Chapter Two

Vectors

Reference : Thomas G.B., Weir, Hass, Giordano. Thomas's calculus

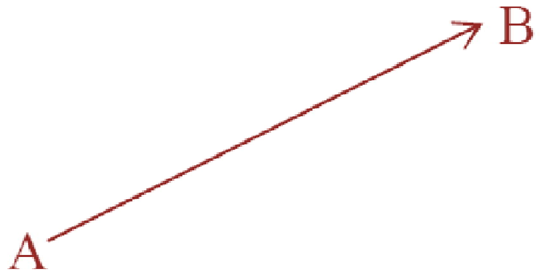
Part one : Vectors Calculus , (chapter twelve)

Quantities

Scalars
Length, Density, Mass,...ect.

Vectors
Forces, velocity, Moment,...ect.

Vectors are often represented by $\overline{a}$, $\overline{b}$ .. or by being (initial) point and end (terminal) point such as $\vec{AB}$, $\vec{AC}$, ...



1-1 Definitions

1- Length of a Vector

The magnitude or length of a vector $\bar{a}$ is called the absolute value of the vector and is usually denoted by $|\bar{a}|$, which may be read "the magnitude of a".

2- Equal Vectors (Equivalent Vectors)

We say that two vectors are equal if they have the same direction and the same length (magnitude), ($\bar{\mathbf{a}} = \bar{\mathbf{b}}$).

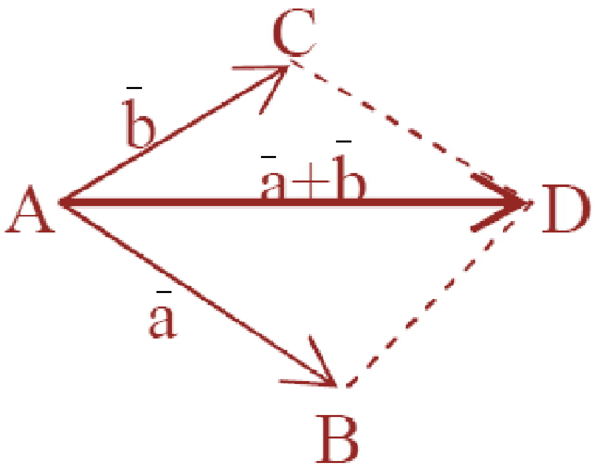
3- Opposite Vector (Negative of Vector)

We say that two vectors are negative of the other if they have the same length but are oppositely directed, and represented by ($-\bar{a}$).

4- Addition

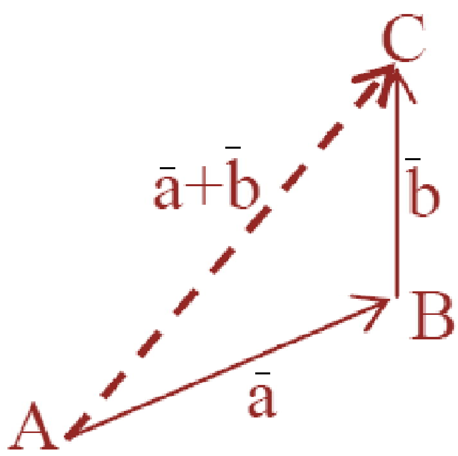
a- if $\bar{a}$ an $\bar{b}$ are drawn from the same point, or origin, then the sum of two vectors $\bar{a}$ and $\bar{b}$ is defined by the familiar parallelogram law; i.e.

$$\bar{a} + \bar{b} = \vec{AB} + \vec{AC} = \vec{AD}$$



b- if ā and b̄ two vectors, and b̄ starting from the terminal point of ā, then the sum of two vectors (ā and b̄) is the vector from the starting point of ā to the terminal point of b̄

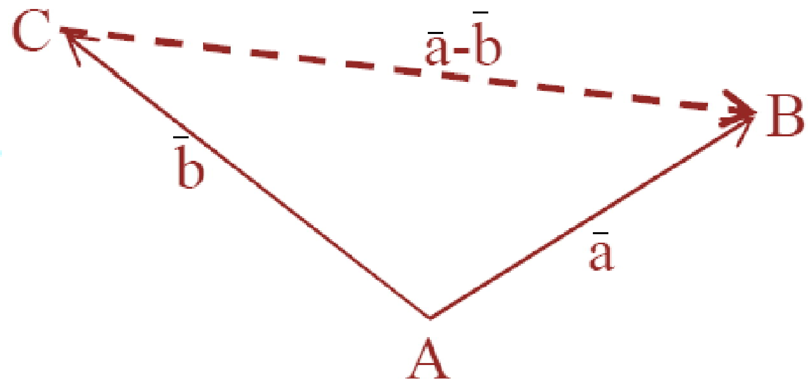
$$\bar{a} + \bar{b} = \vec{AB} + \vec{BC} = \vec{AC}$$



5- Subtraction

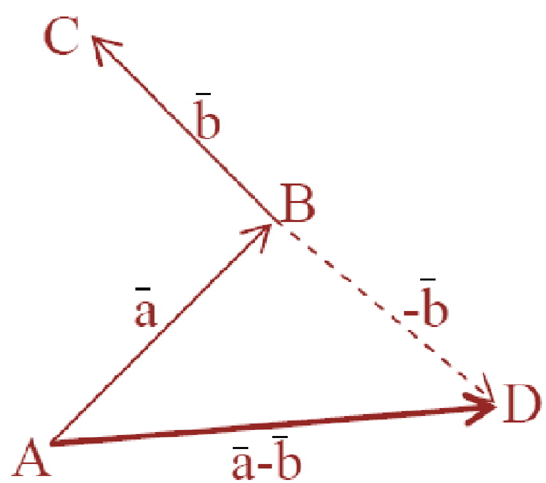
a- if ā an b̄ are drawn from the same point, or origin, then the difference of two vectors ā and b̄ is defined by draw the vector from the tip of b̄ to the tip of ā (triangular law).

$$\overline{a} - \overline{b} = \vec{AB} - \vec{AC} = \vec{CB}$$



b- if ā and b̄ two vectors, and b̄ starting from the terminal point of ā, then the difference of two vectors (ā and b̄) is define by find first the opposite vector of b̄ (-b̄) and then use triangular law.

$$\overline{a} - \overline{b} = \vec{AB} - \vec{BC} = \vec{AD}$$



6- Multiplication by Scalars

If ā is vector and k is scalar, then kā define as follow :-

1- If **k>0** then **k$\bar{a}$** is a vector has same direction of $\bar{\mathbf{a}}$ and its length equal to **k** time of length of $\bar{\mathbf{a}}$.

2- If **k<0** then **k$\bar{a}$** is a vector has opposite direction of $\bar{\mathbf{a}}$ and its length equal to absolute value of **k** time of length of $\bar{\mathbf{a}}$.

Notes :

1- $\bar{a} + \bar{b} = \bar{b} + \bar{a}$

2- $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$

3- $k(\bar{a} + \bar{b}) = k\bar{a} + k\bar{b}$; where k : any number

2-2 Unit Vector

Unite Vector is vector has unit length

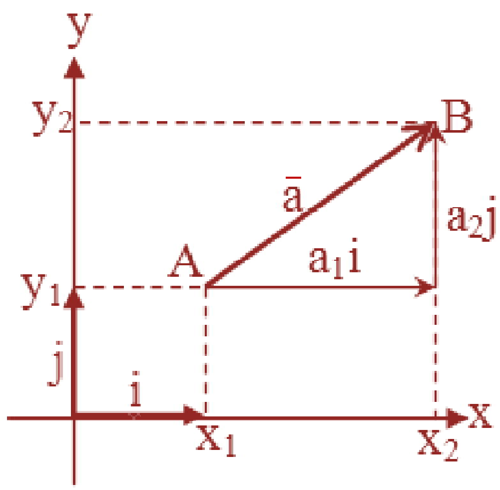
It is often convenient to be able to refer vector expressions to a Cartesian frame of reference. To provide for this we define **i** , **j** and **k** to be vectors of unit length directed, respectively, along the positive **x** , **y** and **z** axes of a right-handed rectangular coordinate system.

Two Dimension

A(x_1,y_1) , B(x_2,y_2)

$$\overrightarrow{AB} = \bar{a} = a_1 i + a_2 j$$

$$= (x_2 - x_1)i + (y_2 - y_1)j$$

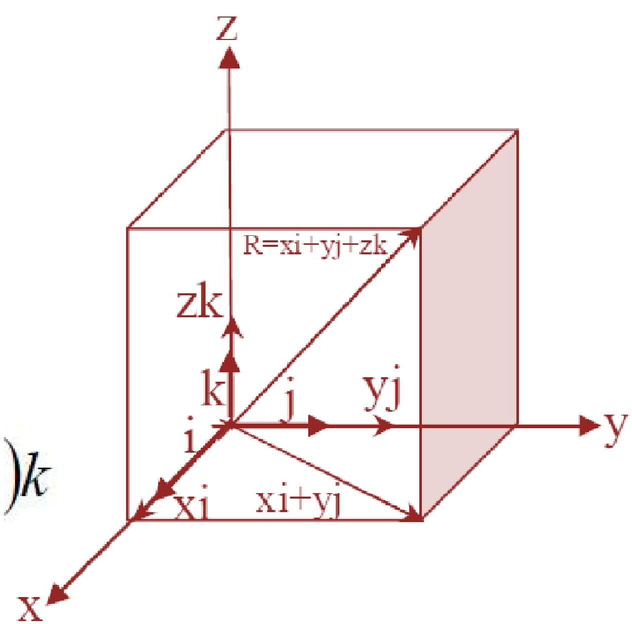


Three Dimension

A(x_1,y_1, z_1) , B(x_2,y_2 ,z_2)

$$\overrightarrow{AB} = \bar{a} = a_1 i + a_2 j + a_3 k$$

$$= (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$



Notes :

if $\bar{a} = a_1 i + a_2 j + a_3 k$ and $\bar{b} = b_1 i + b_2 j + b_3 k$ any vectors :

1- $\bar{a} = \bar{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$

2- $\bar{a} \pm \bar{b} = (a_1 \pm b_1)i + (a_2 \pm b_2)j + (a_3 \pm b_3)k$

3- $c\bar{a} = ca_1 i + ca_2 j + ca_3 k$ (c : any number)

4- vector length $|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

5- for any vector $\bar{a} \neq 0$ there is a unit vector has same direction and find it by relation

$$\frac{\bar{a}}{|\bar{a}|}$$

6- The vector $\bar{u}$ is parallel to vector $\bar{v}$ if there is some scalar $c \neq 0$ such that $\bar{u} = c\bar{v}$

EX1:- Find the unit vector in direction of the vector $\bar{a} = 3i - 4j$

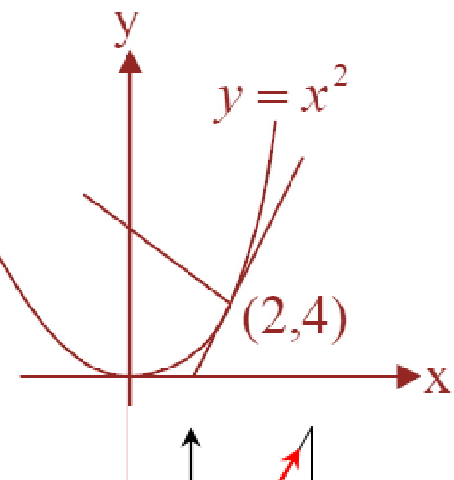
$$|\bar{a}| = \sqrt{a_1^2 + a_2^2} = \sqrt{25} = 5$$

$$\frac{\bar{a}}{|\bar{a}|} = \frac{3}{5}i - \frac{4}{5}j$$

EX2:- Find unit vectors tangent and normal to the curve $y=x^2$ at the point (2,4), in the concavity direction of the curve.

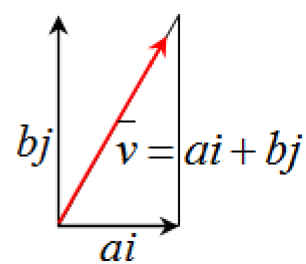
Solution :

The slope of the line tangent to the curve at the point (2,4) is



$$\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx}\Big|_{x=2} = 2 \times 2 = 4 = m_1$$

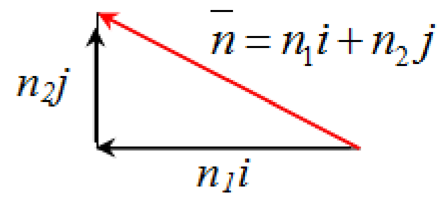
$$slope = m_1 = \frac{b}{a} = \frac{4}{1}$$



then the vector is $\bar{v} = i + 4j$ $\quad |\bar{v}| = \sqrt{1^2 + 4^2} = \sqrt{17}$

$$\frac{\bar{v}}{|\bar{v}|} = \frac{1}{\sqrt{17}}i + \frac{4}{\sqrt{17}}j$$ (the unit vector of the tangent to the curve at (2,4)

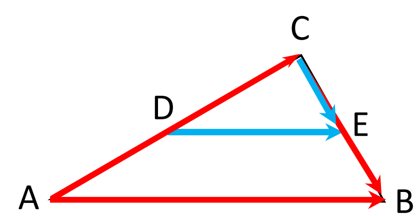
slope of the normal $m_2 = -\frac{1}{4} = \frac{n_2}{n_1}$



$\bar{\text{n}} = n_1 i + n_2 j = -4i + j$

then the unit vector of the normal is $-\frac{4}{\sqrt{17}} i + \frac{1}{\sqrt{17}} j$

EX3 : Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and has half its length.



$\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$

Let $\overrightarrow{DE}$ be the line joining the midpoints of sides $\overrightarrow{AC}$ and $\overrightarrow{CB}$. Then

$$\overrightarrow{DE} = \overrightarrow{DC} + \overrightarrow{CE} = \frac{1}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{CB} = \frac{1}{2}\left(\overrightarrow{AC} + \overrightarrow{CB}\right) = \frac{1}{2}\overrightarrow{AB}$$

EX4: Let $\bar{u}$ be represented by the directed line segment from (0,0) to (3,2), and let $\bar{v}$ be represented by the directed line segment from (1,2) to (4,4). Show that $\bar{u}=\bar{v}$.

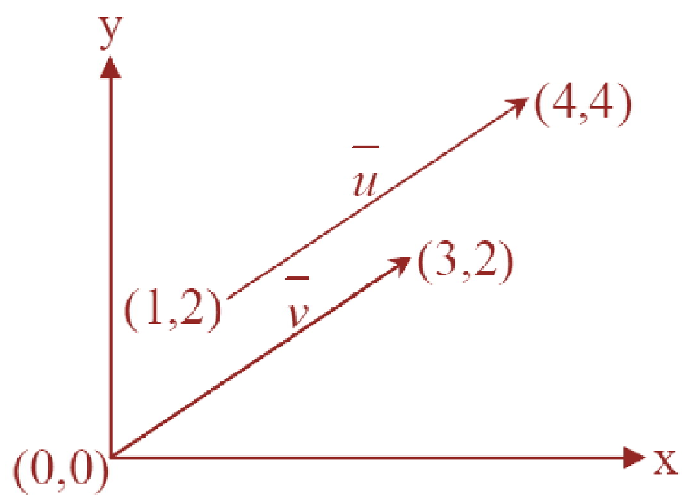
$$\bar{v} = (3-0)i + (2-0)j = 3i + 2j$$

$$\bar{u} = (4-1)i + (4-2)j = 3i + 2j$$

then $\quad \bar{u} = \bar{v}$

$$|\bar{u}| = \sqrt{3^2 + 2^2} = \sqrt{13} \quad , \quad |\bar{v}| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

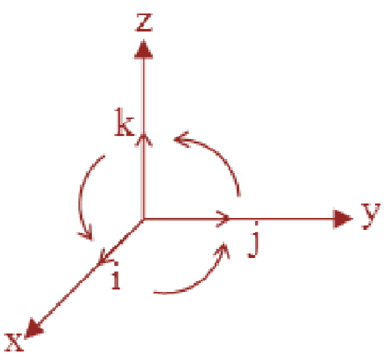
$$\text{slope of } \bar{u} = \frac{2}{3} \quad , \quad \text{slope of } \bar{v} = \frac{2}{3}$$



2-3 Dot or Scalar Product

The dot or scalar product of two vectors $\bar{a}$ and $\bar{b}$ denoted by $\bar{a} \cdot \bar{b}$ (read: $\bar{a}$ dot $\bar{b}$) is defined as the product of the magnitudes of $\bar{a}$ and $\bar{b}$ and the cosine of the angle between them. In symbols,

$$\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos\theta$$



Note that $\bar{a} \cdot \bar{b}$ is a scalar and not a vector.

Notes

1. $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$
2. $\bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}$
3. $m(\bar{a} \cdot \bar{b}) = (m\bar{a}) \cdot \bar{b} = \bar{a} \cdot (m\bar{b}) = (\bar{a} \cdot \bar{b})m$, ***where m is a scalar***
4. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1,$

   $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$ (Orthogonal)
5. if $\bar{a} = a_1 i + a_2 j + a_3 k$ and $\bar{b} = b_1 i + b_2 j + b_3 k$ any vectors :

   then $\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

   $$\bar{a} \cdot \bar{a} = a_1^2 + a_2^2 + a_3^2 = |\bar{a}|^2$$
6. If $\bar{a} \cdot \bar{b} = \bar{0}$ **also $\bar{a}$ and $\bar{b}$ are not null vectors ,**

   then $\bar{a}$ and $\bar{b}$ are perpendicular

EX1 :- Prove that the line connected between points A(1,2) and B(-2,-4) is orthogonal (normal) on the line connected between points C(6,4) and D(12,1).

$$\overrightarrow{AB} = (-2-1)i + (-4-2)j = -3i - 6j$$

$$\overrightarrow{CD} = (12-6)i + (1-4)j = 6i - 3j$$

$$\overrightarrow{AB}.\overrightarrow{CD} = (-3\times 6) + (-6\times(-3)) = -18 + 18 = 0$$

EX2 :- For $\bar{u} = 3i - j + 2k$, $\bar{v} = -4i + 2k$, $\bar{w} = i - j - 2k$ and $\bar{z} = 2i - k$, find the angle between (a) $\bar{u}$ and $\bar{v}$, (b) $\bar{u}$ and $\bar{w}$, (c) $\bar{v}$ and $\bar{z}$.

$$\text{(a)}\ \bar{u}.\bar{v} = |\bar{u}||\bar{v}|\cos\theta \Rightarrow \cos\theta = \frac{\bar{u}.\bar{v}}{|\bar{u}||\bar{v}|} = \frac{3\times(-4)+(-1)\times 0+2\times 2}{\sqrt{3^2+(-1)^2+2^2}\sqrt{(-4)^2+2^2}}$$

$$= \frac{-8}{\sqrt{14}\sqrt{20}}$$

$$\Rightarrow \theta \cong 118.56^\circ \quad \text{or} \quad \theta \cong 2.069 \text{ radians}$$

$$\text{(b)}\ \bar{u}.\bar{w} = |\bar{u}||\bar{w}|\cos\theta \Rightarrow \cos\theta = \frac{\bar{u}.\bar{w}}{|\bar{u}||\bar{w}|} = \frac{3+1-4}{\sqrt{14}\sqrt{6}} = 0$$

Since $\bar{u}.\bar{w} = 0$, $\bar{u}$ and $\bar{w}$ are orthogonal vectors,

and furthermore , $\theta = \pi/2$ radians

$$\text{(c)}\ \bar{v}.\bar{z} = |\bar{v}||\bar{z}|\cos\theta \Rightarrow \cos\theta = \frac{\bar{v}.\bar{z}}{|\bar{v}||\bar{z}|} = \frac{-8+0-2}{\sqrt{20}\sqrt{5}} = \frac{-10}{\sqrt{100}} = -1$$

consequently , $\theta = \pi$ radians

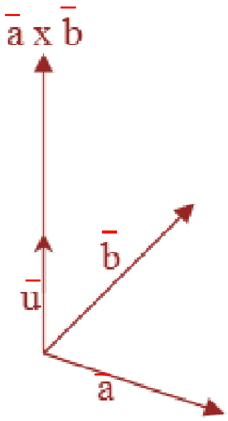
2-4 Cross or Vector Product

The cross or vector product of $\bar{a}$ and $\bar{b}$ is a vector $\bar{c} = \bar{a} \times \bar{b}$ (read: $\bar{a}$ cross $\bar{b}$) and is defined as the product of the magnitudes of $\bar{a}$ and $\bar{b}$ and the sine of the angle between them. In symbols,

$\bar{a} \times \bar{b} = |\bar{a}||\bar{b}|\, \bar{u} \sin\theta$

where $\bar{u}$ is a unit vector indicating the direction of $\bar{a} \times \bar{b}$.

The direction of the vector $\bar{c} = \bar{a} \times \bar{b}$ is perpendicular to the plane of $\bar{a}$ and $\bar{b}$



Notes

1. $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$

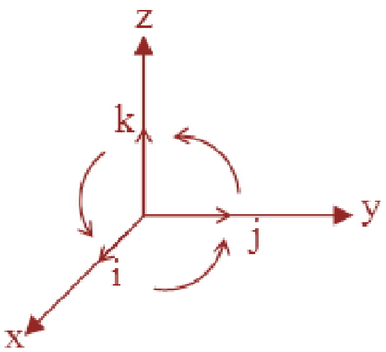
2. $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$

3. $m(\bar{a} \times \bar{b}) = (m\bar{a}) \times \bar{b} = \bar{a} \times (m\bar{b}) = (\bar{a} \times \bar{b})m$, *where m is a scalar*

4. $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0,$

   $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \ \mathbf{j} \times \mathbf{k} = \mathbf{i}, \ \mathbf{k} \times \mathbf{i} = \mathbf{j}$

   $j \times i = -k, \ k \times j = -i, \ i \times k = -j$



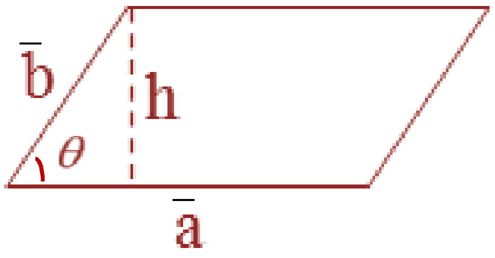
5. if $\bar{a} = a_1 i + a_2 j + a_3 k$ and $\bar{b} = b_1 i + b_2 j + b_3 k$ any vectors :

$$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

6. $|\bar{a} \times \bar{b}|$ **= the area of a parallelogram with sides $\bar{a}$ and $\bar{b}$.**

$$area = |\bar{a}| h$$

$$\sin\theta = \frac{h}{|\bar{b}|} \Rightarrow h = |\bar{b}| \sin\theta$$



$$|\bar{a} \times \bar{b}| = |\bar{u}||\bar{a}||\bar{b}| \sin\theta = |\bar{a}||\bar{b}| \sin\theta$$

7. If $\bar{a} \times \bar{b} = 0$ and neither $\bar{a}$ nor $\bar{b}$ is a null vector, then $\bar{a}$ and $\bar{b}$ are parallel.

2-5 Triple Products

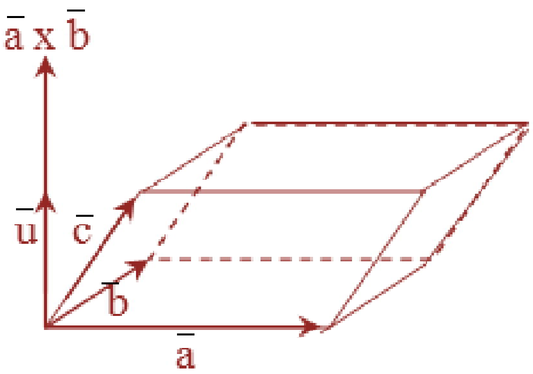
Dot and cross multiplication of three vectors, $\bar{a}$, $\bar{b}$ and $\bar{c}$ may produce meaningful products of the form $(\bar{a} \cdot \bar{b})\bar{c}$, $\bar{a} \cdot (\bar{b} \times \bar{c})$, and $\bar{a} \times (\bar{b} \times \bar{c})$.

Notes

1. $(\bar{a} \cdot \bar{b})\bar{c} \neq \bar{a}(\bar{b} \cdot \bar{c})$

2. $\bar{a} \cdot (\bar{b} \times \bar{c}) = \bar{b} \cdot (\bar{c} \times \bar{a}) = \bar{c} \cdot (\bar{a} \times \bar{b})$ **= volume of a parallelepiped having $\bar{a}$, $\bar{b}$ and $\bar{c}$ as edges. (*scalar triple product)***

$$\bar{a}.(\bar{b} \times \bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



3. $(\bar{a} \times \bar{b}) \times \bar{c} \neq \bar{a} \times (\bar{b} \times \bar{c})$

4. $(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{a} . \bar{c})\bar{b} - (\bar{b} . \bar{c})\bar{a}$

$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} . \bar{c})\bar{b} - (\bar{a} . \bar{b})\bar{c}$

EX1 :- Find $\bar{a}\times\bar{b}$ if $\bar{a}=4i-k$, $\bar{b}=-2i+j+3k$

$$\bar{a}\times\bar{b}=\begin{vmatrix} i & j & k \\ 4 & 0 & -1 \\ -2 & 1 & 3 \end{vmatrix}\begin{matrix} i & j \\ 4 & 0 \\ -2 & 1 \end{matrix} =(0+1)i+(2-12)j+(4-0)k=i-10j+4k$$

EX2:- Find the area of the parallelogram which its two adjacent sides are $\bar{a}=i-2j+k$, $\bar{b}=2i-k$

$$\bar{a}\times\bar{b}=\begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & 0 & -1 \end{vmatrix}\begin{matrix} i & j \\ 1 & -2 \\ 2 & 0 \end{matrix} =2i+3j+4k$$

$$\left|\bar{a}\times\bar{b}\right|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$$

EX3:- Find the volume of the parallelepiped having :

$\bar{a}=i+k$, $\bar{b}=2j+k$ and $\bar{c}=i-j-k$ as adjacent edges.

$$\bar{\text{a}}.\left(\bar{b}\times\bar{c}\right)=(i+k).\begin{vmatrix} i & j & k \\ 0 & 2 & 1 \\ 1 & -1 & -1 \end{vmatrix}\begin{matrix} i & j \\ 0 & 2 \\ 1 & -1 \end{matrix} = (i+k).\left(-i+j-2k\right)=-3$$

volume is $|-3|=3$

or $(\bar{a} \times \bar{b}).\bar{c}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} i & j \\ 1 & 0 \\ 0 & 2 \end{matrix} = -2i - j + 2k = \bar{m}$$

$\bar{m}.\bar{c} = -2 \times 1 + (-1) \times (-1) + (2) \times (-1) = -3$, the volume is $|-3| = 3$

Example

Prove that $(\bar{a} \cdot \bar{b})\bar{c} \neq \bar{a}(\bar{b} \cdot \bar{c})$

Let : $\bar{a} = a_1 i + a_2 j + a_3 k$, $\bar{b} = b_1 i + b_2 j + b_3 k$ and $\bar{c} = c_1 i + c_2 j + c_3 k$

$$(\bar{a} \cdot \bar{b})\bar{c} = [(a_1 i + a_2 j + a_3 k) \cdot (b_1 i + b_2 j + b_3 k)]c_1 i + c_2 j + c_3 k$$

$$= (a_1 b_1 + a_2 b_2 + a_3 b_3)(c_1 i + c_2 j + c_3 k)$$

$$= (a_1 b_1 + a_2 b_2 + a_3 b_3)c_1 i + (a_1 b_1 + a_2 b_2 + a_3 b_3)c_2 j + (a_1 b_1 + a_2 b_2 + a_3 b_3)c_3 k$$

$$\bar{a}(\bar{b} \cdot \bar{c}) = a_1 i + a_2 j + a_3 k[(b_1 i + b_2 j + b_3 k) \cdot (c_1 i + c_2 j + c_3 k)]$$

$$= (a_1 i + a_2 j + a_3 k)(b_1 c_1 + b_2 c_2 + b_3 c_3)$$

$$= (b_1 c_1 + b_2 c_2 + b_3 c_3)a_1 i + (b_1 c_1 + b_2 c_2 + b_3 c_3)a_2 j + (b_1 c_1 + b_2 c_2 + b_3 c_3)a_3 k$$

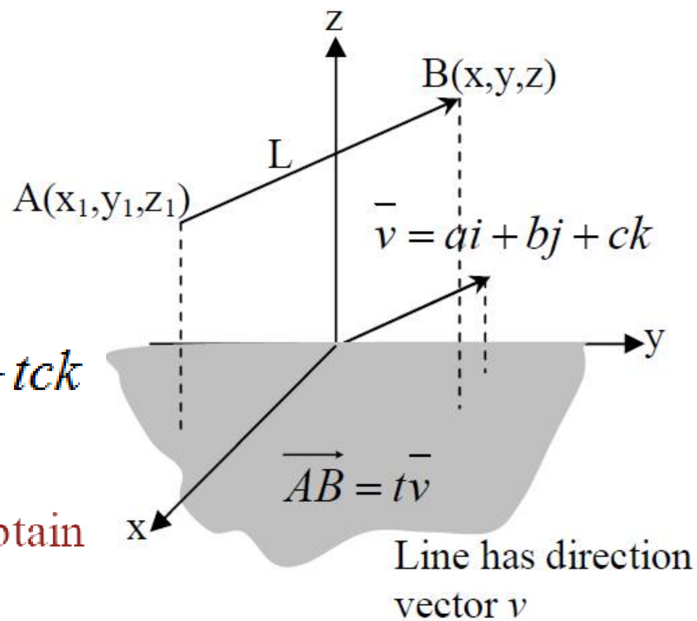
then : $(\bar{a} \cdot \bar{b})\bar{c} \neq \bar{a}(\bar{b} \cdot \bar{c})$

2-6 Lines and Planes in Space

A- Lines

consider the line **L** through the point **A(x_1,y_1,z_1)** and parallel to the vector

$$\bar{v} = ai + bj + ck$$



$\overrightarrow{AB} = t\bar{v}$, where **t** is a scalar

$$\overrightarrow{AB} = (x - x_1)i + (y - y_1)j + (z - z_1)k$$

$$(x - x_1)i + (y - y_1)j + (z - z_1)k = tai + tbj + tck$$

By equation corresponding components, we obtain equations

$$\left.\begin{aligned} x - x_1 &= ta \\ y - y_1 &= tb \\ z - z_1 &= tc \end{aligned}\right\} t = \frac{x - x_1}{a} \;,\; t = \frac{y\text{-}y_1}{b} \;,\; t = \frac{z\text{-}z_1}{c}$$

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}\;.....(a)$$

The above equation (a) represent the line equation through the point $A(x_1,y_1,z_1)$ and parallel to the vector $\bar{v}=ai+bj+ck$

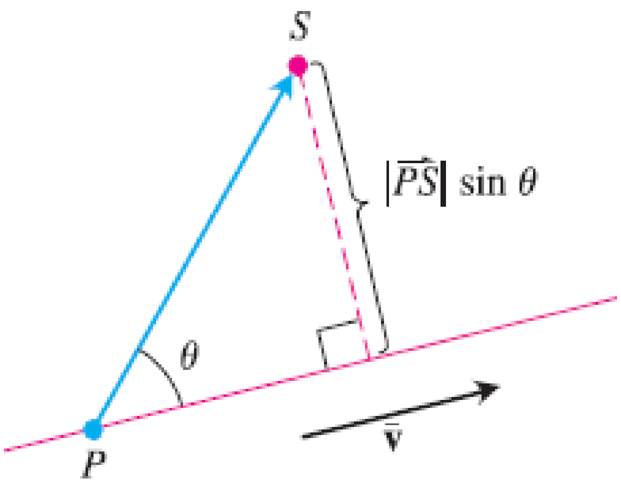
Distance from a Point *S* to a Line Through *P* Parallel to $\bar{v}$

$$d=|\overrightarrow{PS}|\sin\theta$$

$$|\overline{PS}\times\bar{v}|=|\bar{u}||\overline{PS}||\bar{v}|\sin\theta=|\overline{PS}||\bar{v}|\sin\theta$$

$$\sin\theta=\frac{|\overline{PS}\times\bar{v}|}{|\overline{PS}||\bar{v}|}$$

$$d=\frac{|\overrightarrow{PS}\times\bar{\mathbf{v}}|}{|\bar{\mathbf{v}}|}$$



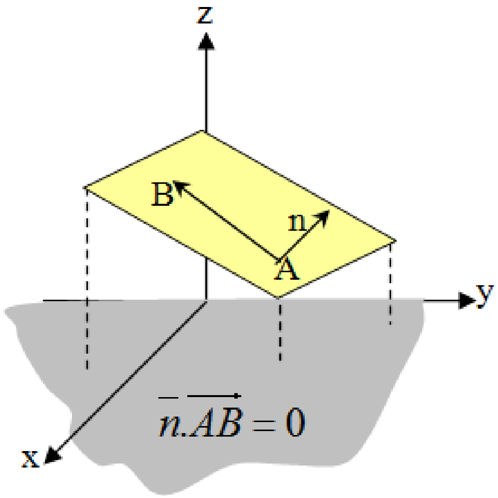
B- Planes

Consider the plane containing the point **A(x_1,y_1,z_1)** having a nonzero normal vector $\bar{n} = ai + bj + ck$ as shown in figure. This plane consists of all points **B(x,y,z)** for which vector $\overrightarrow{AB}$ is orthogonal to $\bar{n}$. Using the dot product, we have

$$\bar{n}.\overrightarrow{AB} = 0$$

$$(ai + bj + ck).[(x - x_1)i + (y - y_1)j + (z - z_1)k] = 0$$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$



The above equation (b) represent the equation of a plane in space, which it containing the point **A(x_1,y_1,z_1)** and having a normal vector $\bar{n} = ai + bj + ck$

Equation (b) can be rewritten to obtain the general form of the equation of a plane in space.

$$ax + by + cz = d$$

$$ax_1 + by_1 + cz_1 = d$$

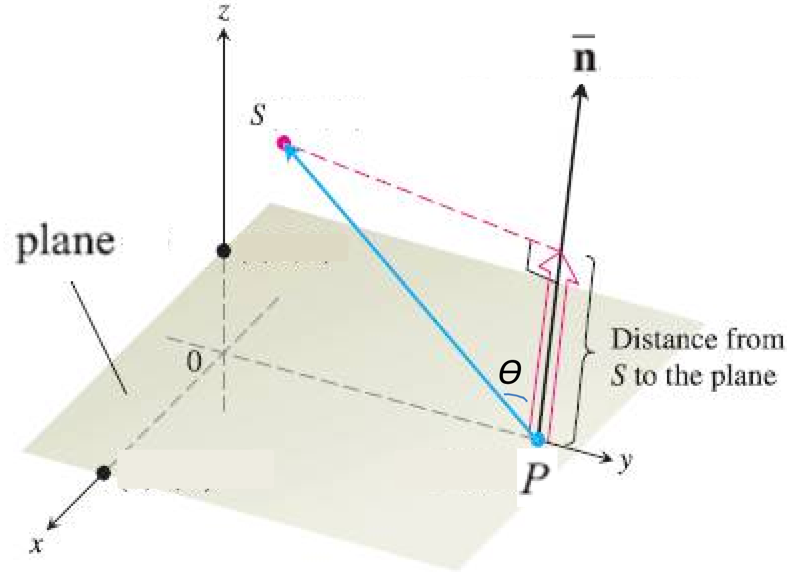
Distance from a Point to a Plane

If P is a point on a plane with normal $\overline{\mathbf{n}}$, then the distance from any point S to the plane is

$$d = \left|\overrightarrow{PS}\right| \cos\theta$$

$$\overrightarrow{PS}.\overline{n} = \left|\overrightarrow{PS}\right|\left|\overline{n}\right| \cos\theta \Rightarrow \cos\theta = \frac{\overrightarrow{PS}.\overline{n}}{\left|\overrightarrow{PS}\right|\left|\overline{n}\right|}$$

$$d = \left|\overrightarrow{PS} \cdot \frac{\overline{\mathbf{n}}}{\left|\overline{\mathbf{n}}\right|}\right|$$



Example

Find the equation of the line containing the point (1,2,3) and parallel to vector

$\bar{v} = i + 7j - 2k$

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \Rightarrow \frac{x-1}{1} = \frac{y-2}{7} = \frac{z-3}{-2}$$

or x=1+t , y=2+7t , z=3-2t

Example

Find the equation of the plane containing the point (3,-1,7) and the vector $3i - 2j + k$ normal on it.

$$a(x\text{-}x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$3(x-3) + (-2)(y+1) + (1)(z-7) = 0$$

$$3x\text{-}9\text{-}2y\text{-}2 + z\text{-}7 = 0$$

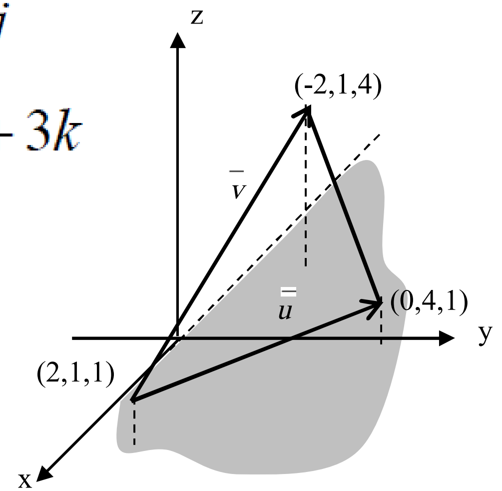
$$3x\text{-}2y + z\text{-}18 = 0 \Rightarrow 3x\text{-}2y + z = 18$$

Example

Find the general equation of the plane containing the points (2,1,1), (0,4,1) and (-2,1,4).

$$\bar{u} = (0-2)i + (4-1)j + (1-1)k \Rightarrow \bar{u} = -2i + 3j$$

$$\bar{v} = (-2-2)i + (1-1)j + (4-1)k \Rightarrow \bar{v} = -4i + 3k$$



$$\bar{n} = \bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix} = 9i + 6j + 12k$$

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

$$9(x-2) + 6(y-1) + 12(z-1) = 0 \Rightarrow 3x + 2y + 4z - 12 = 0$$

$$\text{or} \quad 3x + 2y + 4z = 12$$

Example

Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

$$P(1, 3, 0) \qquad \bar{\mathbf{v}} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}.$$

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

$$\overrightarrow{PS} \times \bar{\mathbf{v}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

$$d = \frac{|\overrightarrow{PS} \times \bar{\mathbf{v}}|}{|\bar{\mathbf{v}}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

Example

Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

$$\bar{\mathbf{n}} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

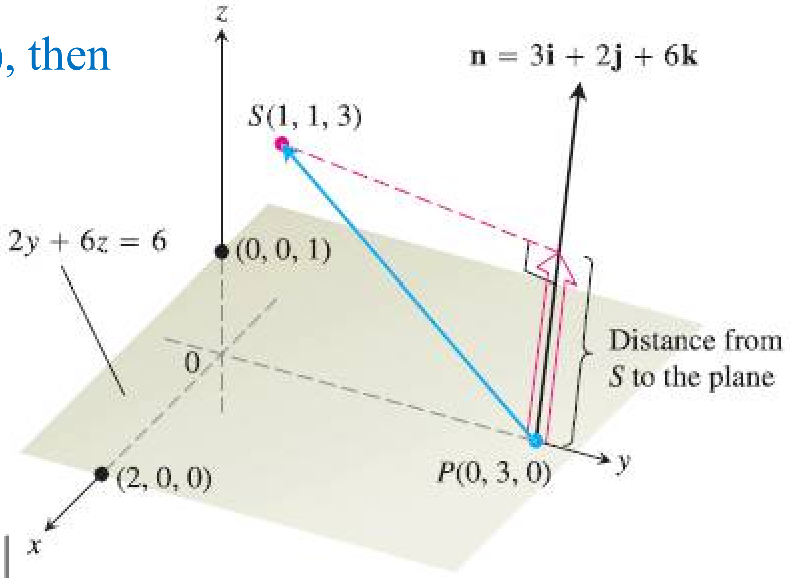
The points on the plane easiest to find from the plane's equation are the intercepts with x-axis or y-axis or z-axis.

If we take P to be the y-intercept (0, 3, 0), then

$$\begin{aligned}\overrightarrow{PS} &= (1-0)\mathbf{i} + (1-3)\mathbf{j} + (3-0)\mathbf{k} \\ &= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},\end{aligned}$$

$$|\bar{\mathbf{n}}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7$$

$$\begin{aligned}d &= \left|\overrightarrow{PS} \cdot \frac{\bar{\mathbf{n}}}{|\bar{\mathbf{n}}|}\right| \\ &= \left|(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)\right| \\ &= \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \frac{17}{7}.\end{aligned}$$



Or

$$\frac{x-x_1}{a}=\frac{y\text{-}y_1}{b}=\frac{z\text{-}z_1}{c}\text{.....(a)}$$

$$\frac{x-1}{3}=\frac{y-1}{2}=\frac{z-3}{6}=t$$

$$x=3t+1\ ,\ y=2t+1\ ,\ z=6t+3$$

$$3(3t+1)+2(2t+1)+6(6t+3)=6\quad,\quad 49t=\text{-}17\ ,\quad t=-\frac{17}{49}$$

$$A\ \left(x=-3*\frac{17}{49}+1\ ,\ y=-2*\frac{17}{49}+1\ ,\ z=-6*\frac{17}{49}+3\right)$$

$$\overrightarrow{AS}=(1+3*\frac{17}{49}-1)i+(1+2*\frac{17}{49}-1)j+(3+6*\frac{17}{49}-3)k$$

$$=3*\frac{17}{49}i+2*\frac{17}{49}j+6*\frac{17}{49}k$$

$$d=\left|\overrightarrow{AS}\right|=\sqrt{\left(\frac{3*17}{49}\right)^2+\left(\frac{2*17}{49}\right)^2+\left(\frac{6*17}{49}\right)^2}=\frac{17}{7}$$

Example

Find the point where the line $x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$

intersects the plane $3x + 2y + 6z = 6$.

From line equations : $\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8$$

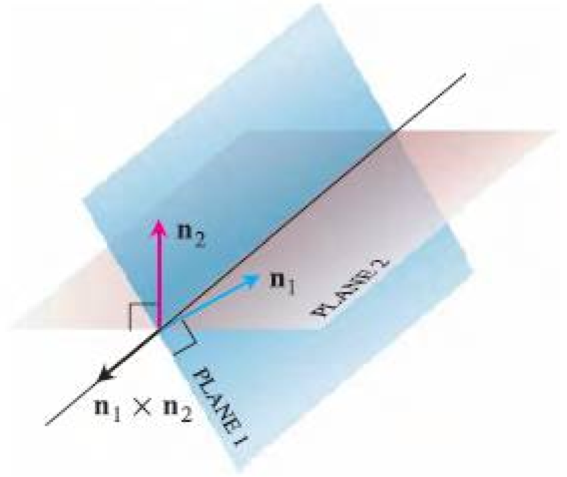
$$t = -1.$$

The point of intersection is

$$(x, y, z)\big|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right)$$

Example

Find parametric equations for the line in which the planes
3x - 6y - 2z = 15 and 2x + y - 2z = 5 intersect.



From planes equations :
$n_1=3i - 6j - 2k$, $n_2=2i + j -2k$

The line of intersection of two planes is perpendicular to both planes' normal vectors n_1 and n_2 and therefore parallel to n_1 x n_2.

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Substituting x=0 or y=0 or *z = 0 in* the plane equations

If z=0 , 3x - 6y=15 , 2x + y=5 , by solving these equations :
(3, -1, 0)

$$\frac{x - x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \text{.....(a)}$$

$$x = 3 + 14t, \qquad y = -1 + 2t, \qquad z = 15t.$$

Example

Find a plane through the points $P_1(1\ 2, 3)$, $P_2(3, 2, 1)$ and perpendicular to the plane $4x - y + 2z = 7$.

$\overrightarrow{P_1P_2} = 2i - 2k$

From plane equ. $\bar{n} = 4i - j + 2k$

A vector normal to the desired plane is

$$\overrightarrow{P_1P_2} \times \bar{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 4 & -1 & 2 \end{vmatrix} = -2\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}$$

choosing P_1 (1 2 3) as a point on the plane

$$(-2)(x-1) + (-12)(y-2) + (-2)(z-3) = 0$$

$$-2x - 12y - 2z = -32 \Rightarrow x + 6y + z = 16$$

Example

The line L: x=3+2t, y=2t, z=t intersect the plane x + 3y - z = -4 in a point P. Find the coordinates of P and find equations for the line in the plane through P perpendicular to L.

From line equations : (3+2t , 2t , t)

$$(3 + 2t) + 3(2t) - t = -4 \Rightarrow t = -1 \; :$$

the point is $(1, -2, -1)$.

The required line must be perpendicular to both the given line and to the normal, and hence is parallel to :

$$\bar{u} = \bar{v} \times \bar{n} = \begin{vmatrix} i & j & k \\ 2 & 2 & 1 \\ 1 & 3 & -1 \end{vmatrix} = -5i + 3j + 4k$$

$$\frac{x - x_1}{a} = \frac{y\text{-}y_1}{b} = \frac{z\text{-}z_1}{c} \;.....\text{(a)}$$

$x = 1 - 5t$, $y = -2 + 3t$, and $z = -1 + 4t$.

Part Two : Vectors Analysis

2-7 Unit Tangent Vector (T) and Unit Normal (N) Vector

Reference : Thomas G.B., Weir, Hass, Giordano. Thomas's calculus (Section 13.3 and 13.4)

$$\bar{v}(t) = v_1(t)i + v_2(t)j + v_3(t)k$$

where $\bar{v}(t)$: vector function and $v_1(t), v_2(t)$ and $v_3(t)$: scalar functions

$$T = \frac{\dfrac{d\bar{v}(t)}{dt}}{\left|\dfrac{d\bar{v}(t)}{dt}\right|}$$

Unit Tangent Vector

$$|T| = 1 \Rightarrow |T|^2 = 1 \Rightarrow T.T = 1 \Rightarrow \frac{d}{dt}[T.T] = T.\frac{dT}{dt} + T.\frac{dT}{dt} = 0$$

$$= 2T.\frac{dT}{dt} = 0 \Rightarrow T.\frac{dT}{dt} = 0$$

$$N = \frac{\dfrac{dT}{dt}}{\left|\dfrac{dT}{dt}\right|}$$

Unit Normal Vector

Example

Find the unit tangent vector and unit normal vector for the curve represented by

(a) $\bar{r}(t) = \frac{t^3}{3} i + \frac{t^2}{2} j$ **at t=2**

$$\bar{r}(t) = \frac{t^3}{3} i + \frac{t^2}{2} j \ ; \ \frac{d\bar{r}(t)}{dt} = t^2 i + tj$$

$$\left| \frac{d\bar{r}(t)}{dt} \right| = \sqrt{t^4 + t^2} = t\sqrt{t^2+1} \ ; \ T = \frac{t^2}{t\sqrt{t^2+1}} i + \frac{t}{t\sqrt{t^2+1}} j$$

$$T = \frac{t}{\sqrt{t^2+1}} i + \frac{1}{\sqrt{t^2+1}} j$$

$$\frac{dT}{\mathrm{dt}} = \frac{\sqrt{t^2+1} - t(\frac{1}{2} * 2t * (t^2+1)^{-\frac{1}{2}})}{t^2+1} i + \frac{(-\frac{1}{2} * 2t * (t^2+1)^{-\frac{1}{2}})}{t^2+1} j$$

$$= \frac{1}{(t^2+1)^{\frac{3}{2}}} i - \frac{t}{(t^2+1)^{\frac{3}{2}}} j$$

$$\left|\frac{dT}{dt}\right| = \sqrt{\left[\frac{1}{(t^2+1)^{\frac{3}{2}}}\right]^2 + \left[\frac{t}{(t^2+1)^{\frac{3}{2}}}\right]^2} = \sqrt{\frac{1+t^2}{(t^2+1)^3}} = \frac{1}{t^2+1}$$

$$N = \frac{1}{\sqrt{t^2+1}} i - \frac{t}{\sqrt{t^2+1}} j$$

at t=2

$$(T)_{t=2} = \frac{2}{\sqrt{5}} i + \frac{1}{\sqrt{5}} j$$

$$(N)_{t=2} = \frac{1}{\sqrt{5}} i - \frac{2}{\sqrt{5}} j$$

(b) $\mathbf{r}(t) = (e^t \sin 2t)\mathbf{i} + (e^t \cos 2t)\mathbf{j} + 2e^t\mathbf{k}$ **at t=0**

$$\frac{d\bar{r}(t)}{dt} = (e^t \sin 2t + 2e^t \cos 2t)\,\mathbf{i} + (e^t \cos 2t - 2e^t \sin 2t)\,\mathbf{j} + 2e^t\mathbf{k}$$

$$\left|\frac{d\bar{r}(t)}{dt}\right| = \sqrt{(e^t \sin 2t + 2e^t \cos 2t)^2 + (e^t \cos 2t - 2e^t \sin 2t)^2 + (2e^t)^2} = 3e^t$$

$$T = \left(\tfrac{1}{3} \sin 2t + \tfrac{2}{3} \cos 2t\right)\mathbf{i} + \left(\tfrac{1}{3} \cos 2t - \tfrac{2}{3} \sin 2t\right)\mathbf{j} + \tfrac{2}{3}\mathbf{k}$$

$$\frac{dT}{dt} = \left(\tfrac{2}{3} \cos 2t - \tfrac{4}{3} \sin 2t\right)\mathbf{i} + \left(-\tfrac{2}{3} \sin 2t - \tfrac{4}{3} \cos 2t\right)\mathbf{j}$$

$$\left|\frac{dT}{dt}\right| = \sqrt{\left(\tfrac{2}{3} \cos 2t - \tfrac{4}{3} \sin 2t\right)^2 + \left(-\tfrac{2}{3} \sin 2t - \tfrac{4}{3} \cos 2t\right)^2}$$

$$N = \frac{\left(\tfrac{2}{3} \cos 2t - \tfrac{4}{3} \sin 2t\right)\mathbf{i} + \left(-\tfrac{2}{3} \sin 2t - \tfrac{4}{3} \cos 2t\right)\mathbf{j}}{\sqrt{\left(\tfrac{2}{3} \cos 2t - \tfrac{4}{3} \sin 2t\right)^2 + \left(-\tfrac{2}{3} \sin 2t - \tfrac{4}{3} \cos 2t\right)^2}}$$

at t=0

$$\mathbf{T}(0) = \tfrac{2}{3}\mathbf{i} + \tfrac{1}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k} \qquad \mathbf{N}(0) = \frac{\left(\tfrac{2}{3}\mathbf{i} - \tfrac{4}{3}\mathbf{j}\right)}{\left(\tfrac{2\sqrt{5}}{3}\right)} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

Example

At what point or points is the tangent to the curve $\bar{a}(t) = t^3 i + 5t^2 j + 10tk$

Perpendicular to the tangent at the point where *t=1*?

$$\frac{d\bar{a}(t)}{dt} = 3t^2 i + 10tj + 10k = \bar{c}$$

$$\left.\frac{d\bar{a}(t)}{dt}\right|_{t=1} = 3(1)^2 i + 10(1) j + 10k = 3i + 10j + 10k = \bar{d}$$

$$\bar{c}.\bar{d} = 0 \quad \Rightarrow 3(3t^2) + 10(10t) + 10(10) = 0 \Rightarrow 9t^2 + 100t + 100 = 0$$

$$t = \frac{-100 \pm \sqrt{10000 - 3600}}{18} = \frac{-100 \pm 80}{18} \Rightarrow \quad t = -10 \text{ or } -\frac{10}{9}$$

at $\text{t} = \text{-}10$, $\text{x} = t^3 = (-10)^3 = -1000$, $y = 5(-10)^2 = 500$, $z = 10(-10) = -100$

at $\text{t} = \frac{\text{-}10}{9}$, $\text{x} = t^3 = \left(\frac{\text{-}10}{9}\right)^3 = \frac{-1000}{729}$, $y = 5\left(\frac{\text{-}10}{9}\right)^2 = \frac{500}{81}$,

$$z = 10\left(\frac{\text{-}10}{9}\right) = \frac{-100}{9}$$

The tangent at (-1000,500,-100) and (-1000/729 , 500/81 , -100/9) are both perpendicular to the tangent at t=1.

2-8 Direction Derivative (D) and Gradient Vector (grad or ∇)

Reference : Thomas G.B., Weir, Hass, Giordano. Thomas's calculus (Section 14.5)

Let ϕ(x,y,z) be a scalar function and $\bar{r}$ any vector

$$D = \nabla \phi \cdot \frac{\bar{r}}{|\bar{r}|}$$

$$\nabla = grad \equiv \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

$$\nabla \phi = grad \ \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

Also, if ϕ is a function of a single variable u which, in turn, is a function of x, y, and z then

$$\nabla \phi = \frac{d\phi}{du}\frac{\partial u}{\partial x} i + \frac{d\phi}{du}\frac{\partial u}{\partial y} j + \frac{d\phi}{du}\frac{\partial u}{\partial z} k$$

$$= \frac{d\phi}{du}\left(\frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j + \frac{\partial u}{\partial z} k\right)$$

$$\nabla \phi = \frac{d\phi}{du} \nabla u$$

EX:- What is the directional derivative of the function $\phi(x,y,z) = xy^2 + yz^3$ at the point (2,-1,1) in the direction of the vector $i+2j+2k$?

Solution :-

$$\nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial (xy^2 + yz^3)}{\partial x} = y^2$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial (xy^2 + yz^3)}{\partial y} = 2xy + z^3$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial (xy^2 + yz^3)}{\partial z} = 3yz^2$$

$$\nabla \phi = y^2 i + \left(2xy + z^3\right) j + 3yz^2 k \;\Big|_{(2,-1,1)}$$

$$\nabla \phi = i - 3j - 3k$$

$$\frac{\bar{r}}{|\bar{r}|} = \frac{i + 2j + 2k}{\sqrt{1+4+4}} = \frac{1}{3} i + \frac{2}{3} j + \frac{2}{3} k$$

$$D = \nabla \phi . \left(\frac{1}{3} i + \frac{2}{3} j + \frac{2}{3} k\right) = (i\text{-}3j\text{-}3k) . \left(\frac{1}{3} i + \frac{2}{3} j + \frac{2}{3} k\right) = -\frac{11}{3}$$

EX $h(x, y, z) = \cos xy + e^{yz} + \ln zx, \quad P_0(1, 0, 1/2), \mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$$\frac{\partial h}{\partial x} = -y \sin xy + \frac{1}{zx} z = -y \sin xy + \frac{1}{x}$$

$$\frac{\partial h}{\partial y} = -x \sin xy + z e^{yz}$$

$$\frac{\partial h}{\partial z} = y e^{yz} + \frac{1}{z}$$

$$\nabla h = (-y\sin xy + \frac{1}{x})i + (-x\sin xy + ze^{yz})j + (ye^{yz} + \frac{1}{z})k \Big|_{(1,0,1/2)}$$

$$\nabla h = i + 1/2\,j + 2k$$

$$\frac{\overline{u}}{|\overline{u}|} = \frac{i + 2j + 2k}{\sqrt{1+4+4}} = \frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k$$

$$D = \nabla h.\left(\frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k\right)$$

$$= (\mathrm{i} + 1/2\mathrm{j} + 2\mathrm{k}).\left(\frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k\right) = 2$$

EX1 If $\bar{a} = xi + yj + zk$ and $\ell = |\bar{a}|$ prove that $\nabla f(\ell) = \frac{f'(\ell)}{\ell}\bar{a}$ where $f'(\ell) = \frac{df}{d\ell}$

$$\nabla f(\ell) = \frac{\partial f(\ell)}{\partial x} i + \frac{\partial f(\ell)}{\partial y} j + \frac{\partial f(\ell)}{\partial z} k$$

$$\ell = \sqrt{x^2 + y^2 + z^2} = |\bar{a}|$$

$$\frac{\partial f(\ell)}{\partial x} = \frac{df(\ell)}{d\ell}\frac{\partial \ell}{\partial x} = f'(\ell)\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}}(2x)$$

$$= \frac{f'(\ell)x}{\sqrt{x^2 + y^2 + z^2}} = \frac{f'(\ell)x}{|\bar{a}|} = \frac{f'(\ell)x}{\ell}$$

in same way :

$$\frac{\partial f(\ell)}{\partial y} = \frac{f'(\ell)y}{\ell}, \quad \frac{\partial f(\ell)}{\partial z} = \frac{f'(\ell)z}{\ell}$$

$$\nabla f(\ell) = \frac{f'(\ell)}{\ell} xi + \frac{f'(\ell)}{\ell} yj + \frac{f'(\ell)}{\ell} zk = \frac{f'(\ell)}{\ell}(xi + yj + zk)$$

$$\nabla f(\ell) = \frac{f'(\ell)}{\ell}\bar{a}$$

2-9 Divergence and Curl

Reference : Thomas G.B., Weir, Hass, Giordano. Thomas's calculus **(Section 16.7 and Section 16.8)**

If $\bar{f} = F_1 i + F_2 j + F_3 k$ is a vector function whose components are differentiable functions of x, y, and z , this leads to the combinations

A- divergence of the vector function $\bar{f}$

$$\nabla.\bar{f} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right).(F_1 i + F_2 j + F_3 k) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

B- curl of the vector function $\bar{f}$

$$\nabla \times \bar{f} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times (F_1 i + F_2 j + F_3 k)$$

$$= \left| \begin{array}{ccc|cc} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 & F_3 & F_1 & F_2 \end{array} \right.$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k$$

Note

(if $\bar{f}$ and $\bar{g}$: vector functions, u and v : scalar functions and the partial derivatives of $\bar{f}$, $\bar{g}$, u and v are assumed to exist, then)

1. $\nabla(u+v)=\nabla u+\nabla v$ or $grad(u+v)=grad\, u+grad\, v$
2. $\nabla\cdot\left(\bar{f}+\bar{g}\right)=\nabla\cdot\bar{f}+\nabla\cdot\bar{g}$ or $div\left(\bar{f}+\bar{g}\right)=div\,\bar{f}+div\,\bar{g}$
3. $\nabla\times\left(\bar{f}+\bar{g}\right)=\nabla\times\bar{f}+\nabla\times\bar{g}$ or $curl\left(\bar{f}+\bar{g}\right)=curl\,\bar{f}+curl\,\bar{g}$
4. $\nabla\cdot\left(u\,\bar{f}\right)=(\nabla u)\cdot\bar{f}+u\left(\nabla\cdot\bar{f}\right)$
5. $\nabla\times\left(u\,\bar{f}\right)=(\nabla u)\times\bar{f}+u\left(\nabla\times\bar{f}\right)$
6. $\nabla\cdot\left(\bar{f}\times\bar{g}\right)=\bar{g}\cdot\left(\nabla\times\bar{f}\right)-\bar{f}\cdot(\nabla\times\bar{g})$
7. $\nabla\times\left(\bar{f}\times\bar{g}\right)=(\bar{g}\cdot\nabla)\bar{f}-\bar{g}\left(\nabla\cdot\bar{f}\right)-\left(\bar{f}\cdot\nabla\right)\bar{g}+\bar{f}(\nabla\cdot\bar{g})$
8. $\nabla\left(\bar{f}\cdot\bar{g}\right)=(\bar{g}\cdot\nabla)\bar{f}+\left(\bar{f}\cdot\nabla\right)\bar{g}+\bar{g}\times\left(\nabla\times\bar{f}\right)+\bar{f}\times(\nabla\times\bar{g})$

9. $\nabla \cdot (\nabla u) = \nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$ is called Laplacian of u

$\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called Laplacian *operator*

10. $\nabla \times (\nabla u) = 0$

11. $\nabla \cdot (\nabla \times \bar{f}) = 0$

EX

If $\phi = x^2yz^3$ and $\mathbf{A} = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$, find

(b) $\nabla \cdot \mathbf{A}$, (c) $\nabla \times \mathbf{A}$, (d) div ($\phi\mathbf{A}$), (e) curl ($\phi\mathbf{A}$).

Solution :

(b) $$\nabla \cdot \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \cdot (xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$$

$$= \frac{\partial}{\partial \mathbf{x}}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) = z - 2y$$

(c) $\nabla \times \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \times (xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$

$$= \left| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & -y^2 & 2x^2y \end{matrix} \right| \begin{matrix} \mathbf{i} & \mathbf{j} \\ \partial/\partial x & \partial/\partial y \\ xz & -y^2 \end{matrix}$$

$$= \left(\frac{\partial}{\partial y}(2x^2y) - \frac{\partial}{\partial z}(-y^2) \right)\mathbf{i} + \left(\frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(2x^2y) \right)\mathbf{j} + \left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(xz) \right)\mathbf{k}$$

$$= 2x^2\mathbf{i} + (x - 4xy)\mathbf{j}$$

(d) div $(\phi\mathbf{A})$:

$\phi = x^2yz^3$ and $\mathbf{A} = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$

$$\text{div}\,(\phi\mathbf{A}) = \nabla \cdot (\phi\mathbf{A}) = \nabla \cdot (x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^3yz^4) + \frac{\partial}{\partial y}(-x^2y^3z^3) + \frac{\partial}{\partial z}(2x^4y^2z^3)$$

$$= 3x^2yz^4 - 3x^2y^2z^3 + 6x^4y^2z^2$$

(e) $\text{curl}\,(\phi \mathbf{A}) = \nabla \times (\phi \mathbf{A}) = \nabla \times (x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3yz^4 & -x^2y^3z^3 & 2x^4y^2z^3 \end{vmatrix} \begin{matrix} \mathbf{i} & \mathbf{j} \\ \partial/\partial x & \partial/\partial y \\ x^3yz^4 & -x^2y^3z^3 \end{matrix}$$

$$= (4x^4yz^3 + 3x^2y^3z^2)\mathbf{i} + (4x^3yz^3 - 8x^3y^2z^3)\mathbf{j} - (2xy^3z^3 + x^3z^4)\mathbf{k}$$

EX2 : Find **Divergence and Curl** of the field function $\bar{f}$

$$\bar{f} = \ln(x^2 + y^2)\mathbf{i} - \left(\frac{2z}{x}\tan^{-1}\frac{y}{x}\right)\mathbf{j} + (5z^3 + e^y\cos z)\mathbf{k}$$

Solution :

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx} \qquad \frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}e^u = e^u\frac{du}{dx}$$

$$\frac{\partial}{\partial x}\left[\ln\left(x^2+y^2\right)\right] = \frac{2x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left(-\frac{2z}{x}\tan^{-1}\frac{y}{x}\right) = \left(-\frac{2z}{x}\right)\left[\frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2}\right] = -\frac{2z}{x^2+y^2}$$

$$\frac{\partial}{\partial z}\left(5z^3 + e^y\cos z\right) = 15z^2 - e^y\sin z$$

$$\nabla.\bar{f} = \frac{2x}{x^2+y^2} - \frac{2z}{x^2+y^2} + 15z^2 - e^y\sin z$$

$$= \frac{2(x-z)}{x^2+y^2} + 15z^2 - e^y\sin z$$

$$curlF = \nabla \times \bar{f} = \left| \begin{array}{ccc} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ln\left(x^2+y^2\right) & -\left(\frac{2z}{x}\tan^{-1}\frac{y}{x}\right) & \left(5z^3+e^y\cos z\right) \end{array} \right| \begin{array}{cc} i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \ln\left(x^2+y^2\right) & -\left(\frac{2z}{x}\tan^{-1}\frac{y}{x}\right) \end{array}$$

$$= \left(e^y \cos z + \left(\frac{2}{x}\tan^{-1}\frac{y}{x} \right) \right) i + \left[\left[\left(\left(\frac{2z}{x} \right) \frac{\left(\frac{y}{x^2} \right)}{1 + \left(\frac{y}{x} \right)^2} \right) + \left(\frac{2z}{x^2}\tan^{-1}\frac{y}{x} \right) \right] - \left(\frac{2y}{x^2+y^2} \right) \right] k$$

$$= \left(e^y \cos z + \left(\frac{2}{x}\tan^{-1}\frac{y}{x} \right) \right) i + 2\left(\frac{y(z-x)}{x\left(x^2+y^2\right)} + \left(\frac{z}{x^2}\tan^{-1}\frac{y}{x} \right) \right) k$$

EX3 Prove div curl $\mathbf{A} = 0$.

$$\overline{A} = A_1 i + A_2 j + A_3 k$$

Where $\overline{A}$ is a vector function and A_1 , A_2 and A_3 are scalar functions

$$\text{div curl } \mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A})$$

$$= \nabla \cdot \left| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{matrix} \right| \begin{matrix} \mathbf{i} & \mathbf{j} \\ \partial/\partial x & \partial/\partial y \\ A_1 & A_2 \end{matrix}$$

$$= \nabla \cdot \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)\mathbf{k} \right]$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)\mathbf{k} \right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)$$

$$= \frac{\partial^2 A_3}{\partial x\, \partial y} - \frac{\partial^2 A_2}{\partial x\, \partial z} + \frac{\partial^2 A_1}{\partial y\, \partial z} - \frac{\partial^2 A_3}{\partial y\, \partial x} + \frac{\partial^2 A_2}{\partial z\, \partial x} - \frac{\partial^2 A_1}{\partial z\, \partial y} = 0$$

EX4 Prove $\nabla \cdot (\phi \mathbf{A}) = (\nabla\phi) \cdot \mathbf{A} + \phi\,(\nabla \cdot \mathbf{A})$.

Where A : vector function and ϕ :scalar function

$$\overline{A} = A_1 i + A_2 j + A_3 k$$

$$\nabla \cdot (\phi\,\mathbf{A}) = \nabla \cdot (\phi\ A_1\mathbf{i} + \phi\ A_2\mathbf{j} + \phi\ A_3\mathbf{k})$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (\phi\ A_1\mathbf{i} + \phi\ A_2\mathbf{j} + \phi\ A_3\mathbf{k})$$

$$= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$

$$= \phi\frac{\partial A_1}{\partial x} + A_1\frac{\partial \phi}{\partial x} + \phi \cdot \frac{\partial A_2}{\partial y} + A_2\frac{\partial \phi}{\partial y} + \phi\frac{\partial A_3}{\partial z} + A_3\frac{\partial \phi}{\partial z}$$

$$= \frac{\partial \phi}{\partial x}A_1 + \frac{\partial \phi}{\partial y}A_2 + \frac{\partial \phi}{\partial z}A_3 + \phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right) \cdot (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$+ \phi\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) = (\nabla\phi) \cdot \mathbf{A} + \phi\,(\nabla \cdot \mathbf{A})$$

Prove $div(\bar{f}+\bar{g}) = div\,\bar{f} + div\,\bar{g}$

Where $\bar{f}$ and $\bar{g}$ are vector functions

Prove $curl(\bar{f}+\bar{g}) = curl\,\bar{f} + curl\,\bar{g}$

Where $\bar{f}$ and $\bar{g}$ are vector functions

Prove $\nabla \times (\nabla u) = 0$

Where u is a scalar function

Prove $\nabla \times (u\,\bar{f}) = (\nabla u) \times \bar{f} + u(\nabla \times \bar{f})$

Where $\bar{f}$: vector function and u :scalar function

Prove $\nabla \cdot (\bar{f} \times \bar{g}) = \bar{g} \cdot (\nabla \times \bar{f}) - \bar{f} \cdot (\nabla \times \bar{g})$

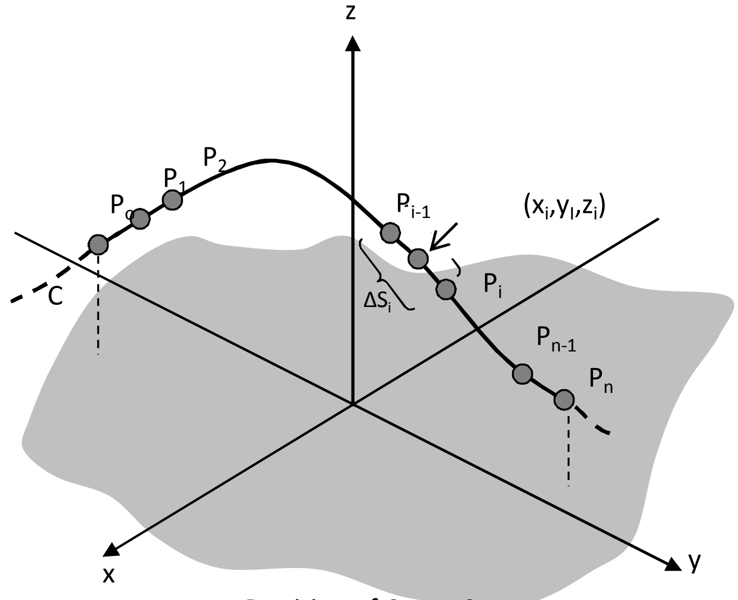
Where $\bar{f}$ and $\bar{g}$ are vector functions

2-10 Line Integral

Reference : Thomas G.B., Weir, Hass, Giordano. Thomas's calculus (12ed.).
Section 16.1 Page 901

Suppose ***f(x,y,z)*** is continuous in some region containing a smooth space curve ***C*** of finite length.

$$\lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i = \int_C f(x, y, z)\,ds$$



Partition of Curve C

1- Evaluation of a Line Integral as a Definite Integral

Let ***f*** be continuous in a region containing a smooth curve ***C*** ***,*** *where* ***C*** is given by ***r(t)=x(t)i+y(t)j+z(t)k*** where ***a≤ t ≤b*** , then

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$$

where

$$ds = \sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$$

$$x'(t) = \frac{dx}{dt}\ , \quad y'(t) = \frac{dy}{dt}\ , \quad z'(t) = \frac{dz}{dt}$$

EX1 :- Evaluate $\int_C (x^2+y^2+z^2)^2 ds$ where C is given by

$x = \cos t$, $y = \sin t$, $z = 3t$ from the point $A(1,0,0)$ to $B(1,0,6\pi)$

Solution :

r(t)=x(t)i+y(t)j+z(t)k=cost i+sint j+3t k

$$f(x,y,z) = (x^2+y^2+z^2)^2$$

$$f(t) = (\cos^2 t + \sin^2 t + (3t)^2)^2 = (1+9t^2)^2$$

$$x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t \;, \quad y = \sin t \Rightarrow \frac{dy}{dt} = \cos t \;,$$

$$z = 3t \Rightarrow \frac{dz}{dt} = 3$$

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt = \sqrt{(-\sin t)^2 + (\cos t)^2 + 3^2}\,dt = \sqrt{10}\,dt$$

$A(1,0,0)$

$$\left.\begin{aligned} 1 = x = \cos t &\Rightarrow t = 0, 2\pi, \ldots \\ 0 = y = \sin t &\Rightarrow t = 0, \pi, 2\pi, \ldots \\ 0 = z = 3t &\Rightarrow t = 0 \end{aligned}\right\} t = 0$$

$B(1,0,6\pi)$

$$\left.\begin{aligned} 1 = x = \cos t &\Rightarrow t = 0, 2\pi, \ldots \\ 0 = y = \sin t &\Rightarrow t = 0, \pi, 2\pi, \ldots \\ 6\pi = z = 3t &\Rightarrow t = 2\pi \end{aligned}\right\} t = 2\pi$$

$$\int_C \left(x^2 + y^2 + z^2\right)^2 ds = \int_0^{2\pi} \left(1 + 9t^2\right)^2 \sqrt{10}\,dt = \sqrt{10} \int_0^{2\pi} \left(1 + 18t^2 + 81t^4\right) dt$$

$$= \sqrt{10}\left[t + 6t^3 + \frac{81}{5}t^5\right]_0^{2\pi} = 506391.931$$

EX2 Integrate $f(x,y,z) = x - \frac{1}{2}(y-3) + 9z$ along the curve

$\bar{r}(t) = \left(\frac{t^2}{2} + t + 1\right)i + \left(t^2+1\right)j + tk$ from $(2.5,2,1)$ to $(5,5,2)$

Solution :

$$f(t) = \left(\frac{t^2}{2} + t + 1\right) - \frac{1}{2}\left(t^2 + 1 - 3\right) + 9t = 10t + 2$$

$$x'(t) = t+1 \ , \ y'(t) = 2t \ , \ z'(t) = 1$$

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

$$ds = \sqrt{(t+1)^2 + (2t)^2 + 1}\, dt = \sqrt{5t^2 + 2t + 2}\, dt$$

at $(2.5,2,1)$

$$x(t) = \frac{t^2}{2} + t + 1 = 2.5 \Rightarrow \frac{t^2}{2} + t + 1 - 2.5 = 0 \Rightarrow t^2 + 2t - 3 = 0$$

$$(t+3)(t-1) = 0 \Rightarrow t = -3 \text{ or } t = 1$$

$$y(t) = t^2 + 1 = 2 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$$

$$z(t) = t = 1$$

$\Big\} \; t = 1$

at $(5,5,2)$

$$\left.\begin{aligned} &x(t)=\frac{t^2}{2}+t+1=5 \Rightarrow t^2+2t-8=0 \\ &(t+4)(t-2)=0 \Rightarrow t=-4 \text{ or } t=2 \\ &y(t)=t^2+1=5 \Rightarrow t^2=4 \Rightarrow t=\pm 2 \\ &z(t)=t=2 \end{aligned}\right\} t=2$$

$$\int_1^2 f(t)\,ds=\int_1^2 (10t+2)\sqrt{5t^2+2t+2}\,dt$$

$$=\left[\frac{2}{3}\left(5t^2+2t+2\right)^{\frac{3}{2}}\right]_1^2=70.38$$

2- Evaluating a Line Integral in Differential Form

If f is a vector field of the form $f(x,y,z)=M(x,y,z)i+N(x,y,z)j+P(x,y,z)k$ and C is a given curve connected between two points $A(a_1,b_1,c_1)$ and $B(a_2,b_2,c_2)$, then the sum over all the subdivisions is :

$$\sum_{i=1}^{n}\left(M(x_i,y_i,z_i)\Delta x_i + N(x_i,y_i,z_i)\Delta y_i + P(x_i,y_i,z_i)\Delta z_i\right)$$

The limits of this sum, as n becomes infinite in such a way that the length of each Δx_i , Δy_i and Δz_i approaches zero, is known as **line integrals** and is written :

$$\int_C \left[M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz\right]$$

A-

$$y = f_1(x) \;,\; z = f_2(x) \quad \Rightarrow dy = f_1'(x)dx \;,\quad dz = f_2'(x)dx$$

then

$$\int_C M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz =$$

$$\int_C M(x,f_1(x),f_2(x))dx + N(x,f_1(x),f_2(x))f_1'(x)dx + P(x,f_1(x),f_2(x))f_2'(x)dx$$

B- if $x = f_1(y)$, $z = f_2(y)$ $\Rightarrow dx = f_1'(y)dy$, $dz = f_2'(y)dy$

then

$$\int_C M(f_1(y),y,f_2(y))f_1'(y)dy + N(f_1(y),y,f_2(y))dy + P(f_1(y),y,f_2(y))f_2'(y)dy$$

C- if $x = f_1(z)$, $y = f_2(z)$ $\Rightarrow dx = f_1'(z)dz$, $dy = f_2'(z)dz$

then

$$\int_C M(f_1(z),f_2(z),z)f_1'(z)dz + N(f_1(z),f_2(z),z)f_2'(z)dz + P(f_1(z),f_2(z),z)dz$$

D-

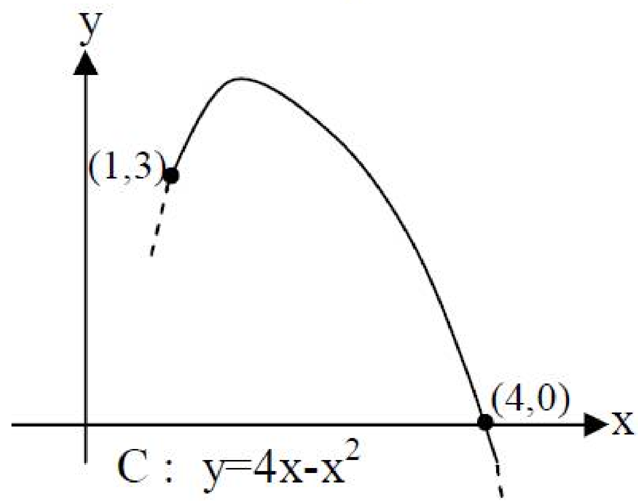
if $x = h(t)$, $y = s(t)$, $z = g(t) \Rightarrow dx = h(t)dt$, $dy = s(t)dt$, $dz = g(t)dt$

then

$$\int_C M\big(h(t), s(t), g(t)\big)h'(t)dt + N\big(h(t), s(t), g(t)\big)s'(t)dt + P\big(h(t), s(t), g(t)\big)g'(t)dt$$

EX1 :- Evaluate $\int_C ydx + x^2dy$ where C is the parabolic arc given by

$y = 4x - x^2$ from $(4,0)$ to $(1,3)$



Solution :

$$y = 4x - x^2 \Rightarrow dy = (4 - 2x)dx$$

$$\int_C ydx + x^2dy = \int_4^1 (4x - x^2)dx + x^2(4 - 2x)dx$$

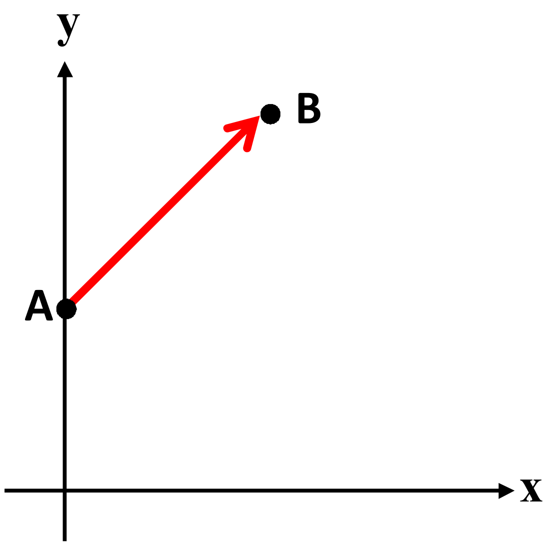
$$= \int_4^1 \left[4x + 3x^2 - 2x^3\right]dx = 2x^2 + x^3 - \frac{x^4}{2}\Bigg|_4^1 = \frac{69}{2}$$

EX2 :- Find the value of the integral

$\int_C (x^2 - y)dx + (y^2 + x)dy$ along each of the following paths

i- AB , *ii-ACB* , *iii- ADB* , *iv-* $x = t$, $y = t^2 + 1$ *from A to B*

$A(0,1)$, $B(1,2)$, $C(1,1)$, $D(0,2)$



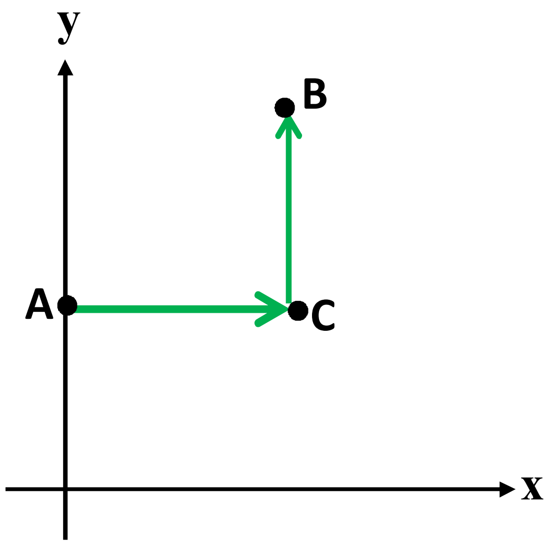
Solution :

i - $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1} \Rightarrow \frac{2-1}{1-0} = \frac{y-1}{x-0}$$

$$y = x + 1 \ , \quad \Rightarrow dy = dx$$

$$\int_A^B \left[x^2 - (x+1)\right]dx + \left((x+1)^2 + x\right)dx = \int_0^1 \left(2x^2 + 2x\right)dx = \frac{2x^3}{3} + 2\frac{x^2}{2}\Bigg|_0^1 = \frac{5}{3}$$

ii - $ACB \Rightarrow \int_{ACB} = \int_{AC} + \int_{CB}$



$AC \Rightarrow y = 1 \ , \ dy = 0$

$$\int_{AC} = \int^{1} \left(x^2 - 1\right)dx + \left(y^2 + x\right) \times 0$$

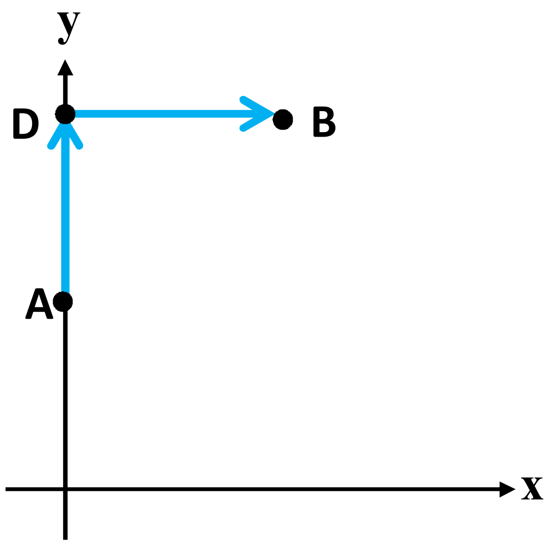
$$= \int_0^1 \left(x^2 - 1\right)dx = \left.\frac{x^3}{3} - x\right|_0^1 = -\frac{2}{3}$$

$\mathrm{CB} \Rightarrow \mathrm{x} = 1 \ , \ \mathrm{dx} = 0$

$$\int_{CB} = \int_1^2 \left(y^2 + 1\right)dy = \left.\frac{y^3}{3} + y\right|_1^2 = \frac{10}{3}$$

$$\int_{ACB} = \int_{AC} + \int_{CB} = -\frac{2}{3} + \frac{10}{3} = \frac{8}{3}$$

iii - $\int\limits_{ADB} = \int\limits_{AD} + \int\limits_{DB}$



$$AD \Rightarrow x = 0 \quad , \quad dx = 0$$

$$\int\limits_{AD} = \int_1^2 y^2 dy = \left.\frac{y^3}{3}\right|_1^2 = \frac{7}{3}$$

$$DB \Rightarrow y = 2 \quad , \quad dy = 0$$

$$\int\limits_{DB} = \int_0^1 \left(x^2 - 2\right) dx = \left.\frac{x^3}{3} - 2x\right|_0^1 = -\frac{5}{3}$$

$$\int\limits_{ADB} = \frac{7}{3} - \frac{5}{3} = \frac{2}{3}$$

iv $-\ x = t \Rightarrow dx = dt$

$y = t^2 + 1 \Rightarrow dy = 2tdt$

$A(0,1)$

$x = 0 = t,\ \ y = 1 = t^2 + 1 \Rightarrow t = 0$

$B(1,2)$

$x = 1 = t,\ \ y = 2 = t^2 + 1 \Rightarrow t = \pm 1\ \ \Rightarrow\ t = 1$

$$\int_C \left(x^2 - y\right)dx + \left(y^2 + x\right)dy$$

$$\int_0^1 \left[t^2 - \left(t^2 + 1\right)\right]dt + \left[\left(t^2 + 1\right)^2 + t\right]2tdt = 2$$

EX **Integrate f(x,y,z) = (3x²-6yz)I + (2y+3xz)j + (1-4xyz²)k along the following paths C :**

(a) $x = t,\ y = t^2,\ z = t^3$. from $(0,0,0)$ to $(1,1,1)$

(b) the straight lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$, and then to $(1,1,1)$.

Solution :

$$\int_C (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$

(a) If $x = t,\ y = t^2,\ z = t^3$, points $(0,0,0)$ and $(1,1,1)$ correspond to $t = 0$ and $t = 1$ respectively. Then

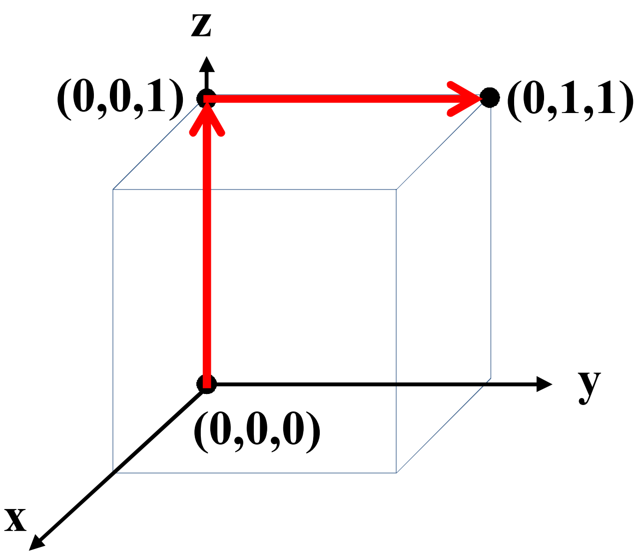
$$\int_{t=0}^{1} \{3t^2 - 6(t^2)(t^3)\}\,dt + \{2t^2 + 3(t)(t^3)\}\,d(t^2) + \{1 - 4(t)(t^2)(t^3)^2\}\,d(t^3)$$

$$\int_{t=0}^{1} (3t^2 - 6t^5)\,dt + (4t^3 + 6t^5)\,dt + (3t^2 - 12t^{11})\,dt = 2$$

(b) Along the straight line from $(0,0,0)$ to $(0,0,1)$

$x = 0,\ y = 0,\ dx = 0,\ dy = 0$

while z varies from 0 to 1. Then



$$\int_{z=0}^{1} \{3(0)^2 - 6(0)(z)\}0 + \{2(0) + 3(0)(z)\}0 + \{1 - 4(0)(0)(z^2)\}\,dz$$

$$= \int_{z=0}^{1} dz = 1$$

Along the straight line from $(0,0,1)$ to $(0,1,1)$,

$x = 0,\ z = 1,\ dx = 0,\ dz = 0$

while y varies from 0 to 1. Then

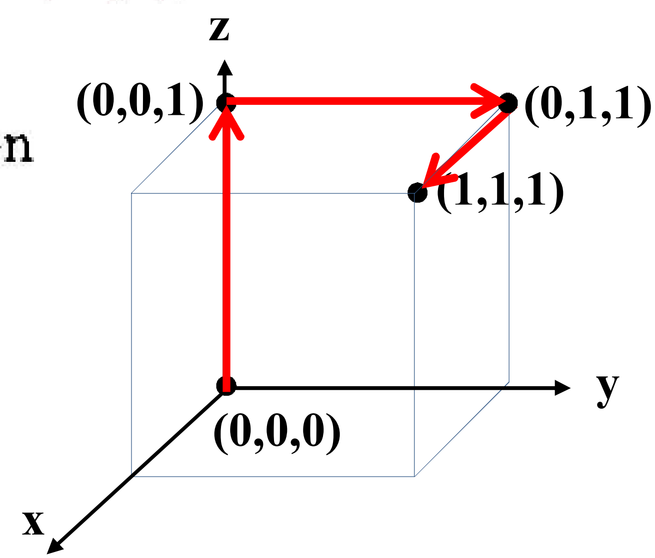
$$\int_{y=0}^{1} \{3(0)^2 - 6(y)(1)\}0 \;+\; \{2y + 3(0)(1)\}\,dy \;+\; \{1 - 4(0)(y)(1)^2\}0$$

$$= \int_{y=0}^{1} 2y\,dy \;=\; 1$$

Along the straight line from $(0,1,1)$ to $(1,1,1)$,

$y = 1,\; z = 1,\; dy = 0,\; dz = 0$

while x varies from 0 to 1. Then



$$\int_{x=0}^{1} \{3x^2 - 6(1)(1)\}\,dx \;+\; \{2(1) + 3x(1)\}0 \;+\; \{1 - 4x(1)(1)^2\}0$$

$$= \int_{x=0}^{1} (3x^2 - 6)\,dx \;=\; -5$$

$$\int_C \;=\; \int_{z=0}^{1} \;+\; \int_{y=0}^{1} \;+\; \int_{x=0}^{1} \;=\; 1 + 1 - 5 \;=\; -3.$$

2-11 Surface Integral

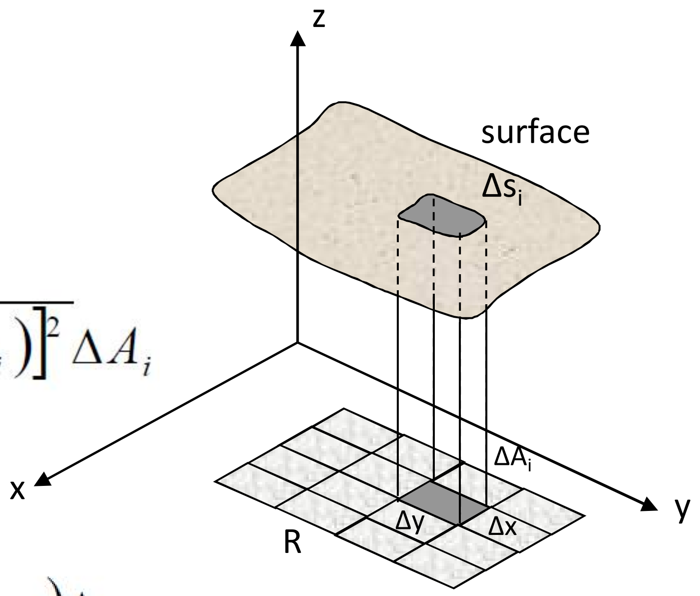
Reference : Thomas G.B., Weir, Hass, Giordano. Thomas's calculus . Section 16.5 and 16.6.

(a) Let s **be a surface given by** ***z=g(x,y) and R its projection on the xy-plane (i.e. you can*** **think of R as the shadow of s on the plane) and** ***f(x,y,z) is*** **defined on s**

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i$$

where

$$\Delta s_i = \sqrt{1 + [g_x(x_i, y_i)]^2 + [g_y(x_i, y_i)]^2}\,\Delta A_i$$



$$\iint_s f(x, y, z)ds = \lim_{\Delta s \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i$$

$$\iint_S f(x,y,z)\,ds = \iint_R f(x,y,g(x,y))\sqrt{1+[g_x(x,y)]^2+[g_y(x,y)]^2}\,dA$$

where

$$g_x(x,y) = \frac{\partial g(x,y)}{\partial x} = \frac{\partial z}{\partial x}\ ,\ g_y(x,y) = \frac{\partial g(x,y)}{\partial y} = \frac{\partial z}{\partial y}\ ,\ dA = dydx$$

(b) If s is the graph of $y=g(x,z)$ and R is its projection onto the xz-plane, then

$$\iint_S f(x,y,z)\,ds = \iint_R f(x,g(x,z),z)\sqrt{1+[g_x(x,z)]^2+[g_z(x,z)]^2}\,dA$$

where

$$g_x(x,z) = \frac{\partial g(x,z)}{\partial x} = \frac{\partial y}{\partial x}\ ,\ g_z(x,z) = \frac{\partial g(x,z)}{\partial z} = \frac{\partial y}{\partial z}\ ,\ dA = dzdx$$

(c) If s is the graph of $x=g(y,z)$ and R is its projection onto the yz-plane, then

$$\iint_S f(x,y,z)\,ds = \iint_R f(g(y,z),y,z)\sqrt{1+[g_y(y,z)]^2+[g_z(y,z)]^2}\,dA$$

where

$$g_y(x,z) = \frac{\partial g(y,z)}{\partial y} = \frac{\partial x}{\partial y}\ ,\ g_z(y,z) = \frac{\partial g(y,z)}{\partial z} = \frac{\partial x}{\partial z}\ ,\ dA = dzdy$$

(d) If s is defined by $g(x,y,z)=c$, then

$$\iint_S f(x,y,z)\,ds = \iint_R f(x,y,z)\frac{\sqrt{[g_x(x,y,z)]^2+[g_y(x,y,z)]^2+[g_z(x,y,z)]^2}}{|g_z(x,y,z)|}\,dxdy$$

where

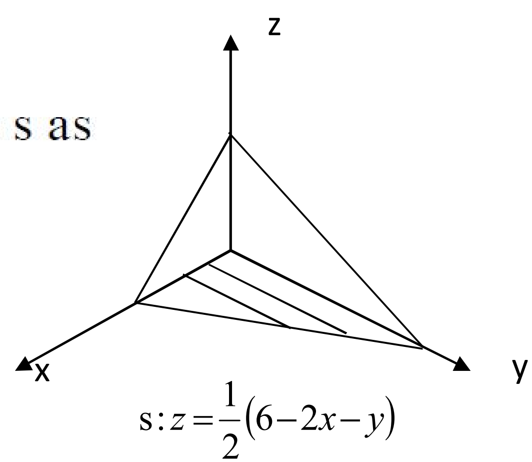
$$g_x(x,y,z)=\frac{\partial g(x,y,z)}{\partial x}=\frac{\partial g}{\partial x}\ ,\quad g_y(x,y,z)=\frac{\partial g(x,y,z)}{\partial y}=\frac{\partial g}{\partial y}\ ,$$

$$g_z(x,y,z)=\frac{\partial g(x,y,z)}{\partial z}=\frac{\partial g}{\partial z}$$

EX1 :- Evaluate the surface integral $\iint_s (y^2+2yz)\,ds$ where **s** is the first-octant portion of the plane $2x+y+2z=6$

Solution :-

By projection s onto the xy-plane, we can write s as

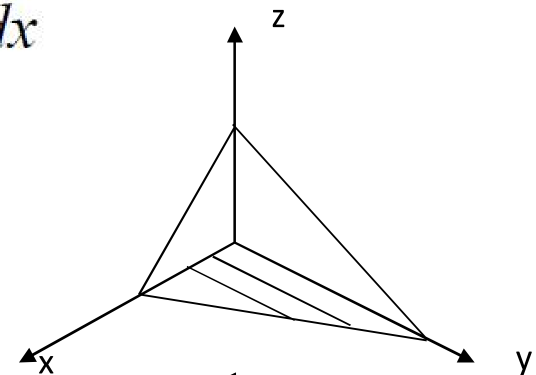


$$z=\frac{1}{2}(6-2x-y)=g(x,y)$$

$$g_x(x,y)=-1 \ \text{ and } \ g_y(x,y)=-\frac{1}{2}$$

$$ds = \sqrt{1 + [g_x(x,y)]^2 + [g_y(x,y)]^2}\,dydx$$

$$= \sqrt{1 + 1 + \frac{1}{4}}\,dydx = \frac{3}{2}\,dydx$$



$s: z = \frac{1}{2}(6 - 2x - y)$

On xy-plane **z=0** , then y=2(3-x)

Along x-axis **y=0** , then x=3

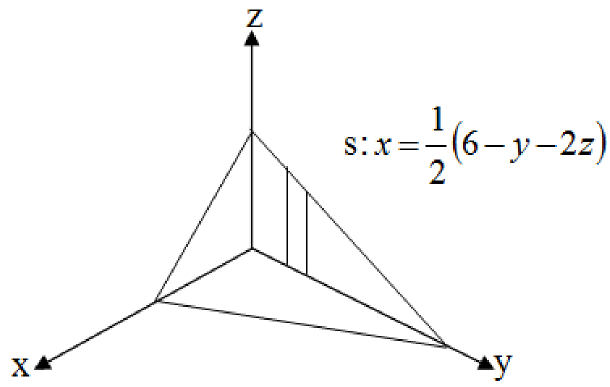
$$\iint_S \left(y^2 + 2yz\right)ds = \iint_R \left[\mathrm{y}^2 + 2y\left(\frac{1}{2}\right)(6 - 2x - y)\right]\left(\frac{3}{2}\right)dydx$$

$$= 3\int_0^3 \int_0^{2(3-x)} y(3-x)\,dydx = 3\int_0^3 \left[\frac{y^2}{2}\right]_0^{2(3-x)} (3-x)\,dx$$

$$= 6\int_0^3 (3-x)^3\,dx = -\frac{3}{2}(3-x)^4\Big|_0^3 = \frac{243}{2}$$

One alterative solution to this example would be to project s onto the yz-plane.

$$x = \frac{1}{2}(6 - y - 2z) = g(y, z)$$



$$ds = \sqrt{1 + [g_y(y,z)]^2 + [g_z(y,z)]^2}\, dzdy$$

$$= \sqrt{1 + \frac{1}{4} + 1}\, dzdy = \frac{3}{2} dzdy$$

On yz-plane **x=0** , then z=(6-y)/2

Along y-axis **z=0** , then y=6

$$\iint_S (y^2 + 2yz)\, ds = \iint_R f(g(y,z), y, z)\sqrt{1 + [g_y(y,z)]^2 + [g_z(y,z)]^2}\, dzdy$$

$$= \int_0^6 \int_0^{(6-y)/2} [y^2 + 2yz]\left(\frac{3}{2}\right) dzdy$$

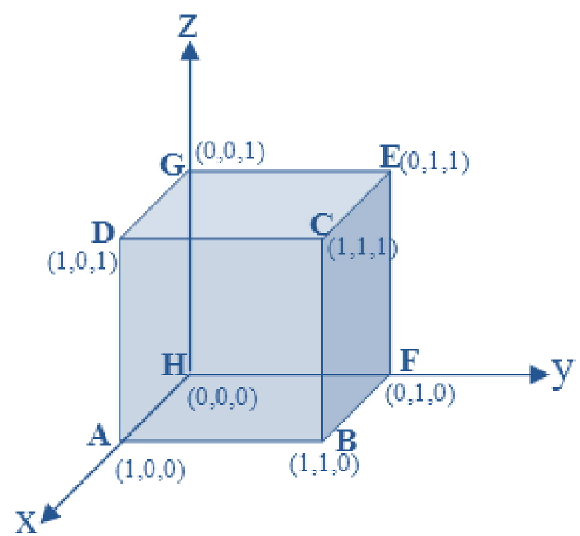
$$= \frac{3}{2}\int_0^6 [y^2 z + yz^2]_0^{(6-y)/2} dy = \frac{3}{2}\int_0^6 \left[y^2\left(\frac{6-y}{2}\right) + y\left(\frac{6-y}{2}\right)^2\right] dy$$

$$= \frac{3}{8}\int_0^6 (36y - y^3)\, dy = \frac{3}{8}\left[18y^2 - \frac{y^4}{4}\right]_0^6 = \frac{243}{2}$$

EX2 :- Evaluate the surface integral $\iint (xz + yz)ds$, where s is a cube, which is its vertices are (0,0,0), (1,0,0), (0,1,0), (0,0,1) , (1,1,0), (1,0,1), (0,1,1), (1,1,1)

Solution :

Let ABFH=s_1 , CDGE=s_2 , ADGH=s_3 , BCEF=s_4 , EFHG=s_5 , ABCD=s_6



1- s_1 since z=0 then ; $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

$$\iint_{s_1} (xz + yz)ds = \iint_R 0\,dxdy = 0 \Rightarrow s_1 = 0$$

$2 - s_2$, $z = 1 = g(x,y)$; $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

$$\iint_{s_2} (xz + yz)ds = \iint_R (x \times 1 + y \times 1)\sqrt{1 + 0 + 0}\,dxdy$$

$$= \int_0^1 \int_0^1 (x + y)dydx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^1 dx = \int_0^1 \left(x + \frac{1}{2} \right) dx = \frac{x^2}{2} + \frac{1}{2}x \Big|_0^1 = 1$$

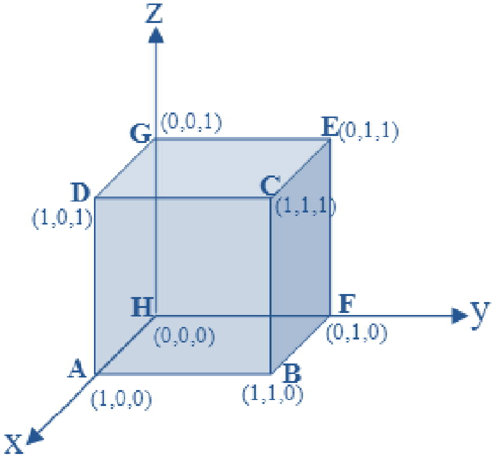
$$3-\ s_3\ ,\ \ y=0=g(x,z)\ ,\ \frac{\partial y}{\partial x}=\frac{\partial y}{\partial z}=0$$

$$\iint_{s_3}(xz+yz)ds=\iint_R xz\sqrt{1+0+0}\,dzdx=\int_0^1\left[x\frac{z^2}{2}\right]_0^1 dx=\frac{1}{2}\int_0^1 xdx=\frac{1}{2}\frac{x^2}{2}\Big|_0^1=\frac{1}{4}$$

$$4-\ s_4\ ,\ \ y=1=g(x,z)\ ,\ \frac{\partial y}{\partial x}=\frac{\partial y}{\partial z}=0$$

$$\iint_{s_3}(xz+yz)ds=\iint_R (xz+z)\sqrt{1+0+0}\,dzdx=\int_0^1\left[x\frac{z^2}{2}+\frac{z^2}{2}\right]_0^1 dx=\frac{1}{2}\int_0^1(x+1)dx$$

$$=\frac{1}{2}\left[\frac{x^2}{2}+x\right]_0^1=\frac{3}{4}$$



$$5-\ s_5\ ,\ \ x=0=g(y,z)\ ,\ \frac{\partial x}{\partial y}=\frac{\partial x}{\partial z}=0$$

$$\iint_{s_5}(xz+yz)ds=\iint_R yz\sqrt{1+0+0}\,dzdy$$

$$=\int_0^1\left[y\frac{z^2}{2}\right]_0^1 dy=\frac{1}{2}\int_0^1 ydy=\frac{1}{2}\frac{y^2}{2}\Big|_0^1=\frac{1}{4}$$

$$6-\ s_6\ ,\ \ x=1=g(y,z)\ ,\ \frac{\partial x}{\partial y}=\frac{\partial x}{\partial z}=0$$

$$\iint_{S_6} (xz + yz)\,ds = \iint_R (z + yz)\sqrt{1 + 0 + 0}\,dzdy$$

$$= \int_0^1 \left[\frac{z^2}{2} + y\,\frac{z^2}{2} \right]_0^1 dy = \frac{1}{2}\int_0^1 (1 + y)dy = \frac{1}{2}\left[y + \frac{y^2}{2} \right]_0^1 = \frac{3}{4}$$

$$s = s_1 + s_2 + s_3 + s_4 + s_5 + s_6 = 3$$

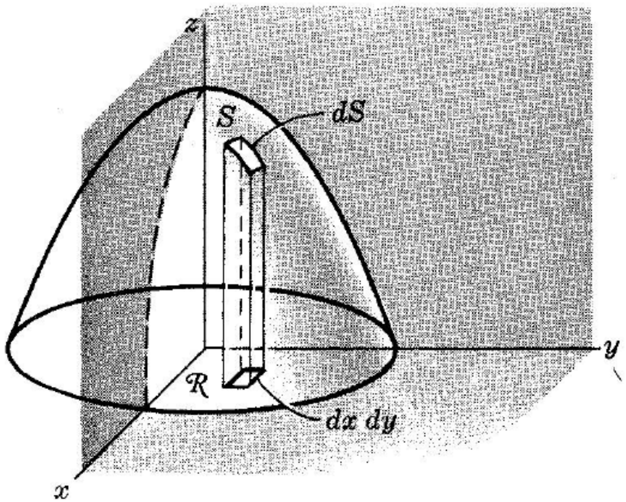
EX3

Evaluate the surface integral $\iint_s (x^2 + y^2)\,ds$ where **s** is the surface of the paraboloid $x^2+y^2+z=2$ above the xy plane.

Solution :-

$$z = 2 - (x^2 + y^2) = g(x, y)$$

$$g_x(x, y) = -2x \quad \text{and} \quad g_y(x, y) = -2y$$



$$ds = \sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\,dydx$$

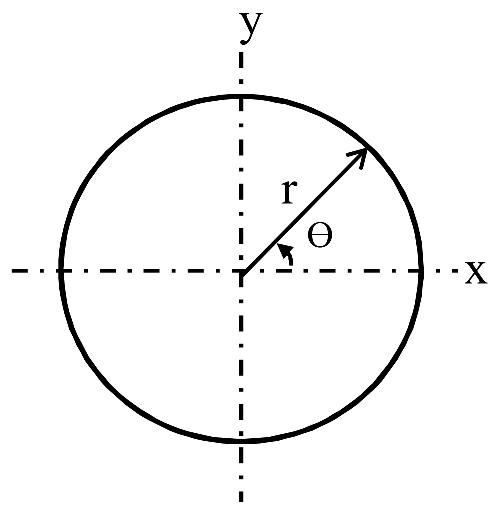
$$= \sqrt{1 + (-2x)^2 + (-2y)^2}\,dydx$$

$$= \sqrt{1 + 4(x^2 + y^2)}\,dydx$$

On xy-plane **z=0** , then

$x^2+y^2=2$

By using polar coordinates



x= r cosΘ , y= r sinΘ

$x^2+y^2= r^2$, dA=r dr dΘ

$r \Rightarrow 0$ to $\sqrt{2}$

$\theta \Rightarrow 0$ to 2π

$$\iint_S \left(x^2+y^2\right)ds = \iint_R \left(x^2+y^2\right)\sqrt{1+4\left(x^2+y^2\right)}\,dydx$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} r^2\sqrt{1+4r^2}\,rdrd\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} r^3\sqrt{1+4r^2}\,drd\theta$$

Let $u = \sqrt{1+4r^2} \Rightarrow u^2 = 1+4r^2 \Rightarrow r^2 = \frac{1}{4}(u^2 - 1)$

$$r = \frac{1}{2}\sqrt{u^2 - 1} \quad , \quad dr = \frac{u}{2\sqrt{u^2-1}}du$$

at $r = \sqrt{2} \quad \Rightarrow \text{u} = 3$

at $r = 0 \quad \Rightarrow \text{u} = 1$

$$\int_0^{2\pi}\int_0^{\sqrt{2}} r^3\sqrt{1+4r^2}\,dr\,d\theta = \int_0^{2\pi}\int_1^3 \frac{1}{8}\left(u^2-1\right)^{\frac{3}{2}} u \frac{u}{2\sqrt{u^2-1}}\,du\,d\theta$$

$$= \frac{1}{16}\int_0^{2\pi}\int_1^3 \left(u^2-1\right) u^2\,du\,d\theta = \frac{1}{16}\int_0^{2\pi}\left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^3 d\theta$$

$$= 2.483\int_0^{2\pi} d\theta = 2.483\left[\theta\right]_0^{2\pi} = 15.603$$

EX4

Integrate $G(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0$, $x = 1$, and $z = 0$.

Solution :-

$$z = \frac{1}{4}\left(16 - y^2\right) = g(x, y)$$

$$g_x(x, y) = 0 \quad \text{and} \quad g_y(x, y) = -\frac{y}{2}$$

$$ds = \sqrt{1 + \left[g_x(x, y)\right]^2 + \left[g_y(x, y)\right]^2}\, dydx$$

$$= \sqrt{1 + \left[\frac{-y}{2}\right]^2}\, dydx = \sqrt{1 + \frac{y^2}{4}}\, dydx = \frac{1}{2}\sqrt{4 + y^2}\, dydx$$

$x = 0$, $x = 1$.

at $z = 0 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$

$$\iint_S x\sqrt{y^2+4}\, ds = \int_{-4}^{4}\int_{0}^{1}\left(x\sqrt{y^2+4}\right)\left(\frac{\sqrt{y^2+4}}{2}\right) dx\, dy$$

$$= \int_{-4}^{4}\int_{0}^{1} \frac{x\,(y^2+4)}{2}\, dx\, dy$$

$$= \int_{-4}^{4} \frac{x^2}{4}(y^2+4)\Big|_0^1\, dy$$

$$= \int_{-4}^{4} \frac{1}{4}\,(y^2+4)\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}+4y\right]_0^4$$

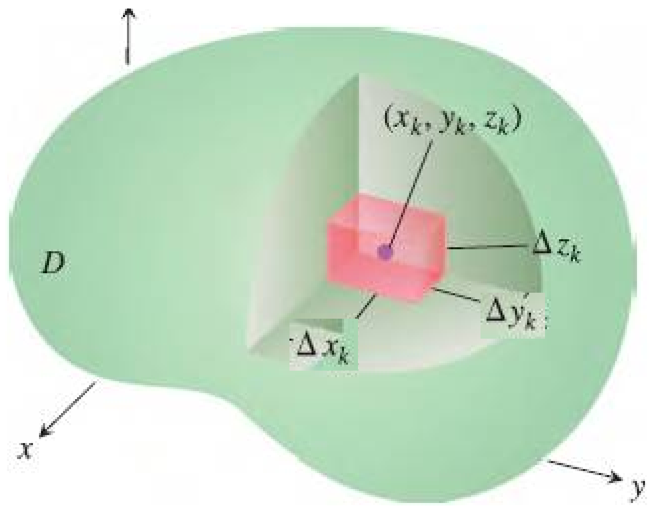
$$= \frac{1}{2}\left(\frac{64}{3}+16\right) = \frac{56}{3}$$

2-12 Volume Integrals (Triple Integrals)

Reference : Thomas G.B., Weir, Hass, Giordano. Thomas's calculus (Section 15.5)

If $f(x,y,z)$ is continuous over abounded solid region D, then the volume integral of over D is defined to be

$$\lim_{\Delta v_k \to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta v_k = \iiint_D f(x,y,z)\,dv$$



Where

$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k.$$

$$dv = dx\,dy\,dz$$

Notes

1- In the special case where $f(x,y,z)=1$ in the solid region D , the volume integral represents the volume of D . That is

$$\text{volume of D} = \iiint_D dv$$

2- Let $f(x,y,z)$ be continuous on a solid region D defined by $a \le x \le b$, $h_1(x) \le y \le h_2(x)$, $g_1(x,y) \le z \le g_2(x,y)$. where h_1 , h_2 , g_1 and g_2 are continuous functions. Then,

$$\iiint_D f(x,y,z)dv = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dzdydx$$

3- dv can be wrote in six different orders as :

$$dv = dxdydz \quad , \quad dv = dydxdz \quad , \quad dv = dzdxdy$$

$$dv = dxdzdy \quad , \quad dv = dydzdx \quad , \quad dv = dzdydx$$

EX1:- Evaluate the iterated integral $\int_0^2 \int_0^x \int_0^{x+y} e^x(y+2z)\,dz\,dy\,dx$

$$\int_0^2 \int_0^x \int_0^{x+y} e^x(y+2z)\,dz\,dy\,dx = \int_0^2 \int_0^x e^x\left(yz+z^2\right)\Big|_0^{x+y} dy\,dx$$

$$= \int_0^2 \int_0^x e^x\left(x^2+3xy+2y^2\right)dy\,dx = \int_0^2 \left[e^x\left(x^2y+\frac{3xy^2}{2}+\frac{2y^3}{3}\right)\right]_0^x dx$$

$$= \frac{19}{6}\int_0^2 x^3e^x\,dx = \frac{19}{6}\left[e^x\left(x^3-3x^2+6x-6\right)\right]_0^2$$

$$= 19\left(\frac{e^2}{3}+1\right) = 65.797$$

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts

$let \quad u = x^3 \;,\; dv = e^x dx \qquad du = 3x^2 dx \;,\; v = e^x$

$x^3e^x - \int 3x^2e^x dx$

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

EX2

Find the volume of a sphere of radius a, which it equation is $x^2+y^2+z^2=a^2$

Solution :-

$$v=\iiint dv=\iiint dzdydx$$

$$x^2+y^2+z^2=a^2$$

$$z=\pm\sqrt{a^2-x^2-y^2}$$

$$\text{at } z=0 \Rightarrow x^2+y^2=a^2 \Rightarrow y=\pm\sqrt{a^2-x^2}$$

$$\text{at } y=0 \Rightarrow x=\pm a$$

$$v=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dzdydx = 8\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\int_{0}^{\sqrt{a^2-x^2-y^2}} dzdydx$$

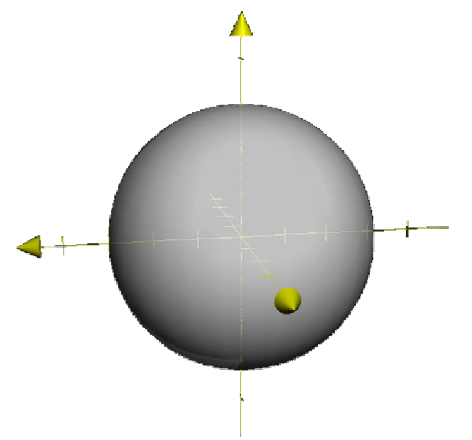
$$v = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} \, dydx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \left(a^2 - x^2 - y^2\right)^{\frac{1}{2}} dydx$$

using polar coordinates

$$x = r\cos\theta \;,\; y = r\sin\theta \;,\; dxdy = rdrd\theta \;,\; \mathrm{r}^2 = x^2 + y^2$$

$r \Rightarrow 0$ to a

$\theta \Rightarrow 0$ to $\pi/2$



$$v = 8\int_0^{\frac{\pi}{2}} \int_0^a \left(a^2 - r^2\right)^{\frac{1}{2}} rdrd\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \left(\frac{-1}{2} \times \frac{2}{3}\right) \left(a^2 - r^2\right)^{\frac{3}{2}} \Big|_0^a \, d\theta = \frac{8}{3} a^3 \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{8}{3} a^3 [\theta]_0^{\frac{\pi}{2}} = \frac{4\pi a^3}{3}$$

EX3

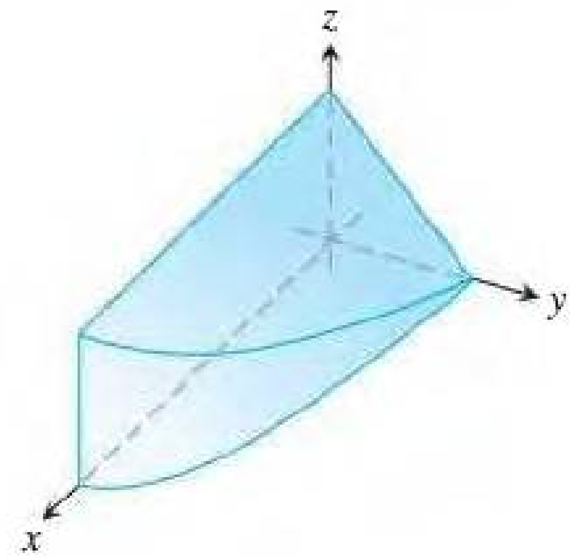
Find the volume of the region in the first octant bounded by the coordinate planes, the plane y + z = 2, and the cylinder $x = 4 - y^2$

Solution :-

$y + z = 2 \Rightarrow z = 2-y$

$x = 4 - y^2 \Rightarrow y = \sqrt{4-x}$

$x = 4$



$$V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz\, dy\, dx$$

$$= \int_0^4 \int_0^{\sqrt{4-x}} [z]_0^{2-y}\, dydx = \int_0^4 \int_0^{\sqrt{4-x}} (2-y)\, dy\, dx$$

$$= \int_0^4 [2y - \frac{y^2}{2}]_0^{\sqrt{4-x}}\, dx = \int_0^4 \left[2\sqrt{4-x} - \left(\frac{4-x}{2}\right)\right] dx$$

$$= \left[-\frac{4}{3}(4-x)^{3/2} + \frac{1}{4}(4-x)^2\right]_0^4 = \frac{20}{3}$$

EX4: Find the volume of the three-dimensional region enclosed by the surfaces $z=8-x^2-y^2$ and $z=x^2+3y^2$

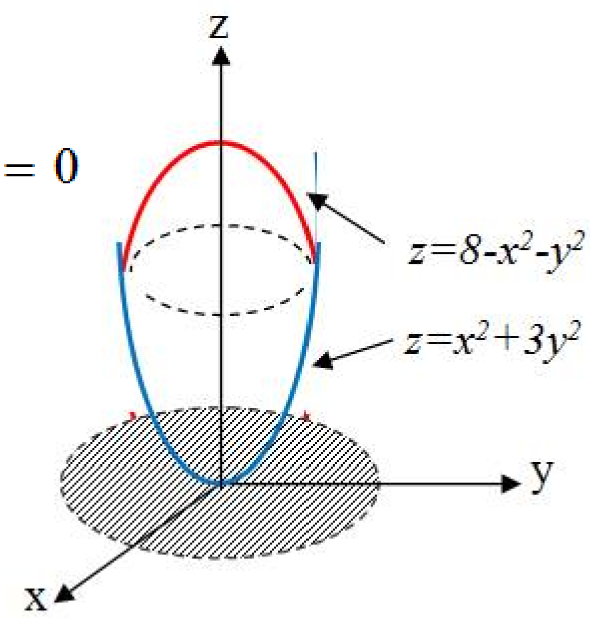
Solution :-

$$z=8-x^2-y^2=x^2+3y^2$$

$$8-x^2-y^2-x^2-3y^2=0 \Rightarrow 8-2x^2-4y^2=0$$

$$x^2=4-2y^2 \Rightarrow x=\pm\sqrt{4-2y^2}$$

$$x=0 \Rightarrow 4\text{-}2y^2=0 \Rightarrow y^2=2 \Rightarrow y=\pm\sqrt{2}$$



$$x^2+3y^2 \le z \le 8-x^2-y^2$$

$$-\sqrt{4-2y^2} \le x \le +\sqrt{4-2y^2}$$

$$-\sqrt{2} \le y \le \sqrt{2}$$

$$V=\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{+\sqrt{4-2y^2}}\int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy = 4\int_{0}^{\sqrt{2}}\int_{0}^{+\sqrt{4-2y^2}}\int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$$

$$=4\int_{0}^{\sqrt{2}}\int_{0}^{\sqrt{4-2y^2}}\left[z\right]_{x^2+3y^2}^{8-x^2-y^2} dxdy = 4\int_{0}^{\sqrt{2}}\int_{0}^{+\sqrt{4-2y^2}}\left[\left(8-x^2-y^2\right)-\left(x^2+3y^2\right)\right]dxdy$$

$$V = 8\int_0^{\sqrt{2}+\sqrt{4-2y^2}}\int_0 \left[\left(4-2y^2\right)-x^2\right]dxdy = 8\int_0^{\sqrt{2}}\left[\left(4-2y^2\right)x-\frac{x^3}{3}\right]_0^{\sqrt{4-2y^2}} dy$$

$$= \frac{16}{3}\int_0^{\sqrt{2}}\left(4-2y^2\right)^{\frac{3}{2}} dy$$

Let $y^2 = 2\sin^2\theta \Rightarrow y = \sqrt{2}\sin\theta \ , \ dy = \sqrt{2}\cos\theta d\theta$

if $y = \sqrt{2} = \sqrt{2}\sin\theta \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

if $y = 0 = \sqrt{2}\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$\cos^2\theta = 1-\sin^2\theta$

$\cos^2\theta = \frac{1}{2}\left(1+\cos 2\theta\right)$

$$V = \frac{16}{3}\int_0^{\frac{\pi}{2}}\left(4-4\sin^2\theta\right)^{\frac{3}{2}}\sqrt{2}\cos\theta d\theta = \frac{128\sqrt{2}}{3}\int_0^{\frac{\pi}{2}}\left(1-\sin^2\theta\right)^{\frac{3}{2}}\cos\theta d\theta$$

$$= \frac{128\sqrt{2}}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta = \frac{128\sqrt{2}}{3}\int_0^{\frac{\pi}{2}}\frac{1}{4}\left(1+\cos 2\theta\right)^2 d\theta$$

$$= \frac{128\sqrt{2}}{12}\int_0^{\frac{\pi}{2}}\left(1+2\cos 2\theta+\cos^2 2\theta\right)d\theta = \frac{128\sqrt{2}}{12}\int_0^{\frac{\pi}{2}}\left(1+2\cos 2\theta+\frac{1}{2}\left(1+\cos 4\theta\right)\right)d\theta$$

$$= \frac{128\sqrt{2}}{3}\left(\theta+\sin 2\theta+\frac{1}{2}\left(\theta+\frac{1}{4}\sin 4\theta\right)\right)\Bigg|_0^{\frac{\pi}{2}} = 8\pi\sqrt{2} = 35.543$$

EX5

Write six different iterated triple integrals for the volume of the region in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ *and the plane y = 3. Evaluate one of the* integrals.

Solution :-

$$v = \iiint dv$$

(1) $v = \int_0^2 \int_0^3 \int_0^{\sqrt{4-x^2}} dz\,dy\,dx$

(2) $v = \int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} dz\,dx\,dy$

(3) $v = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^3 dy\,dz\,dx$

(4) $v = \int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^3 dy\,dx\,dz.$

(5) $v = \int_0^2 \int_0^3 \int_0^{\sqrt{4-z^2}} dx\,dy\,dz.$

(6) $v = \int_0^3 \int_0^2 \int_0^{\sqrt{4-z^2}} dx\,dz\,dy$

(1) $v = \int_0^2 \int_0^3 \int_0^{\sqrt{4-x^2}} dz\,dy\,dx$

$$= \int_0^2 \int_0^3 [z]_0^{\sqrt{4-x^2}} dydx = \int_0^2 \int_0^3 \sqrt{4-x^2}\, dy\, dx$$

$$= \int_0^2 \int_0^3 \sqrt{4-x^2}\, [y]_0^3 dx = \int_0^2 3\sqrt{4-x^2}\, dx$$

Let $x^2 = 4\sin^2\theta \Rightarrow x = 2\sin\theta$, $dx = 2\cos\theta d\theta$

if $x = 2 = 2\sin\theta \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

if $x = 0 = 2\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$$v = 3\int_0^{\frac{\pi}{2}} \left(4 - 4\sin^2\theta\right)^{\frac{1}{2}} 2\cos\theta d\theta = 12\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = 12\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(1+\cos 2\theta\right)d\theta$$

$$= 6\left(\theta + \frac{1}{2}\sin 2\theta\right)_0^{\frac{\pi}{2}} = 3\pi$$