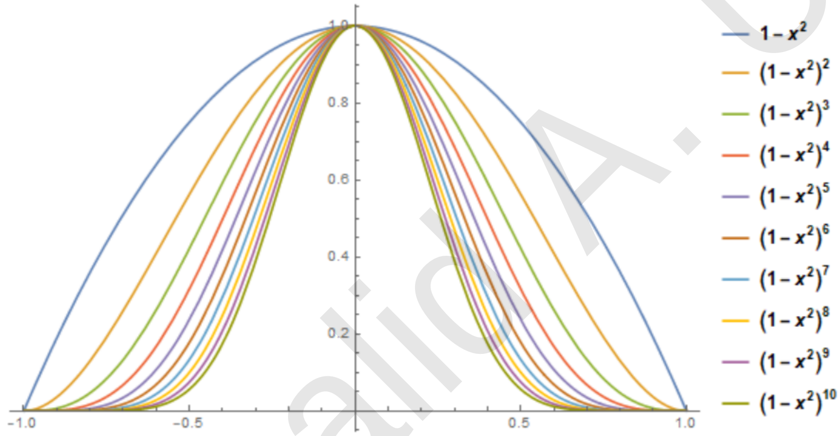

Figure 6.4: The sequence $g_n(x) = (1-x^2)^n$

Theorem 6.2.10. *Let $f_n : [a,b] \longrightarrow \mathbb{R}$ be a sequence of continuous functions. If $f_n \longrightarrow f$ uniformly on $[a,b]$, then*

$$\lim_{n\longrightarrow\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx \tag{6.2}$$

If $f_n \longrightarrow f$ only pointwise on $[a,b]$ but not uniformly then (6.2) may fail.

Example 6.2.11. For $n \geq 2$, define a sequence $f_n : [0,1] \longrightarrow \mathbb{R}$ by

$$f_n(x) = \max\left\{n - n^2\left|x - \frac{1}{n}\right|, 0\right\}.$$

Show that $f_n \longrightarrow 0$ pointwise on $[0,1]$, but (6.2) does not hold.

6.3 Uniform convergence of series of functions.

In this section we extend the notions of pointwise and uniform convergence from sequences to infinite series of functions. Let $(f_n)_{n\in\mathbb{N}}$ denotes a sequence of functions $f_n : [a,b] \longrightarrow \mathbb{R}$. For $m \in \mathbb{N}$, consider the partial sum

$$S_m(x) := \sum_{n=0}^{m} f_n(x) = f_1(x) + f_2(x) + \cdots + f_m(x)$$

Definition 6.3.1. (**Pointwise and uniformly convergent series**). We say that the series of functions

$$\sum_{n=0}^{\infty} f_n(x) \tag{6.3}$$

1. converges pointwise on $[a,b]$ to the function $f : [a,b] \longrightarrow \mathbb{R}$, if

$$S_m \longrightarrow f \qquad \text{pointwise on } [a,b].$$

2. converges uniformly on $[a,b]$ to the function $f : [a,b] \longrightarrow \mathbb{R}$, if

$$S_m \longrightarrow f \qquad \text{uniformly on } [a;b].$$

The limit function $f(x)$ is called the sum of the series and we often write

$$f(x) = \sum_{n=0}^{\infty} f_n(x);$$

pointwise (or uniformly) on $[a,b]$.

Example 6.3.2. (**Geometric series**). Consider the series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

By the formula for the partial sum of geometric series, we know that

$$S_m(x) = \frac{1 - x^{m+1}}{1 - x}.$$

We see that

$$\lim_{m \longrightarrow \infty} S_m(x) = \frac{1}{1-x} \quad \text{if } |x| < 1.$$

Moreover, if $0 < r < 1$ then

$$\max_{x \in [-r,r]} \left| S_m(x) - \frac{1}{1-x} \right| = \max_{x \in [-r,r]} \frac{|x|^{m+1}}{|1-x|} \leq \frac{r^{m+1}}{1-r} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty.$$

We conclude that $S_m \longrightarrow \frac{1}{1-x}$ uniformly on $[-r, r]$. Therefore, for any $0 < r < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

uniformly on the interval $[-r, r]$.

Definition 6.3.3. (**Absolutely convergent series**). We say that the series of functions

$$\sum_{n=0}^{\infty} f_n(x)$$

converges absolutely and uniformly on $[a, b]$ if the series of absolute values

$$\sum_{n=0}^{\infty} |f_n(x)|$$

converges uniformly on $[a, b]$.

The next powerful theorem shows that convergence of the series of

$\|\cdot\|_\infty$ −norms of continuous functions implies absolute and uniform convergence of the series and moreover,the sum of the series remains continuous.

Theorem 6.3.4. *(**Weierstrass M-test for series**). Let $f_n : [a, b] \longrightarrow \mathbb{R}$ be a sequence of continuous functions. Suppose that*

$$\sum_{n=0}^{\infty} \|f_n\|_\infty = \sum_{n=0}^{\infty} \max_{x\in[a,b]} |f_n(x)| < \infty.$$

Then the series $\sum_{n=0}^{\infty} f_n(x)$ converges absolutely and uniformly on $[a, b]$ and the sum of the series

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

is a continuous function on $[a, b]$.

Series of functions are frequently used to define exponential, trigonometric and other special functions.

Example 6.3.5. (**Exponential series**). Consider the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The sum of this series is called the exponential function and is denoted by $\exp(x)$ or e^x. For any $r > 0$, we will apply to this series the Weierstrass M-test. We obtain

$$\sum_{n=0}^{\infty} \|f_n\|_\infty = \sum_{n=0}^{\infty} \max_{x\in[a,b]} |\frac{|x|^n}{n!} < \infty$$

Then we conclude that the series converges absolutely and uniformly on the interval $[-r, r]$. Moreover, the sum of the series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a continuous function.

Exersice. (**Fourier type series**). On the interval $[-\pi, \pi]$, consider the series

$$\sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n}.$$

Prove that

$$f(x) := \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n}$$

defines a continuous function on $[-\pi, \pi]$. $\square$

Solution. By the Weierstrass M-test, using the fact that $|sin(t)| \leq 1$ for any $t \in \mathbb{R}$, we obtain

$$\sum_{n=0}^{\infty} \max_{x \in [-\pi,\pi]} \left| \frac{sin(2^n x)}{2^n} \right| \leq \sum_{n=0}^{\infty} 2^{-n} = 2$$

Hence the series converges uniformly on $[-\pi, \pi]$ to a continuous function $f(x) := \frac{sin(2^n x)}{2^n}$.

The next two theorems provide a justification for formal integration and differentiation of the series "term by term".

Theorem 6.3.6. *Let $f_n : [a, b] \longrightarrow \mathbb{R}$ be a sequence of continuous functions. Assume that*

$$f(x) = \sum_{n=0}^{\infty} f_n(x);$$

uniformly on $[a, b]$. Then the function $f : [a, b] \longrightarrow \mathbb{R}$ is also continuous *and*

$$\int_a^b f(x)dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)dx$$

Theorem 6.3.7. *Let $f_n : [a, b] \longrightarrow \mathbb{R}$ be a sequence of continuously differentiable functions. Assume that*

$$f(x) = \sum_{n=0}^{\infty} f_n(x);$$

pointwise on $[a, b]$. Assume in addition that that the series of derivatives $\sum_{n=0}^{\infty} f_n'(x)$ converges absolutely and uniformly on $[a, b]$. Then the function $f : [a, b] \longrightarrow \mathbb{R}$ is continuously differentiable on $[a, b]$ and

$$f'(x) = \sum_{n=0}^{\infty} f_n'(x)$$

Example 6.3.8. Prove that $(exp(x))' = exp(x)$. Solution. Formally differentiating term by term, we obtain

$$(exp(x))' = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \exp(x)$$

Since the series of derivatives

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

converges absolutely and uniformly to $\exp(x)$ on any interval $[-r, r]$ (see Example 6.3.5), formal differentiation is justified by Theorem 6.3.7.