Differential Equations

3-1 Fundamental Definitions

1- Differential Equation

Is an equation involving a function and one or more of its derivatives, and they are classified by :

a- Type (Namely, Ordinary or Partial)

1- Ordinary Differential Equation

If the derivatives which appear in a differential equation are total derivatives, that is, derivatives of a function of a single variable, is called an ordinary differential equation, for example,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 0 \quad \text{or} \quad y'' + 3y' - 2y = 0$$

in simple term

$$\frac{dy}{dx} = y' \quad , \quad \frac{d^2y}{dx^2} = y'' \quad , \quad \frac{d^3y}{dx^3} = y''' \quad \text{,and so on}$$

2- Partial Differential Equation

A differential equation involving a function of several variables and its partial derivatives is called a partial differential equation, for instance

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

b- Order

That of the highest order derivative that occurs in the equation, for example

$y' = x + 6$ ordinary, first order

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y$$ partial, second order

c- Degree

the exponent of the highest power of the highest order derivative, after the equation has been cleared of fractions and radicals in the dependent variable and its derivatives, for example:

$$(y''')^2 + (y'')^5 + \frac{y}{x^2+1} = e^x$$ ordinary, third order and second degree

d- Linearity

An equation which is linear, i.e., of the first degree, in the dependent variable and its derivatives is called a linear differential equation. From this definition it follows that the most general ordinary linear differential of order n is of the form :

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

Where the a's are function of x

A differential equation which is not linear, i.e., of the second degree and over and cannot be put in the above form, is said to be nonlinear.

$$(y''')^2 + (y'')^5 + \frac{y}{x^2+1} = e^x$$

ordinary, third order and second degree , nonlinear

2- Solution of a Differential Equation

We call a function *y=f(x) a solution of a differential equation if y and its derivatives* satisfy the equation

a- General Solution

A formula that gives all the solutions of a differential equation is called the general solution of the equation. To solve a differential equation means to find its general solution.

EX:-

Show that for any values of the arbitrary constants ***C_1*** *and* ***C_2*** *the function* ***$y=C_1 \cos x+C_2 \sin x$*** *is a solution of the differential equation*

$$\frac{d^2y}{dx^2}+y=0$$

$$y=C_1\cos x+C_2\sin x$$

$$\frac{dy}{dx}=-C_1\sin x+C_2\cos x$$

$$\frac{d^2y}{dx^2}=-C_1\cos x-C_2\sin x$$

$$\frac{d^2y}{dx^2}+y=(-C_1\cos x-C_2\sin x)+(C_1\cos x+C_2\sin x)=0$$

Notice that the equation $\frac{d^2y}{dx^2}+y=0$ has order two and that its general solution has two arbitrary constants.

b- Particular Solution

A particular solution of a differential equation is any solution that is obtained by assigning specific values to the constants in the general solution. In practice, particular solutions to a differential equation are usually obtained from **initial conditions or boundary conditions** that give the value of the ***dependent variable or one of its derivatives*** for a particular value of the independent variable

For example, if *y(0)=1 and y'(0)=0 in the above example*

then the particular solution is

$$y(0) = 1 \ = C_1 \cos 0 + C_2 \sin 0 \Rightarrow C_1 = 1$$

$$y'(0) = 0 \ = -C_1 \sin 0 + C_2 \cos 0 \ \Rightarrow C_2 = 0$$

$y = \cos x$ the particular solution

3-2 Solutions of The First-Order Differential Equations

1- Separation of Variable in First-Order Differential Equations

Reference : Thomas G.B., Weir, Hass, Giordano. Thomas's calculus (12ed.).
Chapter 9 Page 513

A first order differential equation is separable if it can be put in the form

$$M(x)dx + N(y)dy = 0$$

Where ***M*** *is a continuous function of* ***x*** *alone and* ***N*** *is a continuous function of* ***y*** *alone.*

The steps are as follows :

1- Express the given equation in the differential form

$$M(x)dx + N(y)dy = 0 \quad \text{or} \quad M(x)dx = -N(y)dy$$

2- Integrate to obtain the general solution

$$\int M(x)dx + \int N(y)dy = C$$

$$\text{or} \quad \int M(x)dx = -\int N(y)dy + C$$

EX1 :-

Find the general solution to $(x^2+4)\frac{dy}{dx} = xy$

$$(x^2+4)\frac{dy}{dx} = xy \quad \Rightarrow (x^2+4)dy = xydx$$

$$\frac{dy}{y} = \frac{xdx}{(x^2+4)} \qquad \text{(separate variables)}$$

$$\int \frac{dy}{y} = \int \frac{xdx}{(x^2+4)} \qquad \text{(integrate)}$$

$$\ln y = \frac{1}{2}\ln(x^2+4) + C_1 \quad = \ln(x^2+4)^{\frac{1}{2}} + C_1$$

$$y = e^{C_1}\sqrt{x^2+4} \quad \Rightarrow y = C\sqrt{x^2+4}$$

the general solution

EX2 :-

Find the particular solution to the differential equation

$$xydx + e^{-x^2}(y^2-1)dy = 0$$ when $y(0)=1$.

Solution

By multiplying equ. $\left(\frac{e^{x^2}}{y}\right)$

$$\left(\frac{e^{x^2}}{y}\right)xydx + \left(\frac{e^{x^2}}{y}\right)e^{-x^2}(y^2-1)dy = 0$$

$$xe^{x^2}dx + \left(y - \frac{1}{y}\right)dy = 0 \Rightarrow \int xe^{x^2}dx + \int\left(y - \frac{1}{y}\right)dy = 0$$

$$\frac{1}{2}e^{x^2} + \frac{y^2}{2} - \ln y = C_1 \Rightarrow e^{x^2} + y^2 - \ln y^2 = 2C_1 = C$$

$y=1$ when $x=0$ $\Rightarrow 1+1+0=C$

$$e^{x^2} + y^2 - \ln y^2 = 2$$ particular solution

EX Solve the differential equation

$$\left(\frac{\cos^2 3x}{xy}+\frac{2x}{y}\right)dx+\left(\frac{\sin y+5}{x}\right)dy=0$$

Solution

By multiplying equ. (xy)

$$\left(\cos^2 3x+2x^2\right)dx+\left(y\sin y+5y\right)dy=0$$

$$\left(\frac{1}{2}\left(\cos 6x+1\right)+2x^2\right)dx+\left(y\sin y+5y\right)dy=0$$

Use Integration by parts to solve ***ysiny*** $\Rightarrow \int u\,dv = uv - \int v\,du$

u=y ; du=dy ; dv=sinydy ; v=-cosy

$$\int\left(\frac{1}{2}\left(\cos 6x+1\right)+2x^2\right)dx-y\cos y+\int(\cos y+5y)dy=0$$

$$\frac{1}{12}\sin 6x+\frac{1}{2}x+\frac{2}{3}x^3-y\cos y+\sin y+\frac{5}{2}y^2=C$$

2- Homogeneous First-Order Differential Equations

The function given by $z=f(x,y)$ is said to be homogeneous of degree n if

$f(tx,ty)=t^n f(x,y)$ where n is a real number

EX1:-

Verify the function $f(x,y)=x^2y-4x^3+3xy^2$ is a homogeneous function.

Solution

$$f(tx,ty)=(tx)^2(ty)-4(tx)^3+3(tx)(ty)^2$$

$$=t^3x^2y-4t^3x^3+3t^3xy^2$$

$$=t^3\left(x^2y-4x^3+3xy^2\right)$$

$$f(tx,ty)=t^3f(x,y)$$

Then the function $f(x,y)=x^2y-4x^3+3xy^2$ is a homogeneous function of degree 3.

$$M(x,y)dx+N(x,y)dy=0$$

Where M and N are homogeneous functions of the same degree.

A first order differential equation is homogeneous if it can be put into the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \text{.........} (a)$$

$$\frac{y}{x} = v \Rightarrow y = vx \Rightarrow dy = vdx + xdv$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$ substitue in equation (a)

$$v + x\frac{dv}{dx} = F(v)$$ which can be rearranged

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$$

With the variables now separated, we can now solve the equation by integrating with respect to ***x*** *and* ***v****, we can then return to* ***x*** *and* ***y*** *substituting* ***v=y/x****.*

The steps are as follows :

1- Express the given equation in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

2- Let $\frac{y}{x} = v \Rightarrow \frac{dx}{x} + \frac{dv}{v - F(v)} = 0$

3- integrating with respect to ***x*** *and* ***v***

4- return to ***x*** *and* ***y*** *substituting* ***v=y/x*** to obtain the general solution

EX

Show that the equation $(2x^3+y^3)dx-3xy^2dy=0$ is homogeneous and find its general solution.

Solution

$$M(x,y)=2x^3+y^3 \Rightarrow M(tx,ty)=2(tx)^3+(ty)^3=t^3(2x^3+y^3)$$

is homogeneous form degree 3

$$N(x,y)=-3xy^2 \Rightarrow N(tx,ty)=-3(tx)(ty)^2=-3t^3xy^2$$

is homogeneous form degree 3

$$(2x^3+y^3)dx-3xy^2dy=0$$

$$\frac{dy}{dx}=\frac{2x^3+y^3}{3xy^2}=\frac{2+\left(\frac{y}{x}\right)^3}{3\left(\frac{y}{x}\right)^2}=\frac{2+v^3}{3v^2}=F(v)$$

$$\frac{dx}{x}+\frac{dv}{v-F(v)}=0 \Rightarrow \quad \frac{dx}{x}+\frac{dv}{v-\frac{\left(2+v^3\right)}{3v^2}}=0$$

$$\frac{dx}{x}+\frac{3v^2}{2\left(v^3-1\right)}dv=0$$

$$\ln x+\frac{1}{2}\ln\left(v^3-1\right)=C_1 \Rightarrow 2\ln x+\ln\left(v^3-1\right)=2C_1$$

$$\left(\left(\frac{y}{x}\right)^3-1\right)x^2=C$$

$y^3-x^3=Cx$ the general solution

EX Solve the differential equation $x^2 y' = y(3x - 2y)$

Solution

$$y' = \frac{y(3x-2y)}{x^2} = \frac{y^2\left(\frac{3x}{y}-2\right)}{x^2} = v^2\left(\frac{3}{v}-2\right)$$

$$= 3v - 2v^2 = F(v)$$

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0 \Rightarrow \quad \frac{dx}{x} + \frac{dv}{v - (3v - 2v^2)} = 0$$

$$\frac{dx}{x} + \frac{dv}{2v^2 - 2v} = \frac{dx}{x} + \frac{dv}{2v(v-1)} = \frac{dx}{x} + \frac{1}{2}\left(\frac{1}{v-1} - \frac{1}{v}\right)dv = 0$$

$$\ln x + \frac{1}{2}\ln(v-1) - \frac{1}{2}\ln(v) = C_1$$

$$2\ln x+\ln(v-1)-\ln(v)=2C_1 \Rightarrow x^2=C\left(\frac{v}{v-1}\right)$$

$$x^2=C\left(\frac{\frac{y}{x}}{\frac{y}{x}-1}\right)=C\left(\frac{y}{y-x}\right)$$ the general solution

3- Exact First-Order Differential Equations

$$M(x,y)dx+N(x,y)dy=0$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$ **(Exact equation)**

$$M(x,y)dx+N(x,y)dy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=df$$

Steps for solving an equation you know to be exact

1- Match the equation to the form $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ to identify $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

2- Integrate $\frac{\partial f}{\partial x}$ with respect to x, writing the constant of integration as $k(y)$.

3- Differentiate with respect to y and set the result equal to $\frac{\partial f}{\partial y}$ to find $k'(y)$

4- Integrate to find $k(y)$ and determine f.

5- write the solution of the exact equation as $f(x,y)=C$.

EX

Show that the differential equation $(2xy-3x^2)dx+(x^2-2y)dy=0$ is exact and find its general solution.

Solution

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left[2xy-3x^2\right]=2x$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left[x^2-2y\right]=2x$$

$$\Rightarrow\left(\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\right)$$

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=\left(2xy-3x^2\right)dx+\left(x^2-2y\right)dy$$

$$\frac{\partial f}{\partial x}=\left(2xy-3x^2\right) \text{ and } \frac{\partial f}{\partial y}=\left(x^2-2y\right)$$

$$f(x,y) = \int \left(2xy - 3x^2\right)dx = x^2 y - x^3 + k(y)$$

$$\frac{\partial}{\partial y}\left(x^2 y - x^3 + k(y)\right) = x^2 + k'(y) = \frac{\partial f}{\partial y} = \left(x^2 - 2y\right)$$

$$k'(y) = -2y$$

$$k(y) = \int k'(y)dy = \int -2y\,dy = -y^2 + C_1$$

$$f(x,y) = x^2 y - x^3 - y^2 + C_1$$

$$x^2 y - x^3 - y^2 = C$$ the general solution

EX Solve the differential equation

$$\left(y^2-\frac{y}{x(x+y)}+2\right)dx+\left(\frac{1}{(x+y)}+2y(x+1)\right)dy=0$$

Solution

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(y^2-\frac{y}{x(x+y)}+2\right)=2y-\frac{1}{x}\left(\frac{(x+y)-y}{(x+y)^2}\right)$$

$$=2y-\frac{1}{(x+y)^2}$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1}{(x+y)}+2y(x+1)\right)=2y-\frac{1}{(x+y)^2}$$

$$\Rightarrow\left(\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\right)$$

$$f(x,y)=\int\left(\frac{1}{(x+y)}+2y(x+1)\right)dy$$

$$=\ln(x+y)+y^2(x+1)+k(x)$$

$$\frac{\partial f}{\partial x}=\frac{1}{(x+y)}+y^2+k'(x)=\ y^2-\frac{y}{x(x+y)}+2$$

$$k'(x)=2-\left(\frac{y}{x(x+y)}+\frac{1}{(x+y)}\right)=2-\frac{1}{x}$$

$$k(x)=\int k'(x)dx \qquad =2x-\ln x+C_1$$

$$f(x,y)=\ln(x+y)+y^2(x+1)+2x-\ln x=C$$

the general solution

Integrating Factors

If the differential equation ***M(x,y)dx+N(x,y)dy=0*** *is not exact, it may be possible to make* it exact by multiplying by an appropriate factor ***u****, called* ***an integrating factor*** *for the* differential equation.

For instance,

$2ydx+xdy=0$ not exact,

multiplied by the integrating factor ***u=x***,

$2xydx+x^2dy=0$ Exact equations

Finding Integrating Factors

For the differential equation $M(x,y)dx+N(x,y)dy=0$

$$1\text{- If } \frac{1}{N(x,y)}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=h(x)$$

is a function of x alone, then $e^{\int h(x)dx}$ is an integrating factor.

$$2\text{ - If }\quad \frac{1}{M(x,y)}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]=k(y)$$

is a function of y alone, then $e^{\int k(y)dy}$ is an integrating factor.

EX :-

Find the general solution to the differential equation $(y^2\text{-}x)dx+2ydy=0$

Solution

$$\frac{\partial M}{\partial y}=2y \quad \text{and} \quad \frac{\partial N}{\partial x}=0 \quad \text{not exact.}$$

$$\frac{1}{N(x,y)}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=\frac{2y-0}{2y}=1=h(x)$$

$$e^{\int h(x)dx} = e^{\int dx} = e^x$$ integration factor

Multiplying the given differential equation by e^x

$$(y^2 e^x - x e^x)dx + 2y e^x dy = 0$$

$$f(x,y) = \int N(x,y)dy = \int 2ye^x dy = y^2e^x + g(x)$$

$$\frac{\partial}{\partial x}\left(y^2e^x + g(x)\right) = y^2e^x + g'(x) = y^2e^x - xe^x$$

$$g'(x) = -xe^x \Rightarrow g(x) = -xe^x + e^x + C_1$$

$$f(x,y) = y^2e^x - xe^x + e^x + C_1$$

$$(y^2 - x + 1)e^x = C$$

EX Solve the differential equation $y' = \dfrac{3y}{\left(\dfrac{y}{y^3-1} - 6x\right)}$

Solution

$$3y\,dx - \left(\frac{y}{y^3-1} - 6x\right)dy = 0$$

$$\frac{\partial M}{\partial y} = 3 \quad \text{and} \quad \frac{\partial N}{\partial x} = 6 \qquad \text{not exact}$$

$$\frac{1}{M(x,y)}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = k(y) \quad ; \quad \frac{1}{3y}(6-3) = \frac{1}{y}$$

$$e^{\int k(y)dy} = e^{\int \frac{1}{y}dy} = e^{\ln y} = y \qquad \text{integration factor}$$

Multiplying the given differential equation by **y**

$$3y^2dx - \left(\frac{y^2}{y^3-1} - 6xy\right)dy = 0$$

$$f(x,y) = \int 3y^2dx = 3y^2x + k(y)$$

$$\frac{\partial}{\partial y}[3y^2x+k(y)] = 6yx+k'(y) = -\frac{y^2}{y^3-1}+6xy$$

$$k'(y) = -\frac{y^2}{y^3-1} \quad ; \qquad k(y) = -\frac{1}{3}\ln(y^3-1)+C_1$$

$$f(x,y) = 3xy^2 - \frac{1}{3}\ln(y^3-1) = C$$

4- First-Order Linear Differential Equations

A First-order linear differential equation is an equation of the form

$$\frac{dy}{dx}+P(x)y = Q(x) \quad \text{or} \quad y'+P(x)y = Q(x)$$

Where P and Q are continuous functions of x

use an integrating factor **u(x),** which converts the left side into the derivative of the product **u(x)y**. That is, we need a factor **u(x)** such that

$$u(x)\frac{dy}{dx}+u(x)P(x)y=\frac{d\left[u(x)y\right]}{dx}$$

By expanding the right side

$$u(x)y'+u(x)P(x)y=u(x)y'+yu'(x)\Rightarrow u(x)P(x)y=yu'(x)$$

$$P(x)=\frac{u'(x)}{u(x)}$$ Integration with respect to x yields

$$\ln u(x)=\int P(x)dx+C_1\Rightarrow u(x)=Ce^{\int P(x)dx}\Rightarrow u(x)=e^{\int P(x)dx}$$ let C=1

$$y'e^{\int P(x)dx}+yP(x)e^{\int P(x)dx}=Q(x)e^{\int P(x)dx}$$

$$\frac{d}{dx}\left[ye^{\int P(x)dx}\right]=Q(x)e^{\int P(x)dx}$$

whose general solution is given by

$$ye^{\int P(x)dx}=\int Q(x)e^{\int P(x)dx}+C$$

$$y = \frac{1}{u(x)}\left[\int Q(x)u(x)dx + C\right]$$

Steps for solving a linear first-order differential equation

1- put it in standard form $y' + P(x)y = Q(x)$

2- find $\int P(x)dx$ then find $u(x) = e^{\int P(x)dx}$

3- use equation $y = \frac{1}{u(x)}\left[\int Q(x)u(x)dx + C\right]$ to find y

EX Find the general solution to the differential equation

$$xy' - 2y = x^2$$

Solution

$$y' - \left(\frac{2}{x}\right)y = x \Rightarrow P(x) = -\frac{2}{x}, \quad Q(x) = x$$

$$\int P(x)dx = -\int \frac{2}{x}dx = -\ln x^2$$

$$u(x) = e^{\int P(x)\,dx} = e^{-\ln x^2} = \frac{1}{x^2}$$

$$y = \frac{1}{u(x)}\left[\int Q(x)u(x)\,dx + C\right]$$

$$y = \frac{1}{1/x^2}\left[\int x\,\frac{1}{x^2}\,dx + C\right] \Rightarrow y = x^2 \ln x + Cx^2$$ general solution

EX Find the general solution to the differential equation

$$x^2 y'+2 = x(y+2y'+1)$$

Solution

$$(x^2-2x)y'-xy = x-2 \Rightarrow x(x-2)y'-xy = x-2$$

$$y'-\frac{1}{x-2}y = \frac{1}{x} \;;$$

$$P(x) = -\frac{1}{x-2}y \;;\; Q(x) = \frac{1}{x}$$

$$\int P(x)dx = -\int \frac{1}{x-2}dx = -\ln(x-2)$$

$$u(x) = e^{\int P(x)dx} = e^{-\ln(x-2)} = \frac{1}{x-2}$$

$$y = \frac{1}{u(x)}\left[\int Q(x)u(x)dx + C\right]$$

$$y = \frac{1}{1/x-2}\left[\int \frac{1}{x(x-2)}dx + C\right]$$

$$y = (x-2)\left[\int \frac{1}{2}(\frac{1}{(x-2)} - \frac{1}{x})dx + C\right]$$

$$y = (x-2)\left[\frac{1}{2}(\ln(x-2) - \ln(x) + C\right]$$ general solution

Bernoulli Equation

$$y' + P(x)y = Q(x)y^n$$

This equation is linear if $n=0$, and has separable variables if $n=1$.

we assune that $n \neq 0$ and $n \neq 1$.

$$y^{-n}y' + P(x)y^{1-n} = Q(x) \qquad \text{(multiplying by } y^{-n}\text{)}$$

$$(1-n)y^{-n}y' + (1-n)P(x)y^{1-n} = (1-n)Q(x) \quad \text{(multiplying by (1 - n))}$$

$$\frac{d}{dx}\left[y^{1-n}\right] + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$

Which is a linear equation in the variable y^{1-n}

let $z = y^{1-n}$

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x) \quad \text{or} \quad z' + (1-n)P(x)z = (1-n)Q(x)$$

the general solution to the Bernoulli equation is

$$z = \frac{1}{u(x)}\left[\int (1-n)Q(x)u(x)dx + C\right]$$

Where $u(x) = e^{\int (1-n)P(x)dx}$ and $z = y^{1-n}$

EX Find the general solution to the differential equation

$$y' + xy = xe^{-x^2} y^{-3}$$

Solution

n=-3, $z = y^{1-n} = y^{4}$, $P(x) = x$ and $Q(x) = xe^{-x^2}$

$$z' + (1-n)P(x)z = (1-n)Q(x)$$

$$z' + 4xz = 4xe^{-x^2}$$

$$u(x) = e^{\int (1-n)P(x)dx} = e^{\int 4xdx} = e^{2x^2}$$

$$z=\frac{1}{u(x)}\left[\int(1-n)Q(x)u(x)dx+C\right]$$

$$z=\frac{1}{e^{2x^2}}\left[\int 4xe^{-x^2}e^{2x^2}dx+C\right]=e^{-2x^2}\left[2e^{x^2}+C\right]$$

$$y^4=2e^{-x^2}+Ce^{-2x^2}$$ general solution

EX Find the general solution to the differential equation

$$xy'+y=(x+3)(2xy)^2-3y'$$

Solution

$$(x+3)y'+y=(x+3)(2xy)^2\Rightarrow\ y'+\frac{1}{x+3}y=4x^2y^2$$

$$n=2\ ;\ z=y^{1-n}=y^{-1}\ ;\quad P(x)=\frac{1}{x+3}\ ;\ Q(x)=4x^2$$

$$z' + (1-n)P(x)z = (1-n)Q(x)$$

$$z' - \frac{1}{x+3} z = -4x^2$$

$$u(x) = e^{\int (1-n)P(x)\,dx} \quad = e^{-\int \frac{1}{x+3} dx} = e^{-\ln(x+3)} = \frac{1}{x+3}$$

$$z = \frac{1}{u(x)}\left[\int (1-n)Q(x)u(x)\,dx + C\right]$$

$$= \frac{1}{1/(x+3)}\left[\int \frac{-4x^2}{x+3}\,dx + C\right]$$

$$= (x+3)\left[-4\int (x - 3 + \frac{9}{x+3})\,dx + C\right]$$

$$= (x+3)\left[-4(\frac{x^2}{2} - 3x + 9\ln(x+3)) + C\right] = y^{-1}$$

$$\begin{array}{r|l} & x \;\; -3 \\ \hline x+3 & x^2 \\ & -x^2 \mp 3x \\ \hline & -3x \\ & \pm 3x \pm 9 \\ \hline & 9 \end{array}$$

general solution

Summary of First-Order Differential Equations

Method	***Form of Equation***
1- Separable variables	$M(x)dx+N(y)dy=0$
2- Homogeneous	$M(x,y)dx+N(x,y)dy=0$, where M and N are nth-degree homogeneous
3- Exact	$M(x,y)dx+N(x,y)dy=0$ where $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$
4- Integrating factor	$u(x,y)M(x,y)dx+u(x,y)N(x,y)dy=0$ is exact
5- Linear	$y'+P(x)y=Q(x)$
6- Bernoulli	$y'+P(x)y=Q(x)y^n$

3-3 Solutions of The Second-Order Differential Equations

Reference : Thomas G.B., Weir, Hass, Giordano. Thomas's calculus (12ed.).
Chapter 17 Page 989

1- Second Order Differential Equations Reducible to First Order

A- If the second order differential equation has a form

$$F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

as it is from the equation $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2$

$$Let \quad \frac{dy}{dx} = p \quad and \quad \frac{d^2y}{dx^2} = \frac{dp}{dx}$$

$$x\frac{dp}{dx} + p = x^2$$

to find p and solving $\dfrac{dy}{dx} = p$ to find y

EX Find the general solution to the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

Solution

Let $\frac{dy}{dx}=p$ *and* $\frac{d^2y}{dx^2}=\frac{dp}{dx}=\mathrm{p}'$

$$\frac{dp}{dx}+p=0 \Rightarrow dp+pdx=0 \Rightarrow \frac{dp}{p}+dx=0$$

$$\int\frac{1}{p}dp=-\int dx \Rightarrow \ln p=-x+C \Rightarrow p=Ce^{-x}$$

$$\frac{dy}{dx}=p \Rightarrow y=\int pdx \Rightarrow$$

$$y=\int Ce^{-x}dx=-Ce^{-x}+C_1$$ general solution

EX Find the general solution to the differential equation

$$x^2 y'' + (y')^2 - 2xy' = 0$$

Solution

Let $\quad p = y' \ , \quad p' = y'' \quad \left(\frac{dp}{dx} = \frac{d^2y}{dx^2} \quad , \quad p = \frac{dy}{dx} \right)$

$$x^2 p' - 2xp = -p^2 \Rightarrow \frac{dp}{dx} - \frac{2}{x} p = -\frac{1}{x^2} p^2$$ **Bernoulli Equation**

$$n = 2 \ ; \ z = p^{1-n} = p^{-1} \ ; \ P(x) = -\frac{2}{x} \ ; \ Q(x) = -\frac{1}{x^2}$$

$$z' + (1-n)P(x)z = (1-n)Q(x)$$

$$z' + \frac{2}{x} z = \frac{1}{x^2}$$

$$u(x) = e^{\int (1-n)P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$z = \frac{1}{u(x)}\left[\int (1-n)Q(x)u(x)dx + C\right]$$

$$z = \frac{1}{x^2}\left[\int \frac{1}{x^2}x^2 dx + C\right] = \frac{1}{x^2}\left[\int dx + C\right] = \frac{1}{x^2}\left[x + C\right]$$

$$z = \frac{x+C}{x^2} \Rightarrow z = p^{-1} = \frac{1}{p} \Rightarrow p = \frac{x^2}{x+c}$$

$$\frac{dy}{dx} = \frac{x^2}{x+c} \Rightarrow \quad dy = \frac{x^2}{x+c}dx$$

$$y = \int\left[x - c + \frac{c^2}{x+c}\right]dx$$

$$y = \frac{x^2}{2} - Cx + C^2 \ln\left(x + C\right) + C_1$$ general solution

B- If the second order differential equation has a form

$$F\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$$

$$p = \frac{dy}{dx} \quad ; \quad \frac{dp}{dx} = \frac{d^2y}{dx^2} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p\frac{dp}{dy} \Rightarrow F\left(y, p, p\frac{dp}{dy}\right)$$

EX Find the general solution to the differential equation

$$y'' - y = 0$$

Solution

$$y'' = p\frac{dp}{dy} \quad ; \Rightarrow p\frac{dp}{dy} - y = 0$$

$pdp = ydy$ (separation of variables)

$$\frac{p^2}{2} = \frac{y^2}{2} + C \Rightarrow p^2 = y^2 + C_1 \Rightarrow p = \sqrt{y^2 + C_1}$$

$$p = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \sqrt{y^2 + C_1}$$

$$\frac{1}{\sqrt{y^2 + C_1}} dy = dx \Rightarrow \sinh^{-1} \frac{y}{\sqrt{C_1}} = x + C_2$$

EX Find the general solution to the differential equation

$$(y^2 - 1)y'' = y'(1 - 2yy')$$

Solution

$$y'' = p\frac{dp}{dy} \quad ; \quad p = \frac{dy}{dx} \quad ; \quad (y^2 - 1)p\frac{dp}{dy} = p(1 - 2yp)$$

$$\frac{dp}{dy} + \frac{2y}{(y^2 - 1)} p = \frac{1}{(y^2 - 1)}$$

$$P(y) = \frac{2y}{(y^2 - 1)} \quad ; \quad \mathrm{Q(y)} = \frac{1}{(y^2 - 1)}$$

$$u(y) = e^{\int \frac{2y}{y^2-1} dy} = e^{\ln(y^2-1)} = y^2 - 1$$

$$p = \frac{1}{u(y)}\left[\int Q(y)u(y)dy + C\right] = \frac{1}{y^2-1}\left[\int (y^2-1)\frac{1}{y^2-1}dy + C\right]$$

$$= \frac{y+C}{y^2-1}$$

$$p = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y+C}{y^2-1} \ ; \quad \int dx = \int \frac{y^2-1}{y+C} dy$$

$$x = \int \left(\frac{y^2}{y+C} - \frac{1}{y+C}\right) dy = \int \left(y - C + \frac{C^2}{y+C} - \frac{1}{y+C}\right) dy$$

$$x = \frac{y^2}{2} - Cy + (C^2-1)\ln(y+C) + C_1$$

general solution

2- Second-Order Homogeneous Linear Differential Equations

The general form of linear differential equation of order *n* is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots. + a_1(x)\frac{dy}{dx} + a_o(x)y = F(x)$$

$F(x)=0$ homogeneous

$F(x)\neq 0$ nonhomogeneous

For the homogeneous case

$$a_n(x)D^n y + a_{n-1}(x)D^{n-1} y + \ldots. + a_1(x)Dy + a_o(x)y = 0$$

where $D = \frac{d}{dx}$; $D^2 = \frac{d^2}{dx^2}$; $D^n = \frac{d^n}{dx^n}$

and called linear differential operator

For $n=2$ (second order) and a_o , a_1 , ..., a_n are constants then :

$$a_2 D^2 y + a_1 D y + a_o y = 0 \Rightarrow D^2 y + \frac{a_1}{a_2} D y + \frac{a_o}{a_2} y = 0$$

let $\frac{a_1}{a_2} = a$ and $\frac{a_o}{a_2} = b$

$$D^2 y + aDy + by = 0$$

$$\left(D^2 + aD + b\right) y = 0$$ or in simple form

$$\left(m^2 + am + b\right) y = 0$$ since y is never zero , then

$$m^2 + am + b = 0$$ characteristic equation of the differential equation

$$y'' + ay' + by = 0$$

Note that the characteristic equation can be determined from its differential equation by simple replacing

y'' by m^2 y' by m and y by 1

Roots of The $y'' + ay' + by = 0$

1- **Distinct (not Equal) Real Roots : if** $m_1 \neq m_2$ are distinct real roots of the characteristic equation, then the general solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

2- **Equal Real Roots : if** $m_1 = m_2$ are equal real roots of the characteristic equation, then the general solution is

$$y = C_1 e^{m_1 x} + C_2 x e^{m_2 x} = (C_1 + C_2 x) e^{m_1 x}$$

3- **Complex Roots : if** $m_1 = \alpha + i\beta$ ***and*** $m_2 = \alpha - i\beta$ are complex roots of the characteristic equation, then the general solution is

$$y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

Note :

A complex number we mean a number of the form ***z=u+iv***, where ***u*** and ***v*** are real numbers and ***i*** is the so-called imaginary unit whose existence is postulated such that $i^2 = -1$ i.e. $i = \sqrt{-1}$

The real number ***u*** is called the real component or real part of ***z***. The real number ***v*** is called the imaginary component or imaginary part of ***z***.

EX1 :-

Solve the differential equation $y'' + 4y' + 4y = 0$,
subject to the initial conditions $y(0) = 2$ and $y'(0) = 1$

Solution

$$y'' + 4y' + 4y = 0$$

$m^2 + 4m + 4 = (m+2)^2 = 0$ characteristic equation

$m_1 = m_2 = -2$ equal real roots

$$y = C_1 e^{m_1 x} + C_2 x e^{m_2 x} = (C_1 + C_2 x) e^{m_1 x}$$

$y = (C_1 + C_2 x) e^{-2x}$ general solution

$y=2$ when $x=0$, $2 = C_1(1) + C_2(0)(1) = C_1 \Rightarrow C_1 = 2$

$y' = 1$ when $x = 0$,

$y' = -2C_1 e^{-2x} + C_2\left(-2xe^{-2x} + e^{-2x}\right)$

$1 = -2(2)(1) + C_2[-2(0)(1) + 1] \Rightarrow C_2 = 5$

$y = 2e^{-2x} + 5xe^{-2x}$ (particular solution)

EX2 :- Find the general solution to the differential equations

(a) $y'' + 6y' + 12y = 0$ and (b) $y'' + y' - 6y = 0$

Solution

(a) $y'' + 6y' + 12y = 0$

$m^2 + 6m + 12 = 0$ characteristic equation

$$m_{1,2} = \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$m_{1,2}=\frac{-6\pm\sqrt{36-48}}{2}=\frac{-6\pm\sqrt{-12}}{2}=\frac{-6\pm2\sqrt{-3}}{2}$$

$$=-3\pm\sqrt{-3}=-3\pm\sqrt{3}i$$ two complex roots

$\alpha = -3$ and $\beta=\sqrt{3}$

$$y=C_1e^{\alpha x}\cos\beta x+C_2e^{\alpha x}\sin\beta x$$

$$y=C_1e^{-3x}\cos\sqrt{3}x+C_2e^{-3x}\sin\sqrt{3}x$$ general solution

(b) $y''+y'-6y=0$

$m^2+m-6=0$ characteristic equation

$(m-2)(m+3)=0$ not equal roots

$$y=C_1e^{m_1x}+C_2e^{m_2x}$$

$$y=C_1e^{2x}+C_2e^{-3x}$$ general solution

3- Second Order Nonhomogeneous Linear Differential Equations

$$y'' + ay' + by = F(x)$$

second order nonhomogeneous linear differential equation

$y = y_p + y_h$ general solution of the nonhomogeneous equation.

y_p particular solution

y_h general solution of the corresponding homogeneous equation

Particular Solution Methods

A- Opposite Operators Method

$$D^2 y + aDy + by = F(x)$$ or in simple way

$$F(D)y = F(x) \Rightarrow y = \frac{F(x)}{F(D)} = \frac{1}{(D-m_1)(D-m_2)\ldots\ldots(D-m_n)} F(x)$$

the particular solution of this equation

$$y_p = e^{m_1 x}\int e^{(m_2-m_1)x}\int e^{(m_3-m_2)x}\dots\dots\int e^{(m_n-m_{n-1})x}\int e^{-m_n x}F(x)(dx)^n$$

For second order $y_p = e^{m_1 x}\int e^{(m_2-m_1)x}\int e^{-m_2 x}F(x)(dx)^2$

EX Solve the differential equation $y'' + 2y = e^x + 3y'$

Solution

1- Find y_h (homogeneous solution)

$$y'' - 3y' + 2y = 0$$

$m^2 - 3m + 2 = 0$ (the characteristic equation)

$(m-1)(m-2) = 0 \Rightarrow m_1 = 1\ ;\ \ m_2 = 2$ not equal roots.

$$y_h = C_1 e^x + C_2 e^{2x}$$

2- Find y_p (Particular Solution)

$$y_p = e^{m_1 x}\int e^{(m_2-m_1)x}\int e^{-m_2 x}F(x)(dx)^2$$

$$F(x) = e^x$$

$$y_p = e^x \int e^{(2-1)x} \int e^{-2x} e^x (dx)^2 = e^x \int e^x \int e^{-x} (dx)^2$$

$$= e^x \int e^x \left(-e^{-x}\right) dx \quad = -e^x \int dx$$

$y_p = -xe^x$ particular solution

3- Find $y = y_h + y_p$

$y = C_1 e^x + C_2 e^{2x} - xe^x$ (The general solution)

B- Undetermined Coefficients Method

$$y'' + ay' + by = F(x)$$

If F(x) has a term that is a constant multiple of	And if	Then include this expression in the trial function for y_p
e^{mx}	m is not a root of the c. equ.	Ae^{mx}
	m is a single root of the c. equ.	Axe^{mx}
	m is a double root of the c. equ.	$Ax^2 e^{mx}$
$sinkx$, $coskx$	ki is not a root of the c. equ.	$Bcoskx + Csinkx$
	ki is a root of the c. equ.	$Bxcoskx + Cxsinkx$
$ax^2 + bx + c$	0 is not a root of the c. equ.	$Dx^2 + Ex + F$ (chosen to match the degree of $ax^2 + bx + c$)
	0 is a single root of the c. equ.	$Dx^3 + Ex^2 + Fx$ (degree one higher than the degree of $ax^2 + bx + c$)
	0 is a double root of the c. equ.	$Dx^4 + Ex^3 + Fx^2$(degree one higher than the degree of $ax^2 + bx + c$)

EX Solve the differential equation $y'' - y' = 5e^{x} - \sin 2x$

Solution

1- Find y_h (homogeneous solution)

$$y'' - y' = 0$$

$m^2 - m = 0$ (the characteristic equation)

m(m-1)=0 $m_1 = 1$; $m_2 = 0$ not equal roots

$$y_h = C_1 e^{x} + C_2$$

2- Find y_p (Particular Solution)

$$y_p = Axe^{x} + B\cos 2x + C\sin 2x$$

$$y_p' = \left(Axe^{x} + Ae^{x} - 2B\sin 2x + 2\cos 2x\right)$$

$$y_p'' = \left(Axe^{x} + 2Ae^{x} - 4B\cos 2x - 4C\sin 2x\right)$$

Substituting the derivatives of y_p in differential equation gives

$$\left(Axe^{x} + 2Ae^{x} - 4B\cos 2x - 4C\sin 2x\right) - \left(Axe^{x} + Ae^{x} - 2B\sin 2x + 2\cos 2x\right) = 5e^{x} - \sin 2x$$

$$Ae^x - (4B+2C)\cos 2x + (2B-4C)\sin 2x = 5e^x - \sin 2x$$

$$Ae^x = 5e^x \Rightarrow A = 5$$

$$4B+2C=0 \quad ; \quad 2B-4C=-1 \Rightarrow B = -\frac{1}{10} \quad ; \quad C = \frac{1}{5}$$

$$y_p = 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x \quad \text{particular solution}$$

3- Find $y = y_h + y_p$

$$y = y_h + y_p = C_1 e^x + C_2 + 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x$$

(The general solution)

EX Solve the differential equation $y'' - 6y' + 9y = e^{3x}$

Solution

1- Find y_h (homogeneous solution)

$$y'' - 6y' + 9y = 0$$

$$m^2 - 6m + 9 = (m-3)^2 = 0$$ (the characteristic equation)

$$m_1 = m_2 = 3$$ equal real roots

$$y_h = (C_1 x + C_2)e^{3x}$$

2- Find y_p (Particular Solution)

$$y_p = Ax^2 e^{3x}$$

$$y_p' = 3Ax^2 e^{3x} + 2Axe^{3x}$$

$$y_p'' = 9Ax^2 e^{3x} + 12Axe^{3x} + 2Ae^{3x}$$

Substituting y_p and the derivatives of y_p in differential equation gives

$$\left(9Ax^2e^{3x}+12Axe^{3x}+2Ae^{3x}\right)-6\left(3Ax^2e^{3x}+2Axe^{3x}\right)+9Ax^2e^{3x}=e^{3x}$$

$$2Ae^{3x}=e^{3x}\Rightarrow A=\frac{1}{2}$$

$$y_p=\frac{1}{2}x^2e^{3x}$$

3- Find $y=y_h+y_p$

$$y=y_h+y_p=\left(C_1x+C_2\right)e^{3x}+\frac{1}{2}x^2e^{3x}$$

(The general solution)

EX Solve the differential equation $y'' + y = 3x^2 + 4$

Solution

1- Find y_h (homogeneous solution)

$$y'' + y = 0$$

$$m^2 + 1 = 0 \qquad \text{(the characteristic equation)}$$

$$m_{1,2} = \pm i \qquad \alpha = 0 \; ; \; \beta = 1 \qquad \text{complex roots}$$

$$y_h = C_1 \cos x + C_2 \sin x$$

2- Find y_p (Particular Solution)

$$y_p = Dx^2 + Ex + F$$

$$y_p' = 2Dx + E \qquad y_p'' = 2D$$

Substituting y_p and the derivatives of y_p in differential equation gives

$$2D + (Dx^2 + Ex + F) = 3x^2 + 4$$

$$Dx^2 + Ex + 2D + F = 3x^2 + 4$$

$Dx^2 = 3x^2 \Rightarrow D = 3$, $E = 0$, $2D + F = 4 \Rightarrow F = -2$

$y_p = 3x^2 - 2$ particular solution

3- Find $y = y_h + y_p$

$y = y_h + y_p = C_1 \cos x + C_2 \sin x + 3x^2 - 2$ general solution

C- Variation of Parameter

In this method we assume that y_p has the same form as y_h , except that the constants in y_h are replaced by variables.

To find the general solution to the equation $y'' + ay' + by = F(x)$

1- Find $y_h = C_1 y_1(x) + C_2 y_2(x)$ the general solution of the homogeneous equation or in simple way .

$y_h = C_1 y_1 + C_2 y_2$

2- Replace the constants by variables to form

$y_h = v_1(x) y_1(x) + v_2(x) y_2(x)$ or in simple way

$y_h = v_1 y_1 + v_2 y_2$

There are two functions v_1 and v_2 to be determined, and requiring that differential equation be satisfied is only one condition

$$v_1' y_1 + v_2' y_2 = 0$$

$$y = v_1 y_1 + v_2 y_2$$

$$\frac{dy}{dx} = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2 = v_1 y_1' + v_2 y_2'$$

$$\frac{d^2 y}{dx^2} = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'$$

If we substitute these expressions into the left-hand side of differential equation and rearrange terms, we obtain :

$$v_1\left[\frac{d^2 y_1}{dx^2} + a\frac{dy_1}{dx} + by_1\right] + v_2\left[\frac{d^2 y_2}{dx^2} + a\frac{dy_2}{dx} + by_2\right] + v_1' y_1' + v_2' y_2' = F(x)$$

The two bracketed terms are zero, since y_1 and y_2 are solutions of the homogeneous equation.

$$v_1' y_1' + v_2' y_2' = F(x)$$

3 - Solve the following system for v_1' and v_2'

$$v_1' y_1 + v_2' y_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$v_1' y_1' + v_2' y_2' = F(x) \ldots\ldots\ldots\ldots\ldots (2)$$

4 - Integrate to find v_1 and v_2 . The general solution is $y = y_h + y_p$

EX1 :- Solve the differential equation $y'' - 2y' + y = \dfrac{e^x}{2x}$

Solution

1- Find y_h (homogeneous solution)

$$y'' - 2y' + y = 0$$

$$m^2 - 2m + 1 = (m-1)^2 = 0 \quad \text{(the characteristic equation)}$$

$$m_1 = m_2 = 1$$

$$y_h = (C_1 + C_2 x)e^x = C_1 y_1 + C_2 y_2$$

2- Find y_p (Particular Solution)

Replacing C_1 and C_2 by v_1 and v_2

$$y_p = v_1 y_1 + v_2 y_2 = v_1 e^x + v_2 x e^x$$

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' e^x + v_2' x e^x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$v_1' y_1' + v_2' y_2' = F(x).$$

$$v_1' e^x + v_2' \left(x e^x + e^x \right) = \frac{e^x}{2x} \ldots\ldots\ldots\ldots(2)$$

$$v_2' = \frac{1}{2x} \qquad v_1' = -\frac{1}{2}$$

$$v_1 = -\int \frac{1}{2} dx = -\frac{x}{2}$$

$$v_2 = \frac{1}{2} \int \frac{1}{x} dx = \frac{1}{2} \ln x = \ln \sqrt{x}$$

$$y_p = -\frac{1}{2}xe^x + \left(\ln\sqrt{x}\right)xe^x$$

3- Find $y = y_h + y_p$

$$y = C_1e^x + C_2xe^x - \frac{1}{2}xe^x + \left(\ln\sqrt{x}\right)xe^x$$

or $y = \left(C_1 + C_2x - \frac{1}{2}x + \left(\ln\sqrt{x}\right)x\right)e^x$ general solution

EX2 :- Solve the differential equation $y'' + y = \tan x$

Solution

1- Find y_h (homogeneous solution)

$$y'' + y = 0$$

$$m^2 + 1 = 0$$ (the characteristic equation)

$$m_1 = m_2 = \pm i \qquad \alpha = 0 \; ; \; \beta = 1$$ complex roots

$$y_h = C_1 \cos x + C_2 \sin x$$

2- Find y_p (Particular Solution)

Replacing C_1 and C_2 by v_1 and v_2

$$y_p = v_1 y_1 + v_2 y_2 = v_1 \cos x + v_2 \sin x$$

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' \cos x + v_2' \sin x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$v_1' y_1' + v_2' y_2' = F(x).$$

$$-v_1' \sin x + v_2' \cos x = \tan x \ldots\ldots\ldots\ldots\ldots(2)$$

Multiplying the first equation by sinx and the second by cosx

$$v_1' \sin x \cos x + v_2' \sin^2 x = 0$$

$$-v_1' \sin x \cos x + v_2' \cos^2 x = \sin x$$

Adding these two equations

$$v_2' = \sin x \,, \quad v_1' = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

$$v_1 = \int (\cos x - \sec x) dx = \sin x - \ln\left|\sec x + \tan x\right|$$

$$v_2 = \int \sin x dx = -\cos x$$

$$y_p = \sin x \cos x - \cos x \ln\left|\sec x + \tan x\right| - \sin x \cos x$$

$$= -\cos x \ln\left|\sec x + \tan x\right|$$

3- Find $y = y_h + y_p$

$$y = y_h + y_p = C_1 \cos x + C_2 \sin x - \cos x \ln\left|\sec x + \tan x\right|$$

Note For the unknown functions v_1 and v_2 Cramer's rule gives

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ F(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 F(x)}{y_1 y_2' - y_2 y_1'}$$

$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & F(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 F(x)}{y_1 y_2' - y_2 y_1'}$$

3-4 Solutions of The Higher-Order Differential Equations

A- Higher-Order Homogeneous Linear Differential Equations

n > 2 higher-order

general form of the homogeneous linear differential equation

$$a_n D^n y + a_{n-1} D^{n-1} y + \ldots.. + a_1 D y + a_o y = 0$$

characteristic equation

$$m^n + k_1 m^{n-1} + k_2 m^{n-2} + \ldots.. + k_n = 0$$

where $k_1,\ k_2\ ,\ldots.,\ k_n$ are constants

Solutions to the characteristic equation

1- if $m_1,\ m_2,\ \ldots..,m_n$ are not equal real roots, then the general solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots\ldots + C_n e^{m_n x}$$

2 - if $m_1, m_2, ...,m_n$ are equal real roots, then the general solution is

$$y = \left(C_1 + C_2 x + C_3 x^2 + ... + C_n x^{n-1}\right)e^{mx}$$

3 - if $m_1, m_2, ...,m_n$ are repeated complex roots $(m = \alpha \pm \beta i)$

then the general solution is

$$y = \left[\left(C_1 + C_2 x + C_3 x^2 + ... + C_n x^{n-1}\right)\cos \beta x + \left(P_1 + P_2 x + P_3 x^2 + ... + P_n x^{n-1}\right)\sin \beta x\right]e^{\alpha x}$$

EX :- Solve the differential equations

(a) $y''' + 3y'' + 3y' + y = 0$, (b) $y'''' + 2y'' + y = 0$,

(c) $y''' + y' + 6y = 4y''$, (d) $y''' + 2y'' + 8y' + 33y = 0$

Solution

(a) $y''' + 3y'' + 3y' + y = 0$

$m^3 + 3m^2 + 3m + 1 = 0$ (the characteristic equation)

$$(m + 1)(m^2 + 2m + 1) = (m + 1)(m + 1)(m + 1) = (m + 1)^3 = 0$$

$m_1 = m_2 = m_3 = -1$ (three equal roots)

$$y = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x} \quad \text{or} \quad y = \left(C_1 + C_2 x + C_3 x^2\right) e^{-x}$$

(b) $y'''' + 2y'' + y = 0$

$$m^4 + 2m^2 + 1 = 0 \quad \text{(the characteristic equation)}$$

$$\left(m^2 + 1\right)\left(m^2 + 1\right) = 0$$

$$m_{1,2} = \pm i \text{ and } m_{3,4} = \pm i$$

$$y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$$

$$\text{or } y = \left(C_1 + C_3 x\right)\cos x + \left(C_2 + C_4 x\right)\sin x$$

(c) $y''' + y' + 6y = 4y''$

$$y''' - 4y'' + y' + 6y = 0$$

$$m^3 - 4m^2 + m + 6 = 0 \quad \text{(the characteristic equation)}$$

$$(m+1)\left(m^2 - 5m + 6\right) = 0$$

$$(m+1)(m-2)(m-3) = 0$$

$m_1 = -1,\ m_2 = 2, m_3 = 3$ three not equal

$$y = C_1 e^{-x} + C_2 e^{2x} + C_3 e^{3x}$$

(d) $y''' + 2y'' + 8y' + 33y = 0$

$m^3 + 2m^2 + 8m + 33 = 0$ (the characteristic equation)

$(m+3)(m^2 - m + 11) = 0$

$$m_1 = -3\ ,\ m_{2,3} = \frac{1 \pm \sqrt{1-44}}{2} = \frac{1}{2} \pm 3.28i$$

$$y = C_1 e^{-3x} + e^{0.5x}(C_2 \cos 3.28x + C_3 \sin 3.28x)$$

B- Higher-Order Nonhomogeneous Linear Differential Equations

For higher-order nonhomogeneous linear differential equations, we find the general solution in much the same way as we do for second-order equations.

$(y = y_h + y_p)$

Variations of Parameters

For third order (n=3)

$$v_1' y_1 + v_2' y_2 + v_3' y_3 = 0$$

$$v_1' y_1' + v_2' y_2' + v_3' y_3' = 0$$

$$v_1' y_1'' + v_2' y_2'' + v_3' y_3'' = F(x)$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(x) \end{bmatrix}$$

EX1 :- Solve the differential equation $y''' + 3y'' + 3y' + y = x$

Solution

1- Find y_h ; $y''' + 3y'' + 3y' + y = 0$

$$m^3 + 3m^2 + 3m + 1 = 0 \quad \text{(the characteristic equation)}$$

$$(m+1)(m^2 + 2m + 1) = (m+1)^3 = 0$$

$$m_1 = m_2 = m_3 = -1 \quad \text{(three equal roots)}$$

$$y = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x} \quad \text{or} \quad y = \left(C_1 + C_2 x + C_3 x^2\right) e^{-x}$$

2- Find y_p (Particular Solution)

since $F(x) = x$, we let $y_p = A + Bx$

$y_p' = B$, and $y_p'' = y_p''' = 0$

Substituting y_p and the derivatives of y_p in differential equation gives

$$y''' + 3y'' + 3y' + y = x$$

$$0 + 3*0 + 3*B + (A + Bx) = x$$

$$3B + A = 0 \; ; \quad Bx = x \; ; \quad B = 1 \text{ and } A = -3$$

$$3\text{-} \quad y = y_h + y_p$$

$$y = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x} - 3 + x$$

EX2 :- Solve the differential equation $y''' + y' = \csc x$

Solution

$$1\text{- Find } y_h \; ; \quad y''' + y' = 0$$

$$m^3 + m = m(m^2 + 1) = 0$$ (the characteristic equation)

$$m_1 = 0 \text{ and } m_{2,3} = \pm i$$

$$y_h = C_1 e^{m_1 x} + e^{\alpha x}\left(C_2 \cos \beta x + C_3 \sin \beta x\right)$$

$$y_h = C_1 + C_2 \cos x + C_3 \sin x$$

2- Find y_p (Particular Solution)

Replacing C_1 , C_2 and C_3 by v_1 , v_2 and v_3

$$y_p = v_1 + v_2 \cos x + v_3 \sin x = v_1 y_1 + v_2 y_2 + v_3 y_3$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(x) \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(x) = \csc x \end{bmatrix}$$

$$|D| = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} \begin{matrix} 1 & \cos x \\ 0 & -\sin x \\ 0 & -\cos x \end{matrix} = \sin^2 x + \cos^2 x = 1$$

$$|D_1| = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \csc x & -\cos x & -\sin x \end{vmatrix} \begin{matrix} 0 & \cos x \\ 0 & -\sin x \\ \csc x & -\cos x \end{matrix}$$

$$= \cos^2 x \csc x + \sin^2 x \csc x = \csc x$$

$$|D_2| = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \csc x & -\sin x \end{vmatrix} \begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & \csc x \end{matrix} = -\cos x \csc x = -\cot x$$

$$|D_3| = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \csc x \end{vmatrix} \begin{matrix} 1 & \cos x \\ 0 & -\sin x \\ 0 & -\cos x \end{matrix} = -\sin x \csc x = -1$$

$$v_1' = \frac{|D_1|}{|D|} = \frac{\csc x}{1} = \csc x\ , \quad v_2' = \frac{|D_2|}{|D|} = -\cot x\ , \quad v_3' = \frac{|D_3|}{|D|} = -1$$

$$v_1 = \int \csc x dx = -\ln|\csc x + \cot x|$$

$$v_2 = \int -\cot x dx = -\ln|\sin x|$$

$$v_3 = \int (-1) dx = -x$$

$$y_p = -\ln|\csc x + \cot x| - \ln|\sin x| \cos x - x \sin x$$

3- $y = y_h + y_p$

$$y = C_1 + C_2 \cos x + C_3 \sin x - \ln|\csc x + \cot x| - \ln|\sin x| \cos x - x \sin x$$