Chapter 3

Topology of metric spaces

3.1 Open and closed ball

Definition 3.1.1. (Open and closed ball). Let (X, d) be a metric space. We define the open ball of radius $r > 0$ centered at point $a \in X$ to be the set

$$B_r(a) := \{x \in X | d(x, a) < r\}$$

The closed ball of radius $r > 0$ centered at point $a \in X$ is the set

$$\bar{B}_r(a) := \{x \in X | d(x, a) \leq r\}$$

Example 3.1.2. Let $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$. Then the open ball coincides with an open interval in $\mathbb{R}$,

$$B_r(a) = \{x \in \mathbb{R} ||x - a| < r\} = (a - r, a + r).$$

The closed ball is a closed interval in $\mathbb{R}$,

$$\bar{B}_r(a) = \{x \in \mathbb{R} ||x - a| \leq r\} = [a - r, a + r].$$

Example 3.1.3. Let $X = \mathbb{R}^2$ and $a = 0$. Then:

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1. open ball in the Euclidean metric d_2 is the open disc

$$B_r^{d_2}(0) = \{x \in \mathbb{R}^2 | d_2(x, 0) < r\} = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 < r^2\}.$$

2. open ball in the taxi cab metric d_1 is an open diamond

$$B_r^{d_1}(0) = \{x \in \mathbb{R}^2 | d_1(x, 0) < r\} = \{x \in \mathbb{R}^2 || x_1| + |x_2| < r\}.$$

3. open ball in the ∞-metric d_∞ is an open square

$$B_r^{d_\infty}(0) = \{x \in \mathbb{R}^2 | d_\infty(x, 0) < r\} = \{x \in \mathbb{R}^2 | \max\{|x_1|, |x_2| < r\}.$$

Definition 3.1.4. (Interior, exterior, boundary). Let (X, d) be a metric space and $E \subset X$ a subset of X. We say $x \in E$ is an interior point of the set E if

$$\exists \epsilon > 0 : B_\epsilon(x) \subset E, \qquad i.e. \quad B_\epsilon(x) \cap E^c = \emptyset.$$

We say $x \in X$ is an exterior point of the set E if

$$\exists \epsilon > 0 : B_\epsilon(x) \cap E = \emptyset, \qquad i.e. \quad B_\epsilon(x) \subset E^c.$$

We say $x \in X$ is a boundary point of the set E if x is neither an interior point nor an exterior point of E.

Example 3.1.5. Prove that x is a boundary point of the set $E \subset X$ if and only if

$$\forall \epsilon > 0 : B_\epsilon(x) \cap E \neq \emptyset \qquad \text{and} \qquad B_\epsilon(x) \cap E^c \neq \emptyset$$

where $E^c = X \backslash E$ denotes the complement of E in X.

The set of all interior points of E is called the interior of E and is denoted $\operatorname{int}(E)$
The set of all exterior points of E is called the exterior of E and is denoted

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ext(E)
The set of6 all boundary points of E is called the boundary of E and is denoted ∂E

Example Prove that $\text{ext}(E) = \text{int}(E^c)$, $\text{int}(E) = \text{ext}(E^c)$ and $\partial E = \partial(E^c)$.

Example 3.1.6. Let $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ and $E = \mathbb{Q}$, the set of all rational numbers. Since $B_\epsilon(x) = (x - \epsilon, x + \epsilon)$ and every interval in $\mathbb{R}$ contains both rational and irrational numbers, we conclude that $\text{int}(\mathbb{Q}) = \emptyset$, $\quad \partial\mathbb{Q} = \mathbb{R} \quad$ and $\text{ext}(\mathbb{Q}) = \emptyset$.

Remark 3.1.7. Every point $x \in X$ is either interior, or exterior, or boundary point of E. In other words,

$$X = \text{int}(E) \cup \partial E \cup \text{ext}(E)$$

Observe that

$$\text{int}(E) \subset E \quad \text{and} \quad \text{ext}(E) \subset E^c$$

However, if $x \in \partial E$ then x could be an element of E, but it also could be an element of the complement E^c.

Example 3.1.8. Let $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$. Consider half open interval $[1, 2)$. Then $\text{int}(E) = (1{,}2)$, $\text{ext}(E) = (-\infty, 1) \cup (2, +\infty)$ and $\partial E = \{1, 2\}$.

Definition 3.1.9. (Open and closed set). Let (X, d) be a metric space and $E \subset X$ a subset of X. We say E is open if it contains none of its boundary points, i.e.

$$\partial E \cap E = \emptyset.$$

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We say E is closed if it contains all of its boundary points, i.e.

$$\partial E \subset E$$

If E contains some of its boundary points but no others then E is neither open nor closed.

Remark 3.1.10. A set E is open if and only if it contains only interior points, i.e.

$$E = \text{int}(E) \tag{3.1}$$

In other words, E is open if and only if for every $x \in E$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset E$. A set F is closed if and only if it contains all interior and boundary points, i.e.

$$F = \text{int}(F) \cup \partial F$$

Note that each of the sets $\text{int}(F)$ and ∂F might be empty.

Example 3.1.11. Let $X = \mathbb{R}$ with the standard metric $d(x - y) = |x - y|$.

(1) The set $(1, 2)$ does not contain its boundary points 1, 2 and hence it's open.

(2) The set $[1, 2]$ contains both of its boundary points and hence it's closed.

(3) The set $[1, 2)$ contains one of its boundary points 1 but does not contain the other boundary point 2, so it's neither open nor closed.

Example 3.1.12. Let $X = \mathbb{R}^2$ with the standard Euclidean metric $d_2(x, y)$.

(1) The set $(0, 1) \times (0, 1)$ does not contain any boundary points and hence it's open. $(\partial E \nsubseteq E)$

(2) The set $[0, 1] \times [0, 1]$ contains all of its boundary points and hence it's closed. $(\partial F \subset F)$

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(3) The set $G = (0,1) \times [0,1]$ contains some but not all of its boundary points and hence it's neither open nor closed. $(\partial G \cap G \neq \emptyset,\ \partial G \cap G^c \neq \emptyset)$

Example 3.1.13. Let (X, d) be a metric space. Every open ball $B_r(a)$ is an open set in X.

Example 3.1.14. Let (X, d) be a metric space and $a \in X$ a point in X. Then the singleton set $\{a\}$ is closed.

Example 3.1.15. Let (X, d) be a discrete metric space. Prove that any subset from X is open and closed.

Proposition 3.1.16. *Let (X, d) be a metric space*

1. *$\emptyset$ and X are open sets in (X, d).*
2. *The union of any finite, countable, uncountable family of open sets is open.*
3. *The intersection of any finite family of open sets is open.*

Remark 3.1.17. By taking complements we conclude that:

1. X and $\emptyset$ are closed (so X and $\emptyset$ are both open and closed at the same time).
2. The finite union of closed sets is closed.
3. The arbitrary intersection of closed sets is closed.

Example 3.1.18. The intersection of an infinite collection of open sets may be not be open.

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