Chapter 1

Metric space

1.1 Definition and Examples

Definition 1.1.1. (Metric space). Let X be a nonempty set. A metric (or a distance) d on X is a function

$$d \colon X \times X \longrightarrow R$$

which satisfies the following properties:

(M1) $d(x, y) = 0$ if and only if $x = y$,

(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$, (Symmetry)

(M3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$, (Triangle inequality)

The pair $(X d)$ is called a metric space.

Example 1.1.2. (Positivity of the metric). Prove that any metric $d \colon X \times X \longrightarrow \mathbb{R}$ satisfy the following property:

$$(M1') \qquad d(x, y) \geq 0 \quad \forall x, y \in X. \qquad \text{(Positivity)}$$

Example 1.1.3. (The Real line). Let $\mathbb{R}$ be the set of all real numbers.

Define a metric $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$d(x, y) = |x - y| \tag{1.1}$$

Then $(\mathbb{R}, d)$ is a metric space. We refer to this metric as the standard metric on $\mathbb{R}$.

Example 1.1.4. Prove that the set of all positive rational numbers $\mathbb{Q}_+$ with the metric $d(x, y) = |log(\frac{x}{y})|$ is a metric space.

Example

$d(x, y) = |log(\frac{x}{y})|$ is not metric on $\mathbb{Q}$.

Example

$d(x, y) = |log(\frac{|x|}{|y|})|$ is not metric on $\mathbb{Q}$.

Example 1.1.5. (Discreet metric). Let X be an arbitrary set. Define discreet metric $d : X \times X \longrightarrow \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Prove that (X, d) is a metric space.

Example 1.1.6. (Euclidean space $\mathbb{R}^N$). Let $N \in \mathbb{N}$ be a natural number and let $\mathbb{R}^N$ be the space of N-vectors of real numbers:

$$\mathbb{R}^N = \{f(x_1, x_2, \cdots, x_N) | x_1, \cdots, x_N \in \mathbb{R}\}$$

When we write $x \in \mathbb{R}^N$ this means x is an N-vector, that is $x = (x_1, x_2, \cdots, x_N)$. We define the Euclidean metric $d_2 : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$d_2(x, y) = \sqrt{|x_1 - y_1|^2 + \cdots + |x_N - y_N|^2} \tag{1.2}$$

Then $(\mathbb{R}^N, d_2)$ is a metric space, which we call Eucleadean space of dimension N.

Example 1.1.7. (Taxi-cab metric on $\mathbb{R}^N$). Let $N \in \mathbb{N}$ be a natural number and let $\mathbb{R}^N$ be the space of N-vectors as before. We define the taxi-cab metric $d_1 : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$d_1(x, y) = |x_1 - y_1| + \cdots + |x_N - y_N| \tag{1.3}$$

Then $(\mathbb{R}^N, d_1)$ is a metric space.

Example 1.1.8. (∞-metric on $\mathbb{R}^N$). Again let $N \in \mathbb{N}$ be a natural number and let $\mathbb{R}^N$ be as before. We define the sup-norm metric $d_\infty : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$d_\infty(x, y) = \max\{|x_1 - y_1|, \cdots, |x_N - y_N|\} \tag{1.4}$$

Then $(\mathbb{R}^N, d_\infty)$ is a metric space.

Remark 1.1.9. The d_1, d_2 and d_∞ metrics on $\mathbb{R}^N$ are special cases of the more general d_p-metric on $\mathbb{R}^N$,

$$d_p(x, y) = (|x_1 - y_1|^p + \cdots + |x_N - y_N|^p)^{\frac{1}{p}} \tag{1.5}$$

where $p \in [1, \infty)$. Note that $d_\infty < d_2 < d_1$.

Example 1.1.10. (Metric of uniform convergence on $C([a, b])$). Let $C([a, b])$ denote the set of continuous functions $f : [a, b] \longrightarrow \mathbb{R}$,

$$C([a, b]) = \{f : [a, b] \longrightarrow \mathbb{R} | f \text{ is continuous on } [a, b]\}. \tag{1.6}$$

Then

$$d_\infty(f, g) = \max_{x \in [a,b]} |f(x) - g(x)| \tag{1.7}$$

is a metric on $C([a, b])$. This metric is known as metric of uniform convergence, or ∞-metric on $C([a, b])$.

Example 1.1.11. Let $f(x) = x^2$ and $g(x) = x^3$. Find the distances $d_\infty(f, g)$ in $C([0, 1])$, and in $C([-1, 1])$.