Example : Find the derivative of

(A) $y = \sin^{-1}\left(\frac{3}{t^2}\right) = \csc^{-1}\left(\frac{t^2}{3}\right) \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{2t}{3}\right)}{\left|\frac{t^2}{3}\right|\sqrt{\left(\frac{t^2}{3}\right)^2 - 1}} = \frac{-2t}{t^2\sqrt{\frac{t^4-9}{9}}} = \frac{-6}{t\sqrt{t^4-9}}$

(B) $y = x\sin^{-1}x + \sqrt{1-x^2} = x\sin^{-1}x + (1-x^2)^{1/2} \Rightarrow \frac{dy}{dx} = \sin^{-1}x + x\left(\frac{1}{\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)$

$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sin^{-1}x$

The following formulas hold for any constant $a \neq 0$.

1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

EXAMPLE 7 Evaluate

(a) $\int \frac{dx}{\sqrt{4x - x^2}}$ **(b)** $\int \frac{dx}{4x^2 + 4x + 2}$

Solution **(a)** The expression $\sqrt{4x - x^2}$ does not match any of the formulas in Table 7.4, so we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x - 2)^2.$$

Then we substitute $a = 2$, $u = x - 2$, and $du = dx$ to get

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x-2)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}} \qquad a = 2,\ u = x - 2,\ \text{and } du = dx$$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 7.4, Formula 1}$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

(b) We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$$

$$= 4\left(x + \frac{1}{2}\right)^2 + 1 = (2x + 1)^2 + 1.$$

Then,

$$\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x + 1)^2 + 1} = \frac{1}{2}\int \frac{du}{u^2 + a^2} \qquad a = 1, u = 2x + 1, \text{ and } du/2 = dx$$

$$= \frac{1}{2} \cdot \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 7.4, Formula 2}$$

$$= \frac{1}{2} \tan^{-1}(2x + 1) + C \qquad a = 1, u = 2x + 1$$

$$\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2 - u^2}} \qquad a = \sqrt{3}, u = 2x, \text{ and } du/2 = dx$$

$$= \frac{1}{2} \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 7.4, Formula 1}$$

$$= \frac{1}{2} \sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

$$\int \frac{1}{y\sqrt{4y^2 - 1}}\, dy = \int \frac{2}{(2y)\sqrt{(2y)^2 - 1}}\, dy = \int \frac{1}{u\sqrt{u^2 - 1}}\, du, \text{ where } u = 2y \text{ and } du = 2\, dy$$
$$= \sec^{-1}|u| + C = \sec^{-1}|2y| + C$$

Example : Find the derivative of y

$$y = \ln \frac{1}{x\sqrt{x+1}} = -\ln x - \frac{1}{2}\ln(x+1) \Rightarrow y' = -\frac{1}{x} - \frac{1}{2}\left(\frac{1}{x+1}\right) = -\frac{2(x+1)+x}{2x(x+1)} = -\frac{3x+2}{2x(x+1)}$$

Example : Evaluate the integrals

(A) $\int \frac{dx}{2\sqrt{x}+2x} = \int \frac{dx}{2\sqrt{x}(1+\sqrt{x})}$; let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\, dx$; $\int \frac{dx}{2\sqrt{x}(1+\sqrt{x})} = \int \frac{du}{u} = \ln|u| + C$
$= \ln|1 + \sqrt{x}| + C = \ln(1 + \sqrt{x}) + C$

(B) Let $u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x)\, dx = (\sec x)(\tan x + \sec x)\, dx \Rightarrow \sec x\, dx = \frac{du}{u}$;
$\int \frac{\sec x\, dx}{\sqrt{\ln(\sec x + \tan x)}} = \int \frac{du}{u\sqrt{\ln u}} = \int (\ln u)^{-1/2} \cdot \frac{1}{u}\, du = 2(\ln u)^{1/2} + C = 2\sqrt{\ln(\sec x + \tan x)} + C$

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$ **2.** $e^{-x} = \frac{1}{e^x}$

3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$ **4.** $(e^{x_1})^r = e^{r x_1}$, if r is rational

Example : Find the derivative of y

$$y = e^{\sin t}(\ln t^2 + 1) \Rightarrow \frac{dy}{dt} = e^{\sin t}(\cos t)(\ln t^2 + 1) + \frac{2}{t}e^{\sin t} = e^{\sin t}\left[(\ln t^2 + 1)(\cos t) + \frac{2}{t}\right]$$

Example : Evaluate the integrals

Let $u = \tan\theta \Rightarrow du = \sec^2\theta\, d\theta;\ \theta = 0 \Rightarrow u = 0,\ \theta = \frac{\pi}{4} \Rightarrow u = 1$;

$$\int_0^{\pi/4}(1 + e^{\tan\theta})\sec^2\theta\, d\theta = \int_0^{\pi/4}\sec^2\theta\, d\theta + \int_0^1 e^u\, du = [\tan\theta]_0^{\pi/4} + [e^u]_0^1 = \left[\tan\left(\frac{\pi}{4}\right) - \tan(0)\right] + (e^1 - e^0)$$
$$= (1 - 0) + (e - 1) = e$$

$$\int_0^{\ln 2} 4e^x \sinh x\, dx = \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}\, dx = \int_0^{\ln 2}(2e^{2x} - 2)\, dx$$
$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0)$$
$$= 4 - 2\ln 2 - 1 \approx 1.6137$$

Example : Find the derivative of y

$$y = 2\sqrt{t}\tanh\sqrt{t} = 2t^{1/2}\tanh t^{1/2} \Rightarrow \frac{dy}{dt} = \left[\operatorname{sech}^2\left(t^{1/2}\right)\right]\left(\frac{1}{2}t^{-1/2}\right)\left(2t^{1/2}\right) + \left(\tanh t^{1/2}\right)\left(t^{-1/2}\right) = \operatorname{sech}^2\sqrt{t} + \frac{\tanh\sqrt{t}}{\sqrt{t}}$$

Example : Evaluate the integral

$$\int_{-\pi/4}^{\pi/4}\cosh(\tan\theta)\sec^2\theta\, d\theta = \int_{-1}^{1}\cosh u\, du = [\sinh u]_{-1}^{1} = \sinh(1) - \sinh(-1) = \left(\frac{e^1 - e^{-1}}{2}\right) - \left(\frac{e^{-1} - e^1}{2}\right)$$

$= \frac{e - e^{-1} - e^{-1} + e}{2} = e - e^{-1}$, where $u = \tan\theta$, $du = \sec^2\theta\, d\theta$, the lower limit is $\tan\left(-\frac{\pi}{4}\right) = -1$ and the upper limit is $\tan\left(\frac{\pi}{4}\right) = 1$