Chapter 4

Convergence in metric spaces

4.1 Definition of a convergent sequence

Definition 4.1.1. (Convergent sequence). Let (X, d) be a metric space. We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to the limit $x \in X$ if

$$\forall \epsilon > 0 \exists N(\epsilon) \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, n \geq N(\epsilon) \Longrightarrow d(x_n, x) < \epsilon. \tag{4.1}$$

In that case we write $x_n \longrightarrow x$ as $n \longrightarrow \infty$, or $\lim\limits_{n\longrightarrow\infty} x_n = x$

If a sequence has no limit we say that it diverges.

Remarks 4.1.2. 1. Comparing with the definition of a convergent sequence of real numbers from Real Analysis we see that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to the limit $x \in X$ if and only if the distance between x_n and x converges to zero in $\mathbb{R}$, i.e.

$$\lim_{n\longrightarrow\infty} d(x_n, x) = 0$$

in the sense of Real Analysis.

2. The limit of a convergent sequence is unique. Indeed, assume that $(x_n)_{n\in\mathbb{N}}$ converges simultaneously to $x \in X$ and to another limit $y \in$

X. Then using the triangle inequality we obtain

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since the right hand side converges to zero, we conclude that $d(x, y) = 0$ and hence $x = y$.

Definition 4.1.3. (Bounded set). Let (X, d) be a metric space and $E \subset X$ be a subset of X. We say that the set E is bounded in X if there exists a ball $B_R(x) \subset X$ such that $E \subset B_R(x)$. If the set E is not bounded then it is said to be unbounded. We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is bounded if $\{x_n\}_{n\in\mathbb{N}}$ is a bounded subset of X.

Example 4.1.4.

1. Let $X = \mathbb{R}$. The $[a, b]$ is bounded.
2. Let $X = \mathbb{R}^2$. The $(a, b) \times (a, b)$ is bounded.
3. Let $X = \mathbb{R}$. Then $\mathbb{N}$ is unbounded.
4. Let $X = \mathbb{R}^2$. Then $(a, b) \times \mathbb{R}$ is unbounded.

Theorem 4.1.5. *Let (X, d) be a metric space. If a sequence $(x_n)_{n\in\mathbb{N}}$ converges to a limit in X then $(x_n)_{n\in\mathbb{N}}$ is bounded.*

Using the notion of convergence we could give an important characterisation of closed sets in terms of convergent sequences.

Theorem 4.1.6. *Let (X, d) be a metric space. A set $F \subset X$ is closed if and only if every convergent sequence $(x_n)_{n\in\mathbb{N}} \subset F$ has its limit in F, that is*

$$(x_n) \subset F \text{ and } \lim_{n\longrightarrow\infty} x_n = x \Longrightarrow x \in F \tag{4.2}$$

Definition 4.1.7. (Closure of a set). Let (X, d) be a metric space and $E \subset X$ a subset of X. The closure of the set E is the set

$$\mathrm{cl}(E) = \mathrm{int}(E) \cup \partial E$$

It is clear that the closure of a set E is always a closed set. The closure $\mathrm{cl}(E)$ can be characterised as the smallest closed set which contains E. In particular, if E is a closed set then $\mathrm{cl}(E) = E$.

Definition 4.1.8. (Dense set). Let (X, d) be a metric space and $E \subset X$ a nonempty subset of X. The set E is dense in X if $\mathrm{cl}(E) = X$.

Example 4.1.9. Let $X = \mathbb{R}$. Then $\mathrm{cl}((0,1)) = [0,1]$ and $\mathrm{cl}(\mathbb{Q}) = \mathbb{R}$. In particular, the set of all rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example 4.1.10. Let $B_R(a)$ be an open ball in the Euclidean space $\mathbb{R}^N$. Then

$$\mathrm{cl}(B_R(a)) = \bar{B}_R(a) = \{x \in X | d(x,a) \leq r\}$$

In particular, closed ball $\bar{B}_R(a)$ is a closed set in $\mathbb{R}^N$.

4.2 Convergence of sequences in $\mathbb{R}^N$

To avoid confusion between coordinates of the vector in $\mathbb{R}^N$ and elements of the sequence of vectors in $\mathbb{R}^N$, we sometime will be using the upper-script index notation $(x^{(n)})_{n\in\mathbb{N}}$ to denote a sequence of vectors in $\mathbb{R}^N$, where

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \cdots, x_N^{(n)})$$

For each $i = 1, \cdots, N$, the sequence $(x_i^{(n)})_{n\in\mathbb{N}} \subset \mathbb{R}$ is called the sequences of i-coordinates of the sequence $(x^{(n)})$.
The next result shows that a sequence of vectors in $\mathbb{R}^N$ converges if and only if each sequence of coordinates converges individually.

Proposition 4.2.1. *(Convergence in $\mathbb{R}^N$). Let $\mathbb{R}^N$ be the N-dimensional vector space with the standard Euclidean metric $d_2(x,y)$. A sequence*

$$(x^{(n)})_{n\in\mathbb{N}} = (x_1^{(n)}, x_2^{(n)}, \cdots, x_N^{(n)})_{n\in\mathbb{N}}$$

converges in $\mathbb{R}^N$ *to the limit*

$$x = (x_1, x_2, \cdots, x_N)$$

i.e. $\lim_{n\in\mathbb{N}} x^{(n)} = x$, *if and only if*
$\lim_{n\in\mathbb{N}} x_1^{(n)} = x_1$, $\lim_{n\in\mathbb{N}} x_2^{(n)} = x_2, \cdots,$ $\lim_{n\in\mathbb{N}} x_N^{(n)} = x_N$

Remark 4.2.2. The same statement is true if instead of the Euclidean metric $d_2(x, y)$ we consider convergence in $\mathbb{R}^N$ with respect to the taxi-cab metric $d_1(x, y)$ or $\infty-$metric $d_\infty(x, y)$. In fact, one can show that the following three statements are equivalent:

(a) A sequence $(x^{(n)}) \subset \mathbb{R}^N$ converges to a vector $x \in \mathbb{R}^N$ in the metrics d_1

(b) A sequence $(x^{(n)}) \subset \mathbb{R}^N$ converges to a vector $x \in \mathbb{R}^N$ in the metrics d_2

(c) A sequence $(x^{(n)}) \subset \mathbb{R}^N$ converges to a vector $x \in \mathbb{R}^N$ in the metrics d_∞

Because of the equivalence of (a), (b) and (c) we say that the metrics d_1, d_2, d_∞ on $\mathbb{R}^N$ are equivalent, in the sense that they have the same classes of convergent sequences.

Example 4.2.3. Consider the following sequences in $\mathbb{R}^2$. Try to sketch on the plane $\mathbb{R}^2$ geometrical location of several points of the sequences in examples (1)-(5)

1. $(x^{(n)})_{n\in\mathbb{N}} = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$.

2. $(x^{(n)})_{n\in\mathbb{N}} = \left(\frac{1}{n}, \frac{1}{n^2}\right)$.

3. $(x^{(n)})_{n\in\mathbb{N}} = \left(\frac{1}{n}, \sqrt{n}\right)$.

4. $(x^{(n)})_{n\in\mathbb{N}} = \left(\cos(\frac{1}{n}), \sin(\frac{1}{n})\right)$.

5. $(x^{(n)})_{n\in\mathbb{N}} = (\sin(n\pi), \cos(n\pi))$.

4.3 Cauchy Sequences

Definition 4.3.1. (Cauchy sequence). Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called a Cauchy sequence if $\forall\epsilon > 0, \exists N(\epsilon) \in N$ such that

$$\forall n, m \in \mathbb{N}, n \geq N(\epsilon), m \geq N(\epsilon) \Longrightarrow d(x_m, x_n) < \epsilon$$

Theorem 4.3.2. *Let (X, d) be a metric space. Then every convergent sequence is also a Cauchy sequence.*

Definition 4.3.3. (Complete metric space). Let (X, d) be a metric space. We say that X is complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to the limit in X.

Example 4.3.4. The real line $\mathbb{R}$ with the standard metric $d_1(x, y) = |x-y|$ is complete.

Example 4.3.5. The set of all rational numbers $\mathbb{Q}$ with the standard metric $d_1(x, y) = |x - y|$ is not complete.

Example 4.3.6. The $N-$dimensional vector space $\mathbb{R}^N$ with any of the metrics d_1, d_2, d_∞ is complete. This follows from the completeness of the real line $\mathbb{R}$ via Proposition 4.2.1

4.4 Compact sets

Definition 4.4.1. (Compact set). Let (X, d) be a metric space and $K \subset X$ a subset of X. We say a set K is compact if every sequence $(x_n) \subset K$ contains at least one convergent subsequence (x_{n_k}) and

$$\lim_{k\longrightarrow\infty} x_{n_k} \longrightarrow x \in K.$$

Remark 4.4.2. In particular, we say that a metric space (X, d) is compact if every sequence $(x_n) \subset X$ contain at least one convergent subsequence.

Example 4.4.3. Every bounded closed interval $[a, b] \subset \mathbb{R}$ is compact. This is the Bolzano-Weierstrass Theorem. However, the real line $\mathbb{R}$ with the standard metric is not compact. For instance, the sequence $x_n = n$ does not contain any convergent subsequence.

Theorem 4.4.4. *Let (X, d) be a metric space. If a nonempty subset $K \subset X$ is compact then K is bounded and closed.*

Theorem 4.4.5. *(Heine-Borel). Let $\mathbb{R}^N$ be the Euclidean space. A subset $K \subset \mathbb{R}^N$ is compact if and only if K is bounded and closed.*

Corollary 4.4.6. *Let $\mathbb{R}^N$ be the Euclidean space. Then any bounded sequence $(x_n) \subset \mathbb{R}^N$ has a convergent subsequence.*

Remark 4.4.7. The same statement is true if instead of the Euclidean metric $d_2(x, y)$ we consider the taxi-cab metric $d_1(x, y)$ or $\infty-$metric $d_\infty(x, y)$. This is a consequence of the fact that all three metrics on $\mathbb{R}^N$ are equivalent, see Remark 4.2.2

Proposition 4.4.8. *Let (X, d) be a metric space and $K \subset X$ be a compact subset from X. If $M \subset K$ is closed then M is compact.6*

Proposition 4.4.9. *(from H.W-oct-1-sol. page 2)*
In $\mathbb{R}^N$ the intersection of arbitrary many compact set is compact

Proposition 4.4.10. *Let (X, d) be a metric space.*

1. *If $K_1, K_2, \cdots, K_n \subset X$ is a finite collection of compact sets, then the union $\bigcup_{i=1}^{n} K_i$ is also compact.*
2. *If $X \supset K_1 \supset K_2 \supset \cdots \supset K_n$ is a nested sequence of nonempty compact sets then the intersection $\bigcap_{i\in\mathbb{N}} K_i$ is nonempty.*