Chapter One

Matrices

1-1 Properties of Matrices

Definitions

1- An **m** x **n** or (**m,n**) , matrix is a rectangular array of quantities arranged in **m** rows and **n** columns.

2- Matrices are often represented by single capital letters. More explicitly, they are represented by displaying some or all of the constituent quantities between brackets. Thus we write, equivalently,

$$A = \left[a_{i,j}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad ;\ i = 1, 2, \ldots, m \quad j = 1, 2, \ldots, n$$

For example

$$A = [a_{2,3}] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

is a matrix with two rows and three columns (i.e. m=2, n=3)

$a_{11} = 2$, $a_{12} = 1$, $a_{13} = 3$, $a_{21} = 1$, $a_{22} = 0$, $a_{23} = -1$

1-2 Matrices Types

1- Square Matrix

A matrix with the same number of rows and columns is called a square matrix (i.e. **m=n**) .

For example

$$B = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 4 & 6 \\ 0 & 1 & 2 \end{bmatrix} \quad ; m = n = 3$$

Principal diagonal (diagonal) ; (3, 4, 2)

2-Transpose of Matrix

The **n x m** matrix obtained from a given **m x n** matrix **A** by writing its rows as columns and its columns are rows is called the transpose of **A**. The transpose of a matrix **A** is denoted by the symbol $\mathbf{A}^{\mathrm{T}}$ or $\bar{\mathbf{A}}$.

For example

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 & 4 \\ 6 & -9 & 6 & -7 & 0 \\ 3 & -2 & 0 & -1 & 1 \end{bmatrix}_{3X5} \qquad ; \mathrm{m} = 3, \ \mathrm{n} = 5$$

$$A^T = \begin{bmatrix} 2 & 6 & 3 \\ 0 & -9 & -2 \\ 1 & 6 & 0 \\ 3 & 6 & -1 \\ 4 & 0 & 1 \end{bmatrix}_{5X3} \qquad ; \mathrm{m} = 5, \ \mathrm{n} = 3$$

3- Zero Matrix (Null Matrix)

A matrix in which every element is zero is called a null matrix or zero matrix.

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4- Diagonal Matrix

A square matrix in which every element not on the principal diagonal is zero is called a diagonal matrix.

$$A = \begin{bmatrix} a_{11} & & \\ & . & \\ & & a_{nn} \end{bmatrix}$$

for example

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -10 \end{bmatrix}_{4x4}$$

5- Unit Matrix (Identity Matrix)

A diagonal matrix in which each diagonal element is 1, is called a unit matrix. A unit matrix is usually denoted by the symbol **I** .

in general form

$$I = \begin{bmatrix} i_{11} & & & \\ & i_{22} & & \\ & & . & \\ & & & i_{mm} \end{bmatrix} \quad \text{where , } i_{11}=i_{22}=\ldots.=i_{nn}=1$$

for example

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3}$$

6- Symmetric Matrix

A square matrix such that $\mathbf{a_{ij}} = \mathbf{a_{ji}}$; is said to be symmetric, since elements symmetrically located with respect to the principal diagonal are equal (i.e. A matrix equal to its transpose.

for example

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & -9 & 6 & -8 \\ 2 & 6 & 5 & -1 \\ 3 & -8 & -1 & -11 \end{bmatrix}_{4x4}$$

7- Skew-Symmetric Matrix

A square matrix such that $\mathbf{a_{ij}} = - \mathbf{a_{ji}}$ is said to be skew-symmetric (i.e. a matrix equal to the negative of its transpose , $\mathbf{A} = - \mathbf{A}^{\mathbf{T}}$) .

$$B = \begin{bmatrix} 3 & -5 & -4 \\ 5 & 0 & 9 \\ 4 & -9 & 1 \end{bmatrix}$$

8- *Triangular Matrix*

A square matrix in which every element below the principal diagonal is zero is said to be upper triangular. A square matrix in which every element above the principal diagonal is zero is said to be lower triangular.

$$B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 5 & -7 \\ 0 & 0 & -3 \end{bmatrix}$$

Upper triangular

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 4 & 5 & 4 & 0 \\ -7 & 9 & 0 & 8 \end{bmatrix}$$

Lower Triangular

1-3 Operations on Matrices (Algebra of Matrices)

1- Equal Matrices (Identical Matrices)

Two matrices **A** and **B** are equal if and only if they are identical; if and only if they contain the same number of rows and the same number of columns and $\mathbf{a_{ij}} = \mathbf{b_{ij}}$ for all values of **i** and **j** .

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$a_{11} = b_{11} \;,\quad a_{12} = b_{12} \;,\quad a_{21} = b_{21} \;,\quad a_{22} = b_{22}$$

2- Addition and Subtraction of Matrices

Two matrices A and B can be added or subtraction if and only if they have the same number of rows and the same number of columns. Their sum or difference, $A \pm B$ is the matrix we get be adding or subtracting corresponding elements in the two matrices.

In general

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ . & . & . & . \\ b_{m1} & b_{m2} & \ldots & b_{mn} \end{bmatrix}$$

$$A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \ldots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \ldots & a_{2n} \pm b_{2n} \\ . & . & . & . \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \ldots & a_{mn} \pm b_{mn} \end{bmatrix}$$

for example

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 3 & -1 \end{bmatrix}$$

$$A + B = C = \begin{bmatrix} 3 & -1 & 5 \\ 3 & 3 & -3 \end{bmatrix} \qquad A - B = D = \begin{bmatrix} 1 & 3 & 1 \\ -1 & -3 & -1 \end{bmatrix}$$

3- Multiply a Matrix by a Number

To multiply a matrix by a number **k** , we multiply each element in matrix by **k** .

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ . & . & . & . \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

for example

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix} \quad ; k = 7 ;$$

$$kA = \begin{bmatrix} 14 & 7 & 21 \\ 7 & 0 & -14 \end{bmatrix}$$

Notes

1- $(A+B) + C = A + (B + C)$

2- $A + 0 = A$

3- $A + (-A) = 0$

4- $r (A +B)= rA + rB$

5- $A + B = B + A$

6- $(r + s)A = rA + sB$

7- $(rs) A = r (sA)$

3- Multiplication of Matrices

Two matrices **A** and **B** are said to be conformable in the order **AB** if and only if the number of columns in **A** is equal to the number of rows in **B**. In other words, if **A** is an **m x r** matrix and **B** is a **k x n** matrix, **A** and **B** are conformable in the order **AB** if and only if **r=k**.

Notes :

1- A_{mxr} and B_{kxn}; to find **AB** must be **r=k** (number of columns in **A** = number of rows in **B**).

A_{mxr} , B_{rxn} , $AB = C_{mxn}$

$$AB = C = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ . & . & . & . \\ a_{m1} & a_{m2} & \dots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ . & . & . & . \\ b_{r1} & b_{r2} & \dots & b_{rn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ . & . & . & . \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

$$C_{11} = a_{11}b_{11} + a_{12}b_{21} + \dots\dots\dots + a_{1r}b_{r1} = \sum_{k=1}^{r} a_{1k}b_{k1}$$

$$C_{12} = a_{11}b_{12} + a_{12}b_{22} + \dots\dots\dots + a_{1r}b_{r2} = \sum_{k=1}^{r} a_{1k}b_{k2}$$

$$C_{ij} = \sum_{k=1}^{r} a_{ik}b_{kj}$$

2- The resulting matrix **C** is a matrix , where its number of rows equal to number of rows of matrix **A** and its number of columns equal to number of columns of matrix **B**.

3- The elements of the matrix **C** (resulting matrix) are the sum of multiply row from matrix **A** by column from matrix **B**.

EX1:-

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \qquad B=\begin{bmatrix}1 & -1 & 2\\ 3 & 5 & 1\end{bmatrix} \qquad \text{Find C=AB}$$

$$C=AB=\begin{bmatrix}(1+2\times 3) & (-1\times 1+5\times 2) & (1\times 2+2\times 1)\\ (3\times 1+4\times 3) & (3\times(-1)+4\times 5) & (3\times 2+4\times 1)\end{bmatrix}$$

$$=\begin{bmatrix}7 & 9 & 4\\ 15 & 17 & 10\end{bmatrix}$$

EX2:-

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 3 & 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 3 & -1 & 5 \end{bmatrix}$$

$$C_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = \left[1 \times (-1) + 2 \times 0 + (-1) \times 3\right] = -4$$

$$C_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = \left[1 \times 2 + 2 \times 1 + (-1) \times (-1)\right] = 5$$

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$C = AB = \begin{bmatrix} -4 & 5 & -2 \\ 3 & 1 & 5 \\ 12 & 5 & 34 \end{bmatrix}$$

Notes

1- $AB \neq BA$

2- $(AB)\,C = A(BC)$

3- $(B+C)\,A = BA + CA$

4- $k(AB) = (kA)\,B = A\,(k\,B)$ k : any number

5- $(A+B)^T = A^T + B^T$

6- $(A^T)^T = A$

7- $(AB)^T = B^T\,A^T$

8- $IA = A$ and $0.A = 0$

1-4 Partition of Matrices

When the matrix big, we must partition the matrix into small matrices to simplify the solution.

EX:- Find C=AB if $A=\begin{bmatrix}1 & 2 & 1\\ 3 & 4 & 0\\ 0 & 0 & 2\end{bmatrix}$ $B=\begin{bmatrix}1 & 2 & 3 & 1\\ 4 & 5 & 6 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{A}_{11} & \mathrm{A}_{12} \end{array} \\ A= & \left[\begin{array}{cc:c} 1 & 2 & 1\\ 3 & 4 & 0\\ \hdashline 0 & 0 & 2 \end{array}\right] \\ & \begin{array}{cc} \mathrm{A}_{21} & \mathrm{A}_{22} \end{array} \end{array} \qquad A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{B}_{11} & \mathrm{B}_{12} \end{array} \\ B= & \left[\begin{array}{ccc:c} 1 & 2 & 3 & 1\\ 4 & 5 & 6 & 1\\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \\ & \begin{array}{cc} \mathrm{B}_{21} & \mathrm{B}_{22} \end{array} \end{array} \qquad B=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}$$

$$AB=\begin{bmatrix}A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22}\\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22}\end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} + A_{12}B_{22} \\ 0 & A_{22}B_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} 9 & 12 & 15 & 3 \\ 19 & 26 & 33 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note:-

When we partition the matrix, we must get as much as possible zero matrix or unit matrix and in suitable way.

$$A = \left[\begin{array}{ccc:cc} 0 & 2 & 1 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1-5 Determinants

An **n** by **n** matrix is called a matrix of order **n**, for short. with such a matrix we association a number called the determinant of **A** and written sometimes **det(A)** or |**A**| and sometimes |$\mathbf{a_{ij}}$| with vertical bars (which do not mean absolute value).

Let A any square matrix of order n

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad ; m=n$$

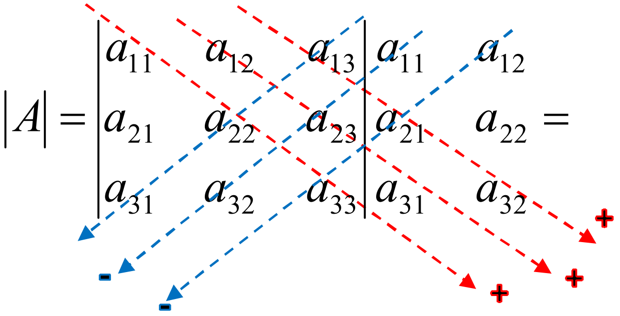
1- When **n=1** (first order) $|A|=|a_{11}| = a_{11}$; $|5| = 5$ or $|-3| = -3$

2- When **n=2** (second order) : the determinant is equal to the difference between the product of the element on the principal diagonal and the product of the elements on other diagonal.

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2\times1-(-1)\times3 = 5$$

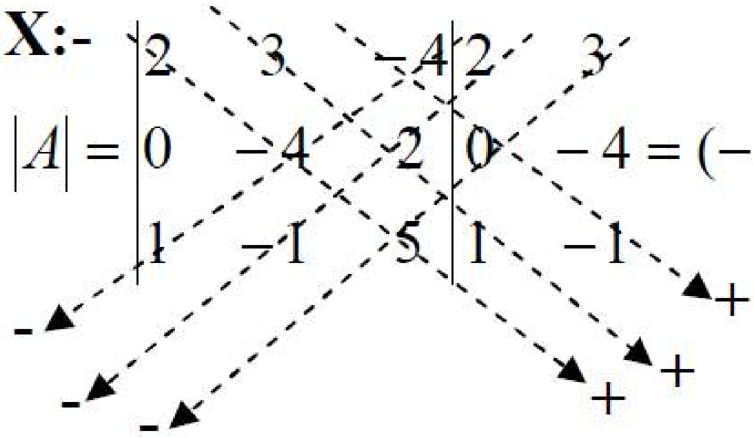
3- When **n=3** (third order) : the determinant can be obtained by diagonal multiplication, by repeating on the right the first two columns of the determinant and often adding the signed products of the element on the various diagonals in the resulting array.



$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

EX:-



$$|A| = \left|\begin{array}{ccc|cc} 2 & 3 & -4 & 2 & 3 \\ 0 & -4 & 2 & 0 & -4 \\ 1 & -1 & 5 & 1 & -1 \end{array}\right. = (-40+6+0)-(16-4+0)$$

$$= -34 - 12 = -46$$

$$|B| = \left|\begin{array}{ccc|cc} 2 & 3 & 4 & 2 & 3 \\ 5 & 6 & 7 & 5 & 6 \\ 8 & 9 & 1 & 8 & 9 \end{array}\right. = 12+168+180-192-126-15 = 27$$

4- When **n > 3** (in general form) : is equal to the sum of the products of the elements of any row or columns and their respective cofactors; i.e.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix}$$

(row definition)

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}$$

or

(column definition)

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}$$

Where : A_{ij} :- any element oh the cofactor matrix

$$C(A)=\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ . & . & . & . \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$$

(Cofactor Matrix)

$$A_{ij}=(-1)^{i+j}\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1(j-1)} & a_{1(j+1)} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(j-1)} & a_{2(j+1)} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \dots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(j-1)} & a_{n(j+1)} & \dots & a_{nn} \end{vmatrix}$$

EX1:- Find the cofactor of the element a_{23} and a_{22}

$$A=\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{bmatrix}$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -(18-24) = 6 \qquad A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{bmatrix}$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 4 \\ 8 & 1 \end{vmatrix} = 2-32 = -30 \qquad A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{bmatrix}$$

EX2: Find the determinant of the matrix

$$|A| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 0 & -1 & 2 & 3 \\ 1 & 6 & 4 & -2 \end{vmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 0 & -1 & 2 & 3 \\ 1 & 6 & 4 & -2 \end{vmatrix}$$

$$|A| = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} + a_{34}A_{34}$$

$$|A| = (0)(-1)^{3+1}\begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 6 & 4 & -2 \end{vmatrix} + (-1)(-1)^{3+2}\begin{vmatrix} 1 & 3 & 4 \\ 4 & 2 & 1 \\ 1 & 4 & -2 \end{vmatrix} + (2)(-1)^{3+3}\begin{vmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 1 & 6 & -2 \end{vmatrix} + (3)(-1)^{3+4}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 1 & 6 & 4 \end{vmatrix}$$

$$|A| = 0 + 75 + 180 - 105 = 150$$

EX3:-

$$|A| = \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix}$$

Multiply second row by (-2) and adding to first row

$$|A| = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix}$$

Multiply second row by (3) and adding to third row

$$|A| = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 1 & -2 & -1 & 4 \end{vmatrix}$$

Adding second row with fourth row

$$|A| = \left|\begin{matrix}1\\2\\1\\3\end{matrix}\quad\begin{matrix}-2\\3\\2\\1\end{matrix}\quad\boxed{\begin{matrix}0\\1\\0\\0\end{matrix}}\quad\begin{matrix}5\\-2\\3\\2\end{matrix}\right|$$

$$|A| = \cancelto{0}{a_{13}} A_{13} + \cancelto{1}{a_{23}} A_{23} + \cancelto{0}{a_{33}} A_{33} + \cancelto{0}{a_{43}} A_{43} = A_{23}$$

$$|A| = (-1)^{2+3}\begin{vmatrix}1 & -2 & 5\\ 1 & 2 & 3\\ 3 & 1 & 2\end{vmatrix} = 38$$

Notes:-

1- If two rows or columns of a matrix are identical, the determinant is zero.

2- The determinant of a matrix is the sum of the products of the elements of the ith row or column by their cofactors, for any **i**.

3- The determinant of the transpose of a matrix is equal to the original determinant $|A|=|A^T|$.

4- If each element of some row or column of a matrix is multiplied by a constant **c**, the determinant is multiplied by **c**.

5- If the matrix is upper or lower triangular matrix, the determinant of the matrix is the product of the elements on the main diagonal.

1-6 Matrix Inverse (Inverse of a Matrix)

$$A^{-1} = \frac{1}{A}$$

$$A^{-1} = \frac{(c(A))^T}{|A|} = \frac{adj(A)}{|A|}$$

Where $\mathbf{(c(A))^T}$ is called the adjoint of the matrix. The adjoint of a square matrix A is sometimes indicated by the notation **adj(A)**.

Steps :

1-Find |**A**| and must be not equal zero (|**A**|**≠0)** (nonsingular matrix)

2- Find c(A) then find adj(A)

$$A^{-1} = \frac{(c(A))^T}{|A|} = \frac{adj(A)}{|A|}$$

EX:- If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ find A^{-1}

1- $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{vmatrix}\begin{matrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{matrix} = (9 + 8 + 12) - (9 + 16 + 6) = -2 \neq 0$

2- $A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -7$ $\quad .A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1$ $\quad A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6$ $\quad A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$ $\quad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -2$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1$ $\quad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$ $\quad A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$

3-

$$c(A) = \begin{bmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}.$$

$$adj(A) = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}.$$

$$A^{-1} = \begin{bmatrix} \frac{7}{2} & -3 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$$

Must be :

$$AA^{-1} = I$$

1-7 Linear Equations

In general form the linear equation is :

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

Where x_1 , x_2 ….,x_n : variables

a_1 , a_2 ….a_n : coefficient

b : constant

A system of linear equations :

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{array}$$

is called as **m** equation for **n** variables (unknowns) and $\mathbf{x_1}$, $\mathbf{x_2}$ **….,**$\mathbf{x_n}$ which solve the system called solution. Which can be written in matrix form as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

where A : Coefficient matrix , X: variables matrix and B: constant matrix of the system

in simple form $AX = B$

Methods of Solution

(1) Inverse Method

$$AX = B \qquad ; \qquad X = A^{-1}B$$

EX : Use inverse method to solve the simultaneous equations

$$3x + 4y - 3z = 2$$

$$-2x + 2y + 2z = 15$$

$$7x - 5y + 4z = 26$$

$$\underbrace{\begin{bmatrix} 3 & 4 & -3 \\ -2 & 2 & 2 \\ 7 & -5 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 2 \\ 15 \\ 26 \end{bmatrix}}_{B}$$

$$X = A^{-1}B$$

$$|A| = \left| \begin{matrix} 3 & 4 & -3 \\ -2 & 2 & 2 \\ 7 & -5 & 4 \end{matrix} \right| \begin{matrix} 3 & 4 \\ -2 & 2 \\ 7 & -5 \end{matrix} \quad = (24 + 56 - 30) - (-42 - 30 - 32) = 154$$

$$A_{11}=(-1)^{1+1}\begin{vmatrix}2 & 2\\ -5 & 4\end{vmatrix}=18\ ,\ A_{12}=(-1)^{1+2}\begin{vmatrix}-2 & 2\\ 7 & 4\end{vmatrix}=22\ ,\ A_{13}=(-1)^{1+3}\begin{vmatrix}-2 & 2\\ 7 & -5\end{vmatrix}=-4$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}4 & -3\\ -5 & 4\end{vmatrix}=-1\ ,\ A_{22}=(-1)^{2+2}\begin{vmatrix}3 & -3\\ 7 & 4\end{vmatrix}=33\ ,\ A_{23}=(-1)^{2+3}\begin{vmatrix}3 & 4\\ 7 & -5\end{vmatrix}=43$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}4 & -3\\ 2 & 2\end{vmatrix}=14\ ,\ A_{32}=(-1)^{3+2}\begin{vmatrix}3 & -3\\ -2 & 2\end{vmatrix}=0\ ,\ A_{33}=(-1)^{3+3}\begin{vmatrix}3 & 4\\ -2 & 2\end{vmatrix}=14$$

$$C(A)=\begin{bmatrix}18 & 22 & -4\\ -1 & 33 & 43\\ 14 & 0 & 14\end{bmatrix}\quad ; \qquad adj(A)=\begin{bmatrix}18 & -1 & 14\\ 22 & 33 & 0\\ -4 & 43 & 14\end{bmatrix}$$

$$A^{-1}=\frac{1}{154}\begin{bmatrix}18 & -1 & 14\\ 22 & 33 & 0\\ -4 & 43 & 14\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{154}\begin{bmatrix}18 & -1 & 14\\ 22 & 33 & 0\\ -4 & 43 & 14\end{bmatrix}\begin{bmatrix}2\\ 15\\ 26\end{bmatrix}=\begin{bmatrix}2.5\\ 3.5\\ 6.5\end{bmatrix}$$

(2) Cramer's Rule

Let we have the system **AX = B**, where **|A|≠0** and **B≠0** then the solution of the system is :

$$x_1 = \frac{|D_1|}{|A|} \qquad x_2 = \frac{|D_2|}{|A|}......, \quad x_n = \frac{|D_n|}{|A|}$$

Where **D_i** is the matrix obtained from **A** by replacing the ith column of **A** by the column matrix **B**.

$$D_1 = \begin{bmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_m & a_{m2} & \dots & a_{mn} \end{bmatrix}. \; ; \; D_2 = \begin{bmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & b_m & \dots & a_{mn} \end{bmatrix}. \; ; \; D_n = \begin{bmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & b_m \end{bmatrix}.$$

EX:- Find the solution of the linear equations below

$$x_1 + 2x_2 + x_3 = 0$$

$$3x_1 - x_2 - 2x_3 = 9$$

$$4x_1 + 3x_2 - 3x_3 = 3$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -2 \\ 4 & 3 & -3 \end{bmatrix} \qquad .X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \qquad B = \begin{bmatrix} 0 \\ 9 \\ 3 \end{bmatrix} \qquad AX = B$$

$$|A| = 24$$

$$|D_1| = \begin{vmatrix} 0 & 2 & 1 \\ 9 & -1 & -2 \\ 3 & 3 & -3 \end{vmatrix} = 72 \qquad x_1 = \frac{|D_1|}{|A|} = \frac{72}{24} = 3$$

$$|D_2| = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 9 & -2 \\ 4 & 3 & -3 \end{vmatrix} = -48 \qquad x_2 = \frac{|D_2|}{|A|} = \frac{-48}{24} = -2$$

$$|D_3| = \begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 9 \\ 4 & 3 & 3 \end{vmatrix} = 24 \qquad x_3 = \frac{|D_3|}{|A|} = \frac{24}{24} = 1$$

$$X = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Example

Use Cramer's rule to solve the following linear equations

$$\begin{array}{l} x+2z+5=y \\ 3z=x \\ 2x+y \quad =1 \end{array}$$

$$\begin{array}{l} x-y+2z=-5 \\ -x \quad +3z=0 \\ 2x+y \quad =1 \end{array} \quad ; \quad \overset{A}{\begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} \overset{=}{=} \overset{B}{\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}} ; \quad |A|=-11$$

$$|D_1|=\begin{vmatrix} -5 & -1 & 2 \\ 0 & 0 & 3 \\ 1 & 1 & 0 \end{vmatrix}=12 ; \quad |D_2|=\begin{vmatrix} 1 & -5 & 2 \\ -1 & 0 & 3 \\ 2 & 1 & 0 \end{vmatrix}=-35 ; \quad |D_3|=\begin{vmatrix} 1 & -1 & -5 \\ -1 & 0 & 0 \\ 2 & 1 & 1 \end{vmatrix}=4$$

$$x=\frac{|D_1|}{|A|}=-\frac{12}{11} \quad ; \quad y=\frac{|D_2|}{|A|}=\frac{35}{11} \quad ; \quad z=\frac{|D_3|}{|A|}=-\frac{4}{11}$$

Example If $\frac{1}{2}(A-2C)+3E=5B$, find E where

$$A=\begin{bmatrix}1 & 2 & -3\\ 3 & 4 & 5\\ 2 & -1 & 7\end{bmatrix}\ ;\ B=\begin{bmatrix}3 & 1 & 5\\ 1 & -4 & 7\\ 2 & 1 & 3\end{bmatrix}\text{ and }C=\begin{bmatrix}1 & 0 & 2\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$E=\frac{1}{3}\left(5B-\frac{1}{2}(A-2C)\right)$$

$$E=\frac{1}{3}\left(5\begin{bmatrix}3 & 1 & 5\\ 1 & -4 & 7\\ 2 & 1 & 3\end{bmatrix}-\frac{1}{2}\left(\begin{bmatrix}1 & 2 & -3\\ 3 & 4 & 5\\ 2 & -1 & 7\end{bmatrix}-2\begin{bmatrix}1 & 0 & 2\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\right)$$

$$E=\frac{1}{3}\left(\begin{bmatrix}15 & 5 & 25\\ 5 & -20 & 35\\ 10 & 5 & 15\end{bmatrix}-\frac{1}{2}\begin{bmatrix}-1 & 2 & -7\\ -1 & 2 & 5\\ 2 & -1 & 5\end{bmatrix}\right)=\frac{1}{3}\begin{bmatrix}15.5 & 4 & 28.5\\ 5.5 & -21 & 32.5\\ 9 & 5.5 & 12.5\end{bmatrix}$$

$$E = \begin{bmatrix} 5.1667 & 1.334 & 9.5 \\ 1.8334 & -7 & 10.8334 \\ 3 & 1.8334 & 4.1667 \end{bmatrix}$$

Example

If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, verify that $A^2 - 4A + 5I = 0$

$$A^2 - 4A + 5I = 0$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-2 & 2+6 \\ -1-3 & -2+9 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ -4 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example

Prove without open the determinant that, (using determinant properties)

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

Subtracting first row from second row

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = (x_2 - x_1)\begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & x_2 + x_1 \\ 1 & x_3 & x_3^2 \end{vmatrix}$$

Subtracting first row from third row

$$(x_2 - x_1)\begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & x_2 + x_1 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1)\begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & x_2 + x_1 \\ 0 & 1 & x_3 + x_1 \end{vmatrix}$$

Subtracting second row from third row

$$(x_2-x_1)(x_3-x_1)\begin{vmatrix}1 & x_1 & x_1^2\\0 & 1 & x_2+x_1\\0 & 0 & x_3-x_2\end{vmatrix}$$

$$=(x_2-x_1)(x_3-x_1)(x_3-x_2)\begin{vmatrix}1 & x_1 & x_1^2\\0 & 1 & x_2+x_1\\0 & 0 & 1\end{vmatrix}$$

$$=(x_2-x_1)(x_3-x_1)(x_3-x_2)$$

Example

For what values of c do the following matrix has inverse?

$$\begin{bmatrix}3 & 1 & 0\\-4 & 2 & 5\\c^2 & c & 1\end{bmatrix}$$

$$\begin{vmatrix} 3 & 1 & 0 \\ -4 & 2 & 5 \\ c^2 & c & 1 \end{vmatrix} = 0$$

$$\left|\begin{matrix} 3 & 1 & 0 \\ -4 & 2 & 5 \\ c^2 & c & 1 \end{matrix}\right| \begin{matrix} 3 & 1 \\ -4 & 2 \\ c^2 & c \end{matrix} \quad = 6 + 5c^2 - 15c + 4 = 5c^2 - 15c + 10$$

$$5c^2 - 15c + 10 = 0 \ , \ c^2 - 3c + 2 = 0$$

$$(c-1)(c-2) = 0 \ , \ c = 1 \ \text{or} \ c = 2$$

All numbers except c=1 or c=2