Chapter 5

Continuous mappings on metric spaces

5.1 Definition and properties of a continuous mapping

In the previous sections we studied a single metric space (X, d) and properties of subsets and sequences in X. Now we consider a pair of metric spaces (X, d_X) and (Y, d_Y) and continuous mapping (or function) from X to Y, denoted

$$f : X \longrightarrow Y$$

Definition 5.1.1. (Continuous mapping). Let (X, d_X) and (Y, d_Y) be a pair of metric spaces. We say that a mapping $f : X \longrightarrow Y$ is continuous at point $x_0 \in X$ if

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \text{ such that } d_X(x, x_0) < \delta(\epsilon) \Longrightarrow d_Y(f(x), f(x_0)) < \epsilon.$$

We say f is continuous on a set $E \subset X$ if f is continuous at every point $x_0 \in E$.

We first observe that continuous mapping maps convergent sequences into convergent sequences.

Theorem 5.1.2. *(Continuity preserves convergence). Let* (X, d_X) *and* (Y, d_Y) *be a pair of metric spaces. A mapping* $f : X \longrightarrow Y$ *is continuous at point* $x_0 \in X$ *if and only if for every sequence* $(x_n)n \in \mathbb{N} \subset X$ *it holds*

$$x_n \xrightarrow{X} x_0 \Longrightarrow f(x_n) \xrightarrow{Y} f(x_0)$$

Another important characterisation of continuity involves the inverse image of open and closed sets.

Proposition 5.1.3. *Let* (X, d_X) *and* (Y, d_Y) *be a pair of metric spaces. A mapping* $f : X \longrightarrow Y$ *is continuous on the space* X *if and only if one of the following two equivalent properties hold:*

(i) For any open set $V \subset Y$ *, the preimage set*

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

is an open set in X*.*

(ii) For any closed set $F \subset Y$ *, the preimage set*

$$f^{-1}(F) = \{x \in X : f(x) \in F\}$$

is a closed set in X*.*

Remark 5.1.4. Note that the forward image of an open set under a continuous mapping might be closed. Consider, for example a function $f : \mathbb{R} \longrightarrow R$, $f(x) = 0$ where $\mathbb{R}$ is equipped with the standard metric $d(x, y) = |x-y|$. Clearly, f is a continuous function but for any open interval (a, b), the set $f((a, b)) = \{0\}$ is a closed set!

Theorem 5.1.5. *(Continuity preserved by composition). Let* $(X, d_X), (Y, d_Y)$ *and* (Z, d_Z) *be metric spaces. If a mapping* $f : X \longrightarrow Y$ *is continuous at* *point* $x_0 \in X$ *and a mapping* $g : Y \longrightarrow Z$ *is continuous at point* $f(x_0) \in Y$ *then the composition mapping* $g \circ f : X \longrightarrow Z$*, defined by*

$$(g \circ f)(x) = g(f(x))$$

is continuous at point $x_0 \in X$*.*

Example 5.1.6. Let (X, d) be a metric space and $z_0 \in X$ a fixed point in X. Then $f : X \longrightarrow \mathbb{R}$,

$$f(x) = (d(x, z_0))^2$$

is continuous function on the space X.

5.2 Continuity and compactness

While continuous functions do not preserve open and closed sets, they interact well with compact sets.

Theorem 5.2.1. *(Continuity preserves compactness). Let* $(X, d_X), (Y, d_Y)$ *be a pair of metric spaces. If a mapping* $f : X \longrightarrow Y$ *is continuous on space* X *and* $K \subset X$ *is a compact subset of* X *then the forward image* $f(K) = \{f(x) | x \in X\}$ *is a compact subset of* Y*.*

The next result is an extension of the Weierstrass Theorem of Analysis to general metric spaces.

Theorem 5.2.2. *(Maximum principle). Let* (X, d) *be a metric spaces,* $f : X \longrightarrow \mathbb{R}$ *a continuous function and* $K \subset X$ *a compact subset of* X*. Then* f *is bounded on* K*, i.e* $f(K)$ *is a bounded subset of* $\mathbb{R}$*, and* f *attains its maximum and minimum on* K*, i.e. there are points* $x_{min}, x_{max} \in K$ *such that*

$$f(x_{max}) = \max_{x \in K} f(x) \text{ and } f(x_{min}) = \min_{x \in K} f(x)$$

Remark 5.2.3. All assumptions of the theorem are essential.

5.3 Further properties of continuous mappings

Theorem 5.3.1. *(Arithmetic of continuous functions). Let (X, d) be a metric space, $f, g : X \longrightarrow \mathbb{R}$ two functions continuous at a point $x_0 \in X$ and $\lambda \in \mathbb{R}$ a real number. Then the functions*

$$f + g, \qquad \lambda f, \qquad f \cdot g$$

are continuous at x_0. In addition, if $g(x_0) \neq 0$ then $\frac{f}{g}$ is continuous at x_0.

Remark 5.3.2. In particular, the set of continuous functions $f : X \longrightarrow \mathbb{R}$ on a metric space (X, d) is a vector space.

Example 5.3.3. (Polynomials on $\mathbb{R}^N$). Denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. An N-dimensional multiindex is a vector $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}_0^N$. The absolute value of the multi-index is

$$|\alpha| = \alpha_1 + \cdots + \alpha_N.$$

Given a multi-index $\alpha \in \mathbb{N}_0^N$, we define the multi-power function $x \longmapsto x^\alpha$ on $\mathbb{R}^N$ by

$$x^\alpha := x_1^{\alpha_1} \cdot \cdots \cdot x_N^{\alpha_N}$$

Clearly, $x \longmapsto x^\alpha$ is a continuous function from $\mathbb{R}^N$ to $\mathbb{R}$. Then for any collection of multiindexes $\alpha^{(1)}, \cdots, \alpha^{(k)} \in \mathbb{N}_0^N$ and coefficients $c_1, \cdots, c_k \in \mathbb{R}$, a polynomial $p_k : \mathbb{R}^N \longrightarrow \mathbb{R}$ is an expression of the form

$$p_k(x) = c_1 x^{\alpha^{(1)}} + c_2 x^{\alpha^{(2)}} + \cdots + c_k x^{\alpha^{(k)}}.$$

By Theorem 5.3.1, p_k is a continuous function on $\mathbb{R}^N$. The degree of p_k is the maximum of the absolute values of the multi-indexes.

Theorem 5.3.4. *Let (X, d) be a metric space. A mapping $f : X \longrightarrow \mathbb{R}^m$,*

$$f(x) = (f_1(x), \cdots, f_m(x))$$

is continuous at $x_0 \in X$ if and only if each of its components $f_j : X \longrightarrow \mathbb{R}, (j = 1, \cdots, m)$, is a continuous function at $x_0 \in X$.

Example 5.3.5. (Matrices as mappings from RN into Rm). Every $m \times N$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mN} \end{pmatrix}$$

defines a mapping $A : \mathbb{R}^N \longrightarrow \mathbb{R}^m$, by the formula

$$x \longmapsto Ax.$$

where Ax is the usual multiplication of a matrix by a vector (from the right). In the next section we will show that $A : \mathbb{R}^N \longrightarrow \mathbb{R}^m$ is a continuous mapping.

Example 5.3.6. (Quadratic functions on $\mathbb{R}^N$). Every $N \times N$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NN} \end{pmatrix}$$

defines a quadratic function $Q_A : \mathbb{R}^N \longrightarrow \mathbb{R}$, by the formula.

$$Q_A(x) = x \cdot Ax$$

where $x \cdot y$ is the dot product of two vectors. Quadratic functions Q_A are also called quadratic forms. A quadratic function Q_A is a polynomial on $\mathbb{R}^N$ of degree two, and hence it is a continuous function on $\mathbb{R}^N$.

Example 5.3.7. Consider a quadratic function $Q_A : \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$$Q_A(x) = x_1^2 + 2x_2^2 + 3x_3^2 - 4x_1x_2 + 6x_2x_3$$

Find a symmetric matrix A which generated Q_A.

5.4 Linear mappings in normed spaces

Definition 5.4.1. (Linear mapping). Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be two normed spaces. We say $L : X \longrightarrow Y$ is a linear mapping if for any two vectors $x, y \in X$ and a scalar $\lambda \in \mathbb{R}$:

$L(x + y) = Lx + Ly$

$L(\lambda x) = \lambda Lx$

Instead of linear mapping we often say linear operator.

Notation 5.4.2. For linear mappings we usually do not use the brackets around a single argument, i.e. we write Lx instead of $L(x)$.

Definition 5.4.3. (Bounded mapping). Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. We say a linear mapping $L : X \longrightarrow Y$ is bounded if there exists $M \geq 0$ such that

$$\|Lx\|_Y \leq M\|x\|_X, \qquad \forall x \in X \tag{5.1}$$

We shall emphasise that the size of the constant M in Equation (5.1) may depend on the choice of the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$.

Theorem 5.4.4. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two normed spaces and* $L : X \longrightarrow Y$ *a linear mapping. Then* L *is continuous if and only if* L *is bounded.*

Example 5.4.5. (Matrices as bounded linear mappings). Every $m \times N$ matrix $A = (a_{ij}) \in \mathbb{R}^{m\times N}$ defines a linear mapping $A : \mathbb{R}^N \longrightarrow \mathbb{R}^m$, by the formula

$$x \longmapsto Ax$$

It is not difficult to see the mapping A is bounded. For example,

$$\|Ax\|_2 \leq M\|x\|_2 \qquad \forall x \in \mathbb{R}^N \tag{5.2}$$

where $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^m$ and $\mathbb{R}^N$ respectively, and where

$$M = \left(\sum_{i=1}^{m}\sum_{j=1}^{N}|a_{ij}|^2\right)^{\frac{1}{2}}$$

Note that the value of the constant M in Equation (5.1) depends on the choice of the norms in $\mathbb{R}^N$ and $\mathbb{R}^m$,

Remark 5.4.6. Note that

$$\|A\|_\infty = \max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}|$$

we define $\|A\|_1$ as the infimum of all constants $C > 0$ such that

$$\|Ax\|_1 \leq C|x|$$

Example 5.4.7. (Quadratic functions are continuous). Every $N \times N$ matrix $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ defines the quadratic function

$$QA(x) = x \cdot Ax$$

see Example 5.3.6 Using equation (5.2), it is easy to see that Q_A is continuous at $x_0 = 0$. Indeed, if $\|x_n\|_2 \longrightarrow 0$ then using the Cauchy-Schwarz inequality and the fact that the linear mapping A is bounded (Example 5.4.5), we obtain

$$|Q_A(x_n)| = |x_n \cdot Ax_n| \leq \|x_n\|_2 \|Ax_n\|_2 \leq M\|x_n\|_2^2 \longrightarrow 0 = Q_A(0)$$

This means that Q_A is continuous at $x_0 = 0$.

Example 5.4.8. (Integral as a bounded linear function on $C([a, b])$. Let $C([a, b])$ be the vector space of continuous functions $f : [a, b] \longrightarrow \mathbb{R}$ with the norm of uniform convergence

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|,$$

proof that the function $\mathcal{I} : C([a, b]) \longrightarrow \mathbb{R}$,

$$\mathcal{I}(f) := \int_a^b f(x)dx$$

is continuous.

Example 5.4.9. (Differentiation as a linear map from $C^1([a, b])$ into $C([a, b])$). Let $C^1([a, b])$ be the vector space of continuously differentiable functions $f : [a, b] \longrightarrow \mathbb{R}$. Proof that the mapping

$$\frac{d}{dx} : C^1([a, b]) \longrightarrow C([a, b]);$$

defined by $f \longmapsto \frac{d}{dx} f := f'$ is unbounded.