Laplace Transformation

A very powerful technique for solving

quantities vary with time, such as the current in an electrical circuit,

the oscillations of a vibrating membrane,

These equations are generally coupled with

initial conditions that describe the state of the system at time $t = 0$.

The **Laplace transform** converts linear ordinary differential equations into algebraic equations, making them easy to solve .

4-1 Laplace Transformation Definition

Suppose that ***f*** *is a real or complex-valued function of the (time) variable* ***t > 0*** *and* ***s*** *is a* real or complex parameter. We define the Laplace transform of ***f*** *as*

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

Common notations used for the Laplace transform

(i) $\mathcal{L}\{f(t)\}$ or $L\{f(t)\}$

(ii) $\mathcal{L}(f)$ or Lf

4.2 Basic Properties of The Laplace Transformation

1- Linearity

$$\mathcal{L}\{c_1 f_1(t) \pm c_2 f_2(t)\} = \int_0^\infty [c_1 f_1(t) \pm c_2 f_2(t)] e^{-st}\,dt$$

$$= c_1 \int_0^\infty f_1(t) e^{-st}\, dt \pm c_2 \int_0^\infty f_2(t) e^{-st}\, dt$$

$$= c_1 \mathcal{L}\{f_1(t)\} \pm c_2 \mathcal{L}\{f_2(t)\}$$

$$\mathcal{L}\{c_1 f_1(t) \pm c_2 f_2(t) \pm \ldots\ldots \pm c_n f_n(t)\} = c_1 \mathcal{L}\{f_1(t)\} \pm c_2 \mathcal{L}\{f_2(t)\} \pm \ldots\ldots \pm c_n \mathcal{L}\{f_n(t)\}$$

2- The Laplace Transforms of Derivatives

(a) First derivative

Let the first derivative of ***f (t)*** be ***f ′(t)*** then, from the definition of the Laplace transform

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt \qquad \text{integrating by parts}$$

$$\text{let } u = e^{-st} \text{ and } \frac{dv}{dt} = f'(t)$$

$$\frac{du}{dt} = -se^{-st} \text{ and } v = \int f'(t)\, dt = f(t)$$

$$\int_0^\infty e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)(-se^{-st})\,dt$$

$$= [0 - f(0)] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

$$\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)}$$

(b) Second derivative

Let the second derivative of *f (t) be f''(t), then*

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt \quad \text{integrating by parts}$$

let $u = e^{-st}$ and $\frac{dv}{dt} = f''(t)$

$\frac{du}{dt} = -se^{-st}$ and $v = \int f''(t)\,dt = f'(t)$

$$\int_0^\infty e^{-st} f''(t)\,dt = \left[e^{-st} f'(t)\right]_0^\infty + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= [0 - f'(0)] + s\mathcal{L}\{f'(t)\}$$

$$= -f'(0) + s[s(f(t)) - f(0)]$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

if ***f(t)*** *and its derivative* $f'(t), f''(t), \ldots\ldots f^n(t)$ *are continuous function and have Laplace* transformation for all values of *t ≥ 0 then* in general form *:*

$$\mathcal{L}\left(f^n(t)\right) = s^n\mathcal{L}\left(f(t)\right) - \sum_{j=0}^{n-1} s^{n-1-j} f^{(j)}(0)$$

For example if n=3 (third derivative) :

$$\mathcal{L}\left(f'''(t)\right) = s^3\mathcal{L}\left(f(t)\right) - \sum_{j=0}^{3-1} s^{3-1-j} f^{(j)}(0)$$

$$\mathcal{L}(f'''(t)) = s^3\mathcal{L}(f(t)) - \sum_{j=0}^{2} s^{2-j} f^{(j)}(0)$$

$$= s^3\mathcal{L}(f(t)) - s^2 f(0) - sf'(0) - f''(0)$$

3-Laplace Integration

$$\mathcal{L}\left(\int_a^t f(t)dt\right) = \int_0^\infty e^{-st}\{\int_a^t f(t)dt\}dt$$ integrating by parts

$$u = \int_a^t f(t)dt \Rightarrow du = f(t)dt \; , \; dv = e^{-st}dt \Rightarrow v = -\frac{1}{s}e^{-st}$$

$$\int_0^\infty e^{-st}\{\int_a^t f(t)dt\}dt = \left[-\frac{1}{s}e^{-st}\int_a^t f(t)dt\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)dt$$

$$= \frac{1}{s}\int_a^0 f(t)dt + \frac{1}{s}\mathcal{L}(f(t))$$

$$\boxed{\mathcal{L}\left[\int_a^t f(t)dt\right] = \frac{1}{s}\mathcal{L}(f(t)) + \frac{1}{s}\int_a^0 f(t)dt}$$

4-3 The Laplace Transformation of Elementary Functions

1- $f(t) = k$ where k is constant

$$F(s) = \mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} k\,dt = -k\left.\frac{e^{-st}}{s}\right|_0^{\infty} = \frac{k}{s}$$

$$\mathcal{L}(k) = \frac{k}{s} \qquad \text{for } s > 0$$

2- $f(t) = t^n$

$$F(s) = \mathcal{L}(t^n) = \begin{cases} \dfrac{\Gamma(n+1)}{s^{n+1}} & n > -1 \\ \dfrac{n!}{s^{n+1}} & n \text{ a positive integer} \end{cases}$$

$\Gamma(x)$ gamma function or generalized factorial function

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \ldots\ldots..(a)$$ integrating by parts

$$u = e^{-t} \qquad du = -e^{-t}dt$$

$$dv = t^{x-1}dt \qquad v = \frac{t^x}{x}$$

$$\Gamma(x) = \left.\frac{t^x e^{-t}}{x}\right|_0^{\infty} + \frac{1}{x}\int_0^{\infty} e^{-t} t^x dt$$

the integral portion vanishes at both limits. By comparison with equation **(a)**, it is clear that the integral which remains is simply **Γ(x+1)**.

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \qquad x > 0$$

or $\quad x\,\Gamma(x) = \Gamma(x+1)$

Moreover, we have specifically

$$\Gamma(1) = \int_0^{\infty} e^{-t}dt = -\left.e^{-t}\right|_0^{\infty} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(b)}$$

therefore , using equation (b),

$$\Gamma(2) = 1.\Gamma(1) = 1$$

$$\Gamma(3) = 2.\Gamma(2) = 2 \times 1 = 2!$$

$$\Gamma(4) = 3.\Gamma(3) = 3 \times 2 \times 1 = 3!$$

and in general

$$\Gamma(n+1) = n! \qquad n = 1,2,3,....$$

then by using this result we can find Laplace transformation of $f(t) = t^n$

$$\mathcal{L}(t^n) = \int_0^\infty e^{-st} t^n dt$$

let $x = st$, $dx = sdt$, $dt = \frac{dx}{}$

$$\mathcal{L}(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x} dx$$

$$\mathcal{L}\left(t^n\right)=\frac{\Gamma(n+1)}{s^{n+1}}=\frac{n!}{s^{n+1}} \qquad s>0$$

3 – $f(t)=e^{-at}$

$$\mathcal{L}\left(e^{-at}\right)=\int_0^{\infty} e^{-at}e^{-st}dt=\left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_0^{\infty}$$

$$\mathcal{L}\left(e^{-at}\right)=\frac{1}{s+a} \qquad (s+a)>0$$

4 – $f(t)=\sin at$

$$\mathcal{L}(\sin at)=\int_0^{\infty} e^{-st}\sin at\,dt$$

$$u=e^{-st}\Rightarrow du=-se^{-st}dt \quad ,$$

$$dv=\sin at\,dt\Rightarrow v=-\frac{1}{a}\cos at$$

$$\int_0^\infty e^{-st}\sin at\,dt = -\frac{1}{a}e^{-st}\cos at\Big|_0^\infty - \frac{s}{a}\int_0^\infty e^{-st}\cos at\,dt$$

$$u = e^{-st} \Rightarrow du = -se^{-st}dt \quad , \quad dv = \cos at\,dt \Rightarrow v = \frac{1}{a}\sin at$$

$$\int_0^\infty e^{-st}\sin at\,dt = \frac{1}{a} - \frac{s}{a}\left[\left[\frac{1}{a}e^{-st}\sin at\right]_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}\sin at\,dt\right]$$

$$\int_0^\infty e^{-st}\sin at\,dt + \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at\,dt = \frac{1}{a} \Rightarrow$$

$$\frac{s^2+a^2}{a^2}\int_0^\infty e^{-st}\sin at\,dt = \frac{1}{a}$$

$$\int_0^\infty e^{-st}\sin at\,dt = \frac{a}{s^2+a^2} \quad \text{or} \quad \mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$$

in same way we can show that

$$\mathcal{L}(\cos at) = \frac{s}{s^2 + a^2} \qquad \mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}$$

$$\mathcal{L}(\cosh at) = \frac{s}{s^2 - a^2}$$

$$\sinh at = \frac{1}{2}(e^{at} - e^{-at}) \qquad \cosh at = \frac{1}{2}(e^{at} + e^{-at})$$

$$\sin 2\theta = 2 \sin\theta \cos\theta,$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin A \sin B = \frac{1}{2}\cos(A - B) - \frac{1}{2}\cos(A + B)$$

$$\cos A \cos B = \frac{1}{2}\cos(A - B) + \frac{1}{2}\cos(A + B)$$

$$\sin A \cos B = \frac{1}{2}\sin(A - B) + \frac{1}{2}\sin(A + B)$$

Table (1) Elementary Standard Laplace transforms

Function $f(t)$	Laplace transforms $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
(i) 1	$\frac{1}{s}$
(ii) k	$\frac{k}{s}$
(iii) e^{at}	$\frac{1}{s-a}$
(iv) $\sin at$	$\frac{a}{s^2+a^2}$
(v) $\cos at$	$\frac{s}{s^2+a^2}$
(vi) t	$\frac{1}{s^2}$
(vii) t^2	$\frac{2!}{s^3}$
(viii) $t^n \ (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
(ix) $\cosh at$	$\frac{s}{s^2-a^2}$
(x) $\sinh at$	$\frac{a}{s^2-a^2}$

4-4 The Laplace Transform of $e^{at} f(t)$

$$\mathcal{L}\left(e^{-at} f(t)\right) = \mathcal{L}\left(f(t)\right)_{s \to s+a}$$

$$\mathcal{L}\left(e^{-at} f(t)\right) = \int_0^\infty f(t) e^{-(s+a)t} dt$$ By definition

then by using this definition we can show that

a - $\mathcal{L}\left(e^{-at} \cos bt\right) = \dfrac{(s+a)}{(s+a)^2 + b^2}$

b - $\mathcal{L}\left(e^{-at} \sin bt\right) = \dfrac{b}{(s+a)^2 + b^2}$

c - $\mathcal{L}\left(e^{-at} t^n\right) = \dfrac{n!}{(s+a)^{n+1}}$ n a positive integer

4-5 The Laplace Transform of $t^n f(t)$

$$\mathcal{L}\left(t^n f(t)\right) = (-1)^n \frac{d^n f(s)}{ds^n} \qquad ; \mathcal{L} f(t) = f(s)$$

$$\mathcal{L}\left(f(t)\right) = \int_0^\infty f(t) e^{-st} dt$$

$$\frac{d}{ds}\int_0^\infty f(t) e^{-st} dt = -\int_0^\infty t f(t) e^{-st} dt = f'(s) \quad ; f'(s) = \frac{df(s)}{ds}$$

or $\displaystyle \int_0^\infty [t f(t)] e^{-st} dt = \mathcal{L}\left(t f(t)\right) = -f'(s)$

$$\boxed{\mathcal{L}\left(t^n f(t)\right) = (-1)^n \frac{d^n f(s)}{ds^n}}$$

EX What is the Laplace transformation of

$(a)\ e^{4t}t^7 \quad (b)\ 5\sin 2t - 3\cos 2t \quad (c)\ t^2e^{3t} + e^{4t}\cosh 5t + e^{-2t}(3\cos 6t - 5\sin 6t)$

$(d)\sin t\cos t \quad (e)\ \frac{1}{2}\sin\left(\sqrt{2}t+\frac{\pi}{4}\right)-\frac{1}{2}\cos\left(\sqrt{2}t+\frac{\pi}{4}\right)+8\sin^2 t\times\cos^2 t \quad (f)\ t^2 \sin t$

solution

$$(a)\ \mathcal{L}\left(e^{4t}t^7\right)=\frac{7!}{(s-4)^8}$$

$$(b)\ \mathcal{L}(5\sin 2t - 3\cos 2t) = 5\mathcal{L}(\sin 2t) - 3\mathcal{L}(\cos 2t)$$

$$=5\left(\frac{2}{s^2+4}\right)-3\left(\frac{s}{s^2+4}\right)$$

$$=\frac{10}{s^2+4}-\frac{3s}{s^2+4}=\frac{10-3s}{s^2+4}$$

$$(c)\ \mathcal{L}\left(t^2e^{3t}+e^{4t}\cosh 5t+e^{-2t}(3\cos 6t-5\sin 6t)\right)$$

$$=\mathcal{L}\left(t^2e^{3t}\right)+\mathcal{L}\left(e^{4t}\cosh 5t\right)+3\mathcal{L}\left(e^{-2t}\cos 6t\right)-5\mathcal{L}\left(e^{-2t}\sin 6t\right)$$

$$\frac{2!}{(s-3)^3}+\frac{(s-4)}{(s-4)^2-25}+\frac{3(s+2)}{(s+2)^2+36}-\frac{5\times 6}{(s+2)^2+36}=$$

$$\frac{2\left[(s-4)^2-25\right]\left[(s+2)^2+36\right]+(s-4)(s-3)^3\left[(s+2)^2+36\right]+(3s-24)(s-3)^3\left[(s-4)^2-25\right]}{(s-3)^3\left[(s-4)^2-25\right]\left[(s+2)^2+36\right]}$$

$$(d)\mathcal{L}(\sin t\cos t)=\frac{1}{2}\mathcal{L}\sin 2t=\frac{1}{2}\times\frac{2}{s^2+4}=\frac{1}{s^2+4}$$

$$(e)\ \ f(t)=\frac{1}{2}\sin\left(\sqrt{2}t+\frac{\pi}{4}\right)-\frac{1}{2}\cos\left(\sqrt{2}t+\frac{\pi}{4}\right)+8\sin^2 t\times\cos^2 t=$$

$$\frac{1}{2}\left(\sin\sqrt{2}t\cos\frac{\pi}{4}+\cos\sqrt{2}t\sin\frac{\pi}{4}\right)-\frac{1}{2}\left(\cos\sqrt{2}t\cos\frac{\pi}{4}-\sin\sqrt{2}t\sin\frac{\pi}{4}\right)$$

$$+8\left[\frac{1}{2}(1-\cos 2t)\times\frac{1}{2}(1+\cos 2t)\right]=$$

$$\frac{1}{2}\times\frac{1}{\sqrt{2}}\left[\sin\sqrt{2}t+\cos\sqrt{2}t\right]-\frac{1}{2}\times\frac{1}{\sqrt{2}}\left[\cos\sqrt{2}t-\sin\sqrt{2}t\right]+2\left[1-(\cos 2t)^2\right]$$

$$=\frac{1}{\sqrt{2}}\sin\sqrt{2}t+2\left[1-\left(\frac{1}{2}+\frac{1}{2}\cos 4t\right)\right]$$

$$\Rightarrow f(t) = \frac{1}{\sqrt{2}} \sin\sqrt{2}t + 1 + \cos 4t$$

$$\mathcal{L}\left(\frac{1}{\sqrt{2}} \sin\sqrt{2}t + 1 + \cos 4t\right) = \frac{1}{\sqrt{2}} \mathcal{L}\left(\sin\sqrt{2}t\right) + \mathcal{L}(1) + \mathcal{L}(\cos 4t)$$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{s^2 + 2} + \frac{1}{s} + \frac{s}{s^2 + 16}$$

$$= \frac{s\left(s^2 + 16\right) + \left(s^2 + 2\right)\left(s^2 + 16\right) + s^2\left(s^2 + 2\right)}{s\left(s^2 + 2\right)\left(s^2 + 16\right)}$$

(f) $t^2 sint$

$$\mathcal{L}\left(t^2 \sin t\right) = (-1)^2 \frac{d^2 f(s)}{ds^2} = \frac{d^2 f(s)}{ds^2}$$

$$\mathcal{L} \sin t = \frac{1}{s^2 + 1} = f(s)$$

$$\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=\frac{-2s}{\left(s^2+1\right)^2}$$

$$\frac{d}{ds}\left(\frac{-2s}{\left(s^2+1\right)^2}\right)=-2\left[s(-2)\left(s^2+1\right)^{-3}(2s)+(1)\left(s^2+1\right)^{-2}\right]$$

$$=\frac{6s^2-2}{\left(s^2+1\right)^3}=\mathcal{L}\left(t^2\sin t\right)$$

4-6 Inverse Laplace transforms

If the Laplace transform of a function $f(t)$ is $f(s)$, i.e.

$\mathcal{L}\{f(t)\}=f(s)$, then $f(t)$ is called

the **inverse Laplace transform** of $f(s)$ and is written as

$f(t)=\mathcal{L}^{-1}\{f(s)\}$

<u>EX</u> Find the following inverse Laplace transforms:

(a) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$ (b) $\mathcal{L}^{-1}\left\{\frac{5}{3s-1}\right\}$

(a) $\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\sin at$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \frac{1}{3}\sin 3t$$

(b) $$\mathcal{L}^{-1}\left\{\frac{5}{3s-1}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{3\left(s-\frac{1}{3}\right)}\right\}$$

$$= \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{\left(s-\frac{1}{3}\right)}\right\} = \frac{5}{3}e^{\frac{1}{3}t}$$

EX Find Inverse Laplace transform of

(a) $\mathcal{L}^{-1}\left[\frac{4}{(s-2)} - \frac{3s}{(s^2+16)} + \frac{5}{(s^2+4)}\right]$ (b) $\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right)$

(a) $\mathcal{L}^{-1}\left[\frac{4}{(s-2)}-\frac{3s}{(s^2+16)}+\frac{5}{(s^2+4)}\right]=4e^{2t}-3\cos 4t+\frac{5}{2}\sin 2t$

(b) $\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right)$

$$\mathcal{L}^{-1}\left[\frac{d^n f(s)}{ds^n}\right]=(-1)^n t^n f(t)$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)}\right)=\text{sint}\Rightarrow\frac{\text{d}}{\text{ds}}\left(\frac{1}{(s^2+1)}\right)=\frac{-2s}{(s^2+1)^2}$$

then

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right)=-\frac{1}{2}t\sin t$$

EX Find Inverse Laplace transform of

(a) $\mathcal{L}^{-1}\left\{\frac{3}{s^2-4s+13}\right\}$ (b) $\mathcal{L}^{-1}\left\{\frac{2(s+1)}{s^2+2s+10}\right\}$

(a) $$\mathcal{L}^{-1}\left\{\frac{3}{s^2-4s+13}\right\} = \mathcal{L}^{-1}\left(\frac{3}{s^2-4s+4-4+13}\right)$$

$$= \mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+3^2}\right\}$$

$$= \mathbf{e^{2t}\sin 3t}$$

(b) $$\mathcal{L}^{-1}\left\{\frac{2(s+1)}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left(\frac{2(s+1)}{s^2+2s+1-1+10}\right)$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+1)}{(s+1)^2+3^2}\right\}$$

$$= \mathbf{2e^{-t}\cos 3t}$$

Inverse Laplace Transforms Using Partial Fractions

By using partial fractions, to resolve the function into simpler fractions which may be inverted on sight. For example,

$$F(s) = \frac{2s-3}{s(s-3)} \equiv \frac{1}{s} + \frac{1}{s-3}$$

$$: \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s-3}\right\} = 1 + e^{3t}$$

Table Summary of Partial Fraction

Type	Denominator containing	Expression	Form of partial fraction
1	Linear factors	$\frac{f(x)}{(x+a)(x-b)(x+c)}$	$\frac{A}{(x+a)}+\frac{B}{(x-b)}+\frac{C}{(x+c)}$
2	Repeated linear factors	$\frac{f(x)}{(x+a)^3}$	$\frac{A}{(x+a)}+\frac{B}{(x+a)^2}+\frac{C}{(x+a)^3}$
3	Quadratic factors	$\frac{f(x)}{(ax^2+bx+c)(x+d)}$	$\frac{Ax+B}{(ax^2+bx+c)}+\frac{C}{(x+d)}$

In general for repeated factors

$$\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}$$

$$\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\cdots+\frac{B_nx+C_n}{(x^2+px+q)^n}$$

EX Find Inverse Laplace transform of

(a) $\dfrac{4s-5}{s^2-s-2}$ (b) $\dfrac{3s^3+s^2+12s+2}{(s-3)(s+1)^3}$ (c) $\dfrac{5s^2+8s-1}{(s+3)(s^2+1)}$

(d) $\dfrac{7s+13}{s(s^2+4s+13)}$

Solution

(a) $$\frac{4s-5}{s^2-s-2} \equiv \frac{4s-5}{(s-2)(s+1)} \equiv \frac{A}{(s-2)} + \frac{B}{(s+1)}$$

$$\equiv \frac{A(s+1)+B(s-2)}{(s-2)(s+1)}$$

$$4s-5 \equiv A(s+1)+B(s-2)$$

The Heaviside "Cover-up" Method for Linear Factors

When $s=2,\ 3=3A,$ $\boxed{A=1}$

When $s = -1$, $-9 = -3B$, $B = 3$.

$$\mathcal{L}^{-1}\left\{\frac{4s-5}{s^2-s-2}\right\} \equiv \mathcal{L}^{-1}\left\{\frac{1}{s-2}+\frac{3}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\}$$

$$= e^{2t} + 3e^{-t}$$

(b)

$$\frac{3s^3+s^2+12s+2}{(s-3)(s+1)^3} \equiv \frac{A}{s-3}+\frac{B}{s+1}+\frac{C}{(s+1)^2}+\frac{D}{(s+1)^3}$$

$$3s^3 + s^2 + 12s + 2 \equiv A(s+1)^3 + B(s-3)(s+1)^2 + C(s-3)(s+1) + D(s-3)$$

The Heaviside "Cover-up" Method for Linear Factors

When $s = 3$, $128 = 64A$, $A = 2.$

When $s = -1$, $-12 = -4D$, $D = 3.$

Equating s^3 terms gives: $3 = A + B$, $B = 1.$

Equating constant terms gives:

$$2 = A - 3B - 3C - 3D$$

$$2 = 2 - 3 - 3C - 9,\ 3C = -12 \quad C = -4$$

$$\mathcal{L}^{-1}\left\{\frac{3s^3 + s^2 + 12s + 2}{(s-3)(s+1)^3}\right\} \equiv \mathcal{L}^{-1}\left\{\frac{2}{s-3} + \frac{1}{s+1} - \frac{4}{(s+1)^2} + \frac{3}{(s+1)^3}\right\}$$

$$= 2e^{3t} + e^{-t} - 4e^{-t}t + \frac{3}{2}e^{-t}t^2$$

(c)

$$\frac{5s^2+8s-1}{(s+3)(s^2+1)} \equiv \frac{A}{s+3} + \frac{Bs+C}{(s^2+1)}$$

$$5s^2+8s-1 \equiv A(s^2+1)+(Bs+C)(s+3).$$

$$=A\,s^2+A+B\,s^2+3Bs+Cs+3C$$

$5 = A + B,$ $\quad s^2$

$8 = 3B + C,$ $\quad s$

-1=A +3C

By solving these equations

$A = 2.$ $\quad B = 3,$ $\quad C = -1.$

$$\mathcal{L}^{-1}\left\{\frac{2}{s+3}+\frac{3s-1}{s^2+1}\right\} \equiv \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\}+\mathcal{L}^{-1}\left\{\frac{3s}{s^2+1}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 2e^{-3t} + 3\cos t - \sin t.$$

(d)

$$\frac{7s+13}{s(s^2+4s+13)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+4s+13}$$

$$7s+13 \equiv A(s^2+4s+13)+(Bs+C)(s)$$

$0 = A + B,$ s^2

$7 = 4A + C,$ s

13=13A , $\boxed{A=1}$ $\boxed{B=-1.}$ $\boxed{C=3.}$

$$\mathcal{L}^{-1}\left\{\frac{7s+13}{s(s^2+4s+13)}\right\} \equiv \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{-s+3}{s^2+4s+13}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-s+3}{(s+2)^2+3^2}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-(s+2)+5}{(s+2)^2+3^2}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{(s+2)^2+3^2}\right\}$$

$$\equiv \mathbf{1 - e^{-2t}\cos 3t + \frac{5}{3}e^{-2t}\sin 3t}$$

4-7 The Solution of Differential Equations Using Laplace Transforms

*Steps

(i) Take the Laplace transform of both sides of the differential equation by applying the formulae for the Laplace transforms of derivatives and, where necessary, using a list of standard Laplace transforms.

(ii) Put in the given initial conditions, i.e. $y(0)$ and $y'(0)$.

(iii) Rearrange the equation to make $L\{y\}$ the subject.

(iv) Determine y by using, where necessary, partial fractions, and taking the inverse of each term.

EX Use Laplace transform to solve the following differential equations

(a) $2y''+5y'-3y=0$, $y(0)=4$ and $y'(0)=9$

(b) $y''-7y'+10y=e^{2x}+20$, $y(0)=0$ and $y'(0)=-1/3$

(c) $y''-3y'-10y=2e^{-2x}(\cos^2 4x - 1/2)+3$, $y(0)=1$ and $y'(0)=3$

(d) $y'''+y''=e^t+t+1$, $y(0)=y'(0)=y''(0)=0$

Solution

(a) $2y''+5y'-3y=0$, $y(0)=4$ and $y'(0)=9$

(i) $$2\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\}+5\mathcal{L}\left\{\frac{dy}{dx}\right\}-3\mathcal{L}\{y\}=\mathcal{L}\{0\}$$

$$2[s^2\mathcal{L}\{y\}-sy(0)-y'(0)]+5[s\mathcal{L}\{y\}-y(0)]-3\mathcal{L}\{y\}=0$$

(ii) $y(0)=4$ and $y'(0)=9$

$$2[s^2\mathcal{L}\{y\}-4s-9]+5[s\mathcal{L}\{y\}-4]-3\mathcal{L}\{y\}=0$$

(iii) Rearranging gives:

$$(2s^2 + 5s - 3)\mathcal{L}\{y\} = 8s + 38$$

$$\mathcal{L}\{y\} = \frac{8s + 38}{2s^2 + 5s - 3}$$

(iv) $y = \mathcal{L}^{-1}\left\{\frac{8s+38}{2s^2+5s-3}\right\}$

$$\frac{8s + 38}{2s^2 + 5s - 3} \equiv \frac{8s + 38}{(2s - 1)(s + 3)}$$

$$\equiv \frac{A}{2s - 1} + \frac{B}{s + 3}$$

$$8s + 38 = A(s + 3) + B(2s - 1)$$

When $s = \frac{1}{2}$, $42 =$ **3.5A,** $A = 12$

$s = -3, 14 = -7B, \quad B = -2.$

$$y = \mathcal{L}^{-1}\left\{\frac{8s+38}{2s^2+5s-3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{12}{2s-1} - \frac{2}{s+3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{12}{2\left(s-\frac{1}{2}\right)}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\}$$

$$\mathbf{y = 6e^{\frac{1}{2}x} - 2e^{-3x}}$$

(b) $y''-7y'+10y=e^{2x}+20,\quad y(0)=0$ and $y'(0)=-1/3$

(i) $\mathcal{L}\left\{\dfrac{d^2y}{dx^2}\right\}-7\mathcal{L}\left\{\dfrac{dy}{dx}\right\}+10\mathcal{L}\{y\}=\mathcal{L}\{e^{2x}+20\}$

$$[s^2\mathcal{L}\{y\}-sy(0)-y'(0)]-7[s\mathcal{L}\{y\}-y(0)]+10\mathcal{L}\{y\}=\frac{1}{s-2}+\frac{20}{s}$$

(ii) $y(0)=0$ and $y'(0)=-\dfrac{1}{3}$

$$s^2\mathcal{L}\{y\}-0-\left(-\frac{1}{3}\right)-7s\mathcal{L}\{y\}+0+10\mathcal{L}\{y\}=\frac{21s-40}{s(s-2)}$$

(iii) $(s^2-7s+10)\mathcal{L}\{y\}=\dfrac{21s-40}{s(s-2)}-\dfrac{1}{3}$

$$=\frac{3(21s-40)-s(s-2)}{3s(s-2)}$$

$$=\frac{-s^2+65s-120}{3s(s-2)}$$

$$\mathcal{L}\{y\} = \frac{-s^2 + 65s - 120}{3s(s-2)(s^2 - 7s + 10)}$$

$$= \frac{1}{3}\left[\frac{-s^2 + 65s - 120}{s(s-2)(s-2)(s-5)}\right]$$

$$= \frac{1}{3}\left[\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}\right]$$

(iv) $y = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}\right\}$

$$\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2} \equiv \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

$$-s^2 + 65s - 120 \equiv A(s-5)(s-2)^2 + B(s)(s-2)^2$$
$$+ C(s)(s-5)(s-2) + D(s)(s-5)$$

When $s=0, -120=-20A,$ $A=6.$

When $s=5, 180=45B,$ $B=4$

When $s=2, 6=-6D,$ $D=-1$

Equating s^3 terms gives: $0=A+B+C,$ $C=-10.$

$$\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{-s^2+65s-120}{s(s-5)(s-2)^2}\right\}=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{6}{s}+\frac{4}{s-5}-\frac{10}{s-2}-\frac{1}{(s-2)^2}\right\}$$

$$=\frac{1}{3}[6+4\,e^{5x}-10\,e^{2x}-x\,e^{2x}]$$

$$\mathbf{y=2+\frac{4}{3}e^{5x}-\frac{10}{3}e^{2x}-\frac{x}{3}e^{2x}}$$

(c) $y''-3y'-10y=2e^{-2x}(\cos^2 4\text{x} - 1/2)+3$, $y(0)=1$ and $y'(0)=3$

$$y''-3y'-10y=2e^{-2x}(\cos^2 4\text{x} - 1/2)+3$$

$$=2e^{-2x}(1/2(1+\cos 8x)-1/2)+3$$

$$=e^{-2x}\cos 8x+3$$

$$\mathcal{L}(y'')-3\mathcal{L}(y')-10\mathcal{L}(y)=\mathcal{L}(e^{-2x}\cos 8x)+\mathcal{L}(3)$$

$$s^2\mathcal{L}(y)-sy(0)-y'(0)-3(s\mathcal{L}(y)-y(0))-10\mathcal{L}(y)=\frac{s+2}{(s+2)^2+64}+\frac{3}{s}$$

$$(s^2-3s-10)\mathcal{L}(y)=\frac{s+2}{(s+2)^2+64}+\frac{3}{s}+s$$

$$\mathcal{L}(y)=\frac{1}{(s^2-3s-10)}[\frac{s(s+2)+3(s^2+4s+68)+s^2(s^2+4s+68)}{s(s^2+4s+68)}]$$

$$\mathcal{L}(y)=\frac{s^4+4s^3+72s^2+14s+204}{s(s+2)(s-5)(s^2+4s+68)}$$

$$\mathcal{L}(y)=\frac{A}{s}+\frac{B}{(s+2)}+\frac{C}{(s-5)}+\frac{Ds+E}{(s^2+4s+68)}$$

$$s^4+4s^3+72s^2+14s+204=A(s+2)(s-5)(s^2+4s+68)+Bs(s-5)(s^2+4s+68)$$
$$+\mathrm{C}s(s+2)(s^2+4s+68)+Ds^2(s+2)(s-5)+\mathrm{E}s(s+2)(s-5)$$

when $\mathrm{s}=0$, $204=-680\mathrm{A}$, $\boxed{\mathrm{A}=-0.3}$

when $\mathrm{s}=-2$, $448=896\mathrm{B}$, $\boxed{\mathrm{B}=0.5}$

when $\mathrm{s}=5$, $3199=3955\mathrm{C}$, $\boxed{\mathrm{C}=0.808}$

Equating s^4 terms gives: $1=\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}$, $\boxed{\mathrm{D}=-0.008}$

Equating s^3 terms gives: $4=\mathrm{A}-\mathrm{B}+6\mathrm{C}-3\mathrm{D}+\mathrm{E}$, $\boxed{\mathrm{E}=-0.072}$

$$y=\mathcal{L}^{-1}\left[\frac{-0.3}{s}+\frac{0.5}{(s+2)}+\frac{0.808}{(s-5)}+\frac{(-0.008s-0.072)}{(s^2+4s+68)}\right]$$

$$y=\mathcal{L}^{-1}\left[\frac{-0.3}{s}+\frac{0.5}{(s+2)}+\frac{0.808}{(s-5)}+\frac{(-0.008s-2*0.008+2*0.008-0.072)}{((s+2)^2+64)}\right]$$

$$y=\mathcal{L}^{-1}\left[\frac{-0.3}{s}+\frac{0.5}{(s+2)}+\frac{0.808}{(s-5)}-0.008\frac{(s+2)}{((s+2)^2+64)}-\frac{0.056}{((s+2)^2+64)}\right]$$

$$y=-0.3+0.5e^{-2x}+.808e^{5x}-.008e^{-2x}\cos 8x-0.007e^{-2x}\sin 8x$$

(d) $y''' + y'' = e^t + t + 1, \quad y(0) = y'(0) = y''(0) = 0$

$$\mathcal{L}(y''') + \mathcal{L}(y'') = \mathcal{L}(e^t) + \mathcal{L}(t) + \mathcal{L}(1)$$

$$[s^3\mathcal{L}(y) - s^2 y(0) - s\,y'(0) - y''(0)]$$

$$+[s^2\mathcal{L}(y) - s\,y(0) - y'(0)] = \frac{1}{s-1} + \frac{1}{s^2} + \frac{1}{s}.$$

$$s^3\mathcal{L}(y) + s^2\mathcal{L}(y) = \frac{2s^2-1}{s^2(s-1)}$$

$$\mathcal{L}(y) = \frac{2s^2-1}{s^4(s+1)(s-1)}.$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s+1} + \frac{F}{s-1}$$

$$2s^2 - 1 = As^3(s+1)(s-1) + Bs^2(s+1)(s-1) + Cs(s+1)(s-1)$$
$$+ D(s+1)(s-1) + Es^4(s-1) + Fs^4(s+1)$$

when $s = 0$, $-1 = -D$, $D = 1$

when $s = -1$, $1 = -2E$, $E = -1/2$

when $s = 1$, $1 = 2F$, $F = 1/2$

Equating s^5 terms gives: $0 = A + E + F$, $A = 0$

Equating s^4 terms gives: $0 = B - E + F$, $B = -1$

Equating s^3 terms gives: $0 = -A + C$, $C = 0$

$$\mathcal{L}(y) = -\frac{1}{s^2} + \frac{1}{s^4} - \frac{1}{2(s+1)} + \frac{1}{2(s-1)}$$

$$y = -\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^4}\right) - \frac{1}{2}\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}\,\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$= -t + \frac{1}{6}t^3 - \frac{1}{2}e^{-t} + \frac{1}{2}e^t.$$