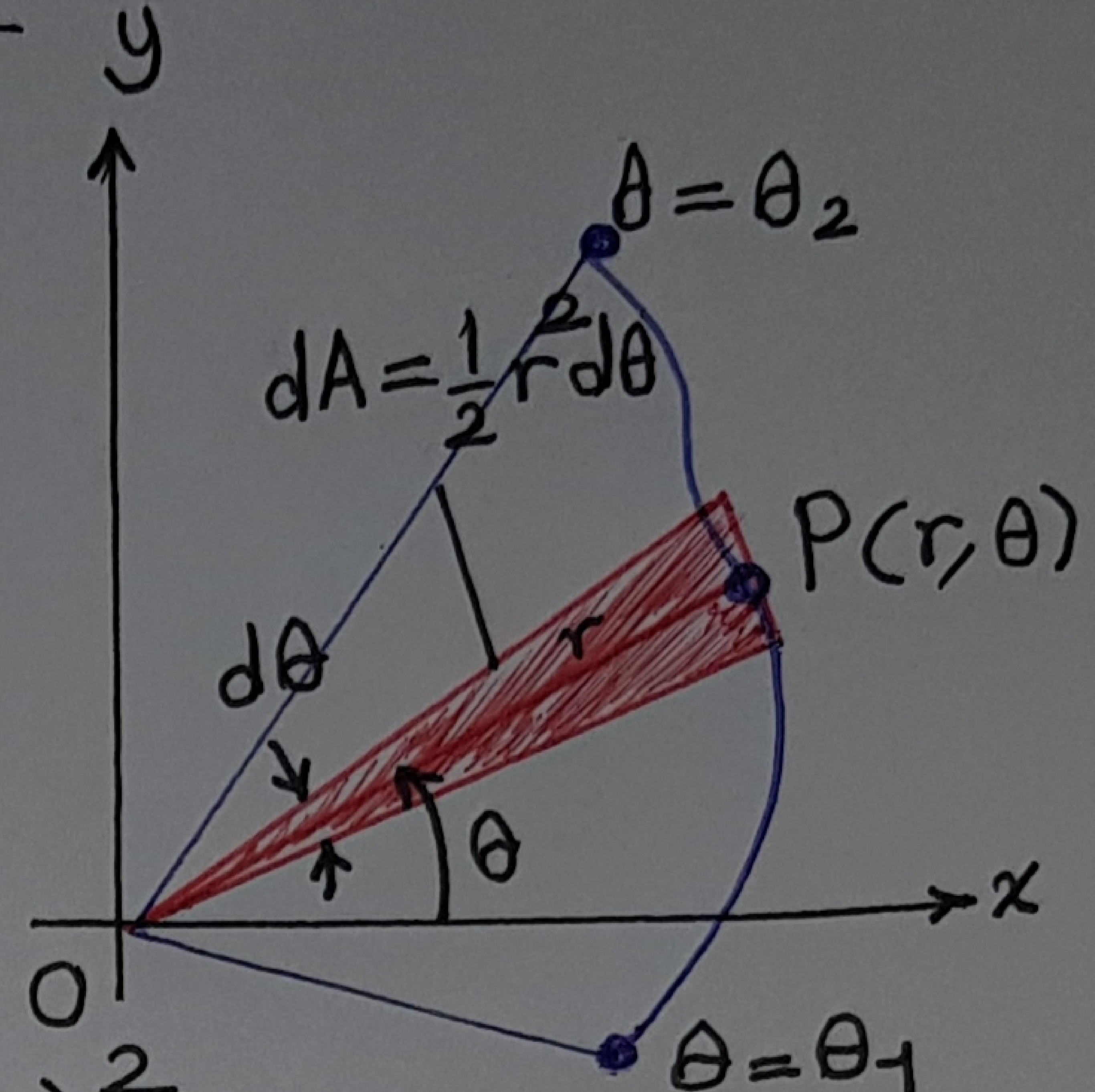
Plane Area in Polar Form

If $r = f(\theta)$, to find the area of the region enclosed by $f(\theta)$:

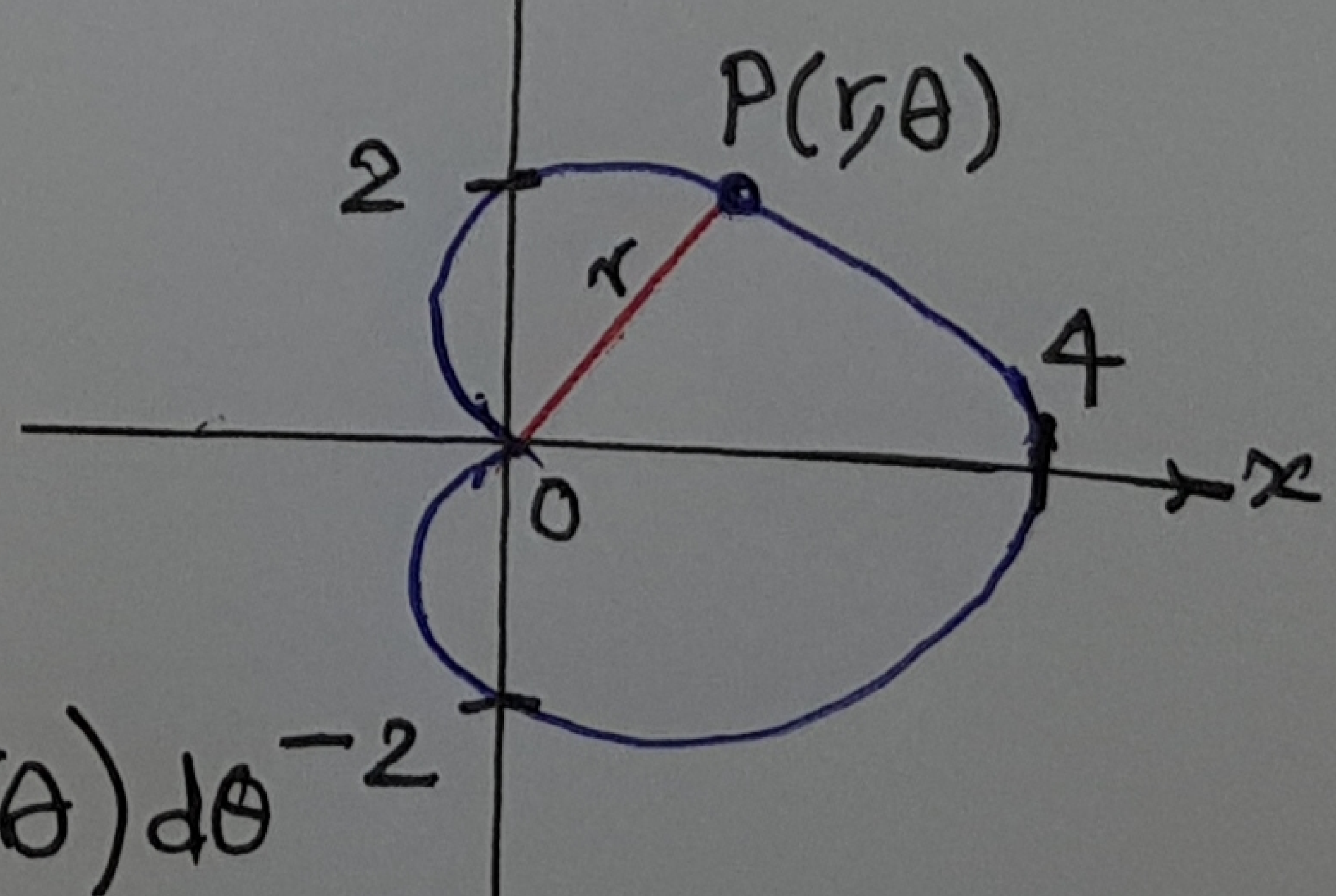


$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} (f(\theta))^2 \, d\theta$$

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta .$$

Example 1: Find the area of the region enclosed by the curve $r = 2(1+\cos\theta)$.

Solution: $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$



$$= \int_0^{2\pi} \frac{1}{2} \left[2(1+\cos\theta)\right]^2 d\theta$$

$$= \int_0^{2\pi} 2\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \int_0^{2\pi} \left(2 + 4\cos\theta + \cancel{2}\,\frac{(1+\cos 2\theta)}{\cancel{2}}\right) d\theta$$

(1)

$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta$$

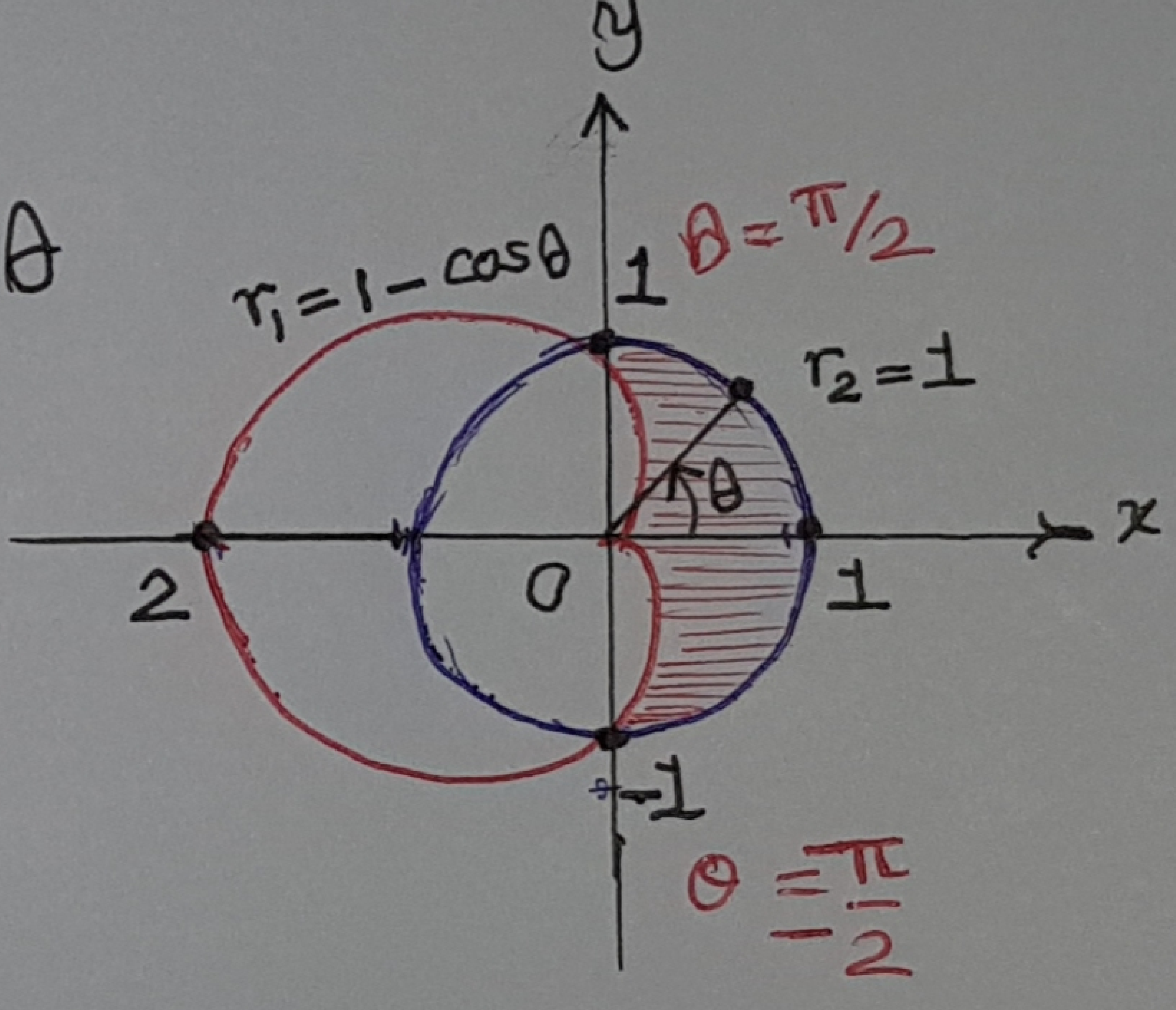
$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2} \right]_0^{2\pi}$$

$$= [6\pi + 0 + 0] - [0] = 6\pi.$$

Example 2 Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Solution

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(r_2^2 - r_1^2)\, d\theta$$

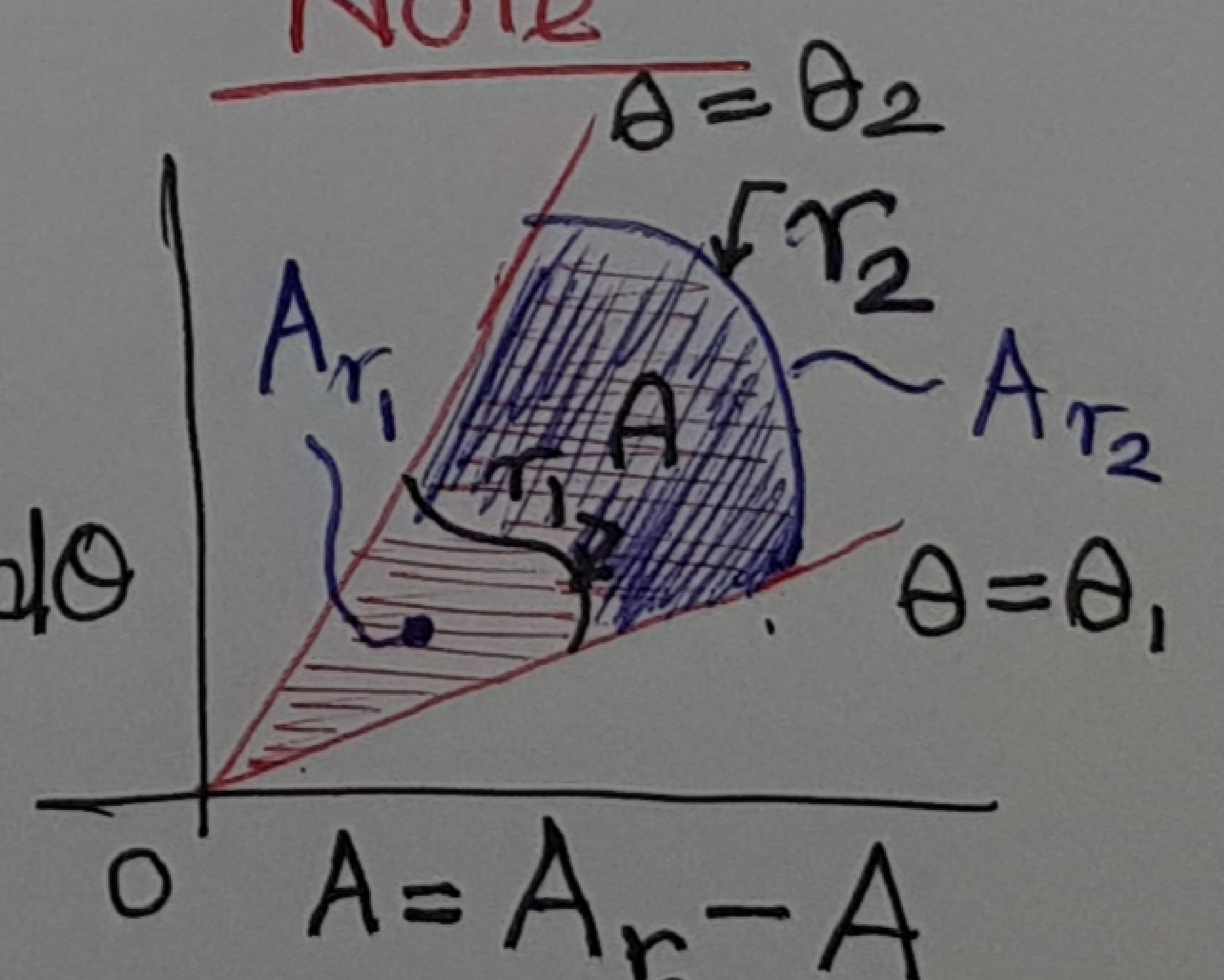


$$= 2 \int_0^{\pi/2} \frac{1}{2}(r_2^2 - r_1^2)\, d\theta$$

$$= \int_0^{\pi/2} \left[(1)^2 - (1 - \cos\theta)^2 \right] d\theta$$

$$= \int_0^{\pi/2} \left(1 - (1 - 2\cos\theta + \cos^2\theta)\right) d\theta$$

Note



$A = A_{r_2} - A_{r_1}$

$$= \int_0^{\pi/2} \left(2\cos\theta - \frac{(1+\cos 2\theta)}{2} \right) d\theta$$

$$= \left[2\sin\theta - \frac{1}{2}\theta - \frac{\sin 2\theta}{4} \right]_0^{\pi/2}$$

$$= \left[2(1) - \frac{1}{2}\left(\frac{\pi}{2}\right) - 0 \right] - \left[0 \right]$$

$$= 2 - \frac{\pi}{4}.$$