Chapter Six: Transcendental Functions

Functions can be classified into two groups called ***algebraic functions*** and ***transcendental functions.***

7.1 Inverse Functions

One-to-One Function

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

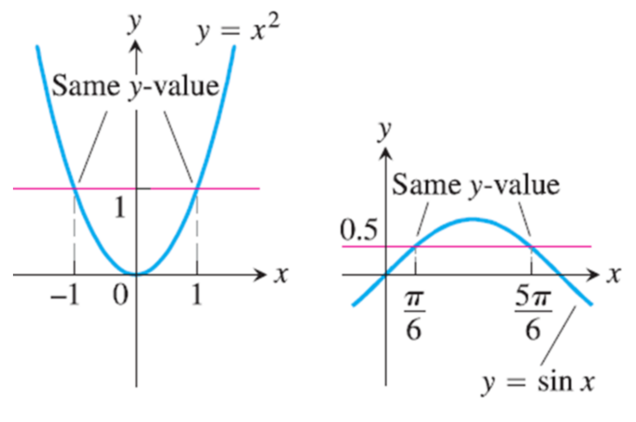


Fig.1 (b) Not one-to-one: Graph intersects one or more horizontal lines **more than once.**

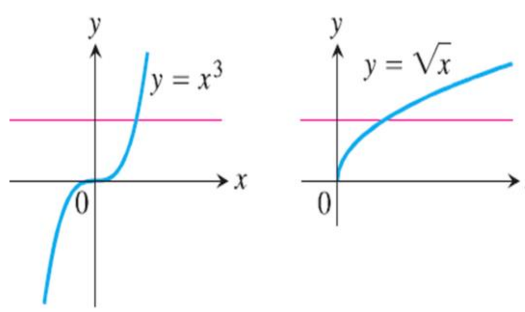


Fig.1 (a) One-to-one: Graph intersects each horizontal line **once**.

Inverse Functions

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

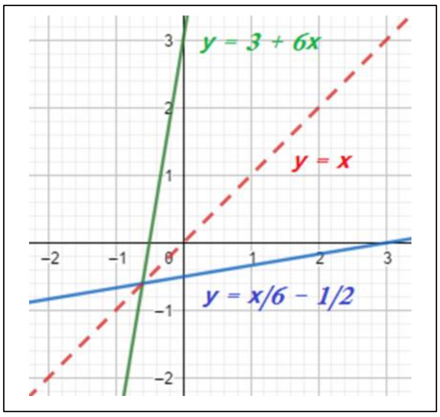
$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

Example 1: Find the inverse of $y = 3 + 6x$.

Solution: 1. Solve for x = *f*(y) : $6x = y - 3 \ \rightarrow x = \frac{y}{6} - \frac{1}{2}$

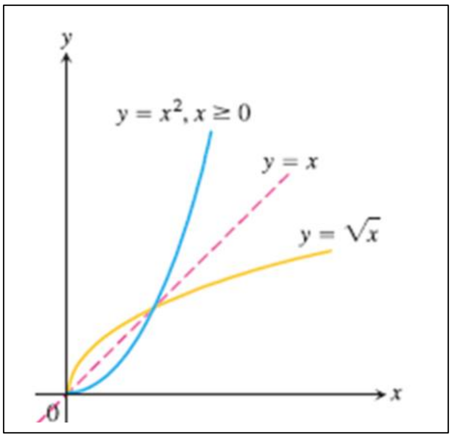
2. Interchange x and y : $y = \frac{x}{6} - \frac{1}{2}$



Example 2: Find the inverse of $y = x^2,\ x \geq 0.$

Solution: 1. Solve for x = *f*(*y*) : $x^2 = y \ \rightarrow x = \sqrt{y}$

2. Interchange x and y : $y = \sqrt{x}\ \ , x \geq 0.$



Inverse Trigonometric Function

The inverse functions for the six basic trigonometric functions

$y = \sin^{-1} x$	or	$y = \arcsin x$
$y = \cos^{-1} x$	or	$y = \arccos x$
$y = \tan^{-1} x$	or	$y = \arctan x$
$y = \cot^{-1} x$	or	$y = \text{arccot}\, x$
$y = \sec^{-1} x$	or	$y = \text{arcsec}\, x$
$y = \csc^{-1} x$	or	$y = \text{arccsc}\, x$

NOTE : $y = \sin^{-1} x \neq \frac{1}{\sin x}$

Derivatives of the Inverse Trigonometric Function

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{du/dx}{1+u^2}$

4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{du/dx}{1+u^2}$

5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

6. $\dfrac{d(\csc^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

Integration Formulas

1. $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + C$

2. $\int \frac{du}{1+u^2} = \tan^{-1} u + C$

3. $\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C$

4. $\int \frac{-du}{\sqrt{1-u^2}} = \cos^{-1}(u) + C$

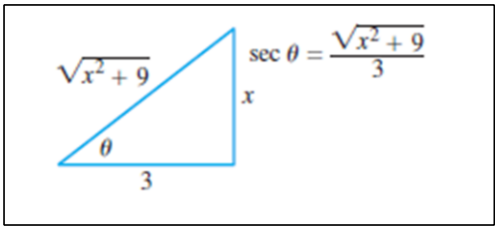
5. $\int \frac{-du}{1+u^2} = \cot^{-1} u + C$

6. $\int \frac{-du}{u\sqrt{u^2-1}} = \csc^{-1} u + C$

Example: What is the angle that has a sine equal to $\sqrt{2}/2$

Solution: $\sin^{-1}\frac{\sqrt{2}}{2} = \pi/4$. $(\sin\frac{\pi}{4} = \sqrt{2}/2$.

Solving using a triangle



Example: Evaluate $\sec(\tan^{-1}\frac{x}{3})$.

Solution:

Let $\theta = \tan^{-1} x/3$

$\tan\theta = x/3 \rightarrow \sec(\tan^{-1}\frac{x}{3}) = \sec\theta = \sqrt{x^2+9}/3$

Examples: Evaluate the following

(a) $\frac{d}{dx}\cos^{-1}x^2 = -\frac{1}{\sqrt{1-(x^2)^2}}(2x).$

(b) $\frac{d}{dx}\sec^{-1}5x^4 = \frac{1}{|5x^4|\sqrt{(5x^4)^2-1}}(20x^3) = \frac{4}{x\sqrt{(5x^4)^2-1}}$

(c) $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x\,\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \pi/4$

Home Work: Exercises 7.7 Pages 530-532.

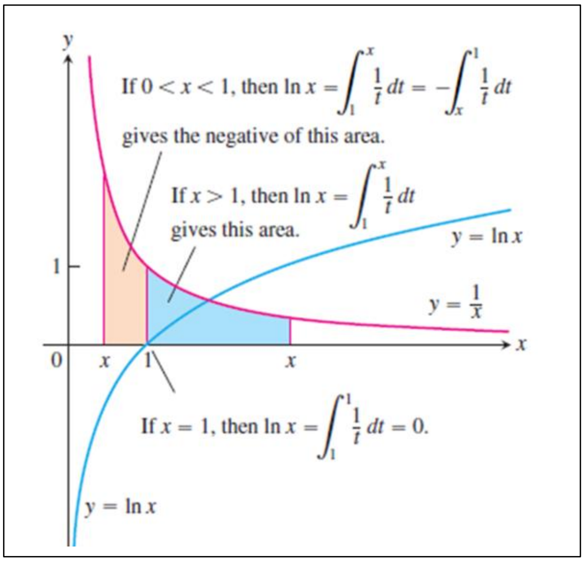
7.2 Natural Logarithms

lnx is the function in the range $(0,\infty)$ and is defined by:

Definition: The Natural Logarithm Function

$$lnx = \int_1^x \frac{1}{t}dt, \qquad x > 0$$

Thus, the natural logarithm is the area under the curve $f(t) = 1/t$ between t = 1 and t = x.



If x = 1,

$$lnx = ln1 = \int_1^1 \frac{1}{t}dt = 0$$

The Derivative of *y = lnx*

$$\frac{d}{dx}lnx = \frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x}.$$

In general, if *y* = ln*u* and *u = f(x),*

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{u}.\frac{du}{dx}.$$

Example: $\frac{d}{dx} ln2x = \frac{1}{2x}(2) = \frac{1}{x}$.

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}(2x) = 2x/(x^2+3).$$

Example: if $y = \ln(3x^2+4)$, find dy/dx.

Solution: $\frac{dy}{dx} = \frac{1}{3x^2+4}.(6x)$.

Example: if y = $\ln(sin7x^3)$, find dy/dx.

Solution: $\frac{dy}{dx} = \frac{1}{sin7x^3}.cos7x^3.(21x^2) = \frac{21x^2cos7x^3}{sin7x^3}$.

Properties of Natural Logarithms (ln*x*)

For any numbers a > 0 and x > 0:

1. ln(ax) = lna + lnx   2. Ln(a/x) = lna – lnx   3. Ln(x^n) = n lnx

4. ln(1/x) = - lnx   5. Ln(0) = - ∞.

NOTE: The Integral $\int \frac{du}{u}$

If u is a differentiable function that is never zero,

$$\int \frac{du}{u} = \ln|u| + C.$$

Examples:

(a) $\int \frac{cos\theta}{1+sin\theta} d\theta = \ln|1+sin\theta| + C$.

u = 1 + sinθ → du = cosθ dθ

$\therefore \int \frac{cos\theta}{1+sin\theta} d\theta = \int \frac{du}{u} = \ln|u| + C$

$= \ln|1+sin\theta| + C$

(b) $\int \frac{x^3+1}{x^4+4x} dx = \frac{1}{4}\int \frac{4(x^3+1)dx}{x^4+4x} = \frac{1}{4}\ln|x^4+4x| + C$

(c) $\int_0^2 \frac{2x}{x^2-5} dx$.   Let u = x^2-5 → ∴ du = 2x dx

∴ x = 0 → u = $(0)^2$ – 5 = - 5

∴ x = 2 → u = $(2)^2$ – 5 = - 1

$\therefore \int_0^2 \frac{2x}{x^2-5} dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u| \Big|_{-5}^{-1} = \ln|-1| - \ln|-5| = 0 - ln5 = -ln5$.

(d) $\int_{-\pi/2}^{\pi/2} \frac{4cosx}{3+2sinx} dx = \int_{1}^{5} \frac{2du}{u}$

$= 2\ln|u| \Big|_1^5 = 2(ln5 - ln1) = 2ln5.$

u=3+2sinx → du = 2cosx dx

$x = -\frac{\pi}{2} \rightarrow u = 3 + 2\sin\left(-\frac{\pi}{2}\right) = 1$

$x = \frac{\pi}{2} \rightarrow u = 3 + 2\sin\left(\frac{\pi}{2}\right) = 5$

OR: $\int_{-\pi/2}^{\pi/2} \frac{4cosx}{3+2sinx} dx = \int_{-\pi/2}^{\pi/2} \frac{2du}{u}$

$= 2\ln|3 + 2sinx| \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\{ln(3+2) - ln(3-2)\}$

$= 2\{\ln 5 - ln1\} = 2ln5.$

Home work: THOMAS'S CALCULUS – 11th Edition, Page 484

Exercises 7.2 : (Using the Properties of Logarithms , Derivative of Logarithms, Integration).

7.3 The Exponential Function $(y = e^x or\ y = exp\ x)$

The function y = lnx : domain (0,∞) and range (- ∞ , ∞).

The inverse of lnx is $\ln^{-1}x$: domain (- ∞ , ∞) and range (0 , ∞).

$y = ln^{-1}x = e^x.$

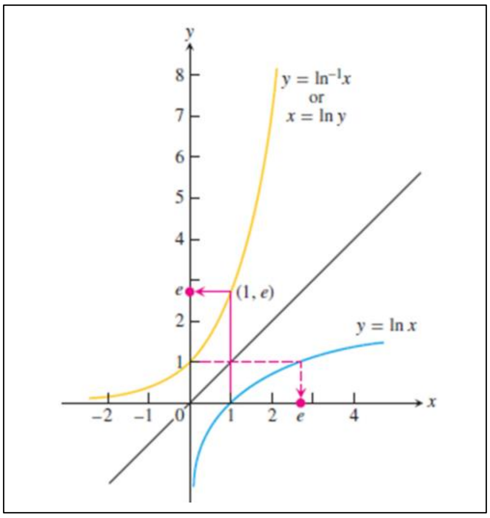
NOTE: $ln^{-1} 1 = e$ $ln^{-1} 1 = e \rightarrow ln\, e = 1$

e = 2.71828

$\because e^x = ln^{-1} x$ (e^x is the inverse function of lnx)

$\therefore \ln e^x = x \ \ (for\ all\ x)$, and

$e^{\ln x} = x\ (for\ all x > 0)$.



Properties of e^x

1. $e^x > 0$ for every $x \in R$
2. $e^0 = 1$
3. $e^1 = e$
4. $e^{x+y} = e^x . e^y$
5. $e^{x-y} = e^x / e^y$
7. $e^{cx} = (e^x)^c$
8. $\lim_{x\to\infty} e^x = \infty$
9. $\lim_{x\to-\infty} e^x = 0$
10. $\ln e^x = x$
11. $e^{\ln x} = x$

6. $e^{-x} = 1 / e^x$ 12. $a^x = (e^{\ln a})^x = e^{x\ln a}$

The Derivative and Integral of e^x

Let $y = e^x$,

Taking ln for both sides → $lny = \ln e^x = xlne = x$

$$\therefore lny = x$$

d. w. r. t. x → $\frac{1}{y} . \frac{dy}{dx} = 1 \rightarrow \frac{dy}{dx} = y = e^x$

$$\therefore \frac{de^x}{dx} = e^x$$

In general, if $y = e^u$, $u = f(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} = e^u \frac{du}{dx}$$

$$\int e^u \, du = e^u + C$$

Examples

(a) $\frac{de^{\tan x}}{dx} = e^{\tan x} \sec^2 x.$

(b) $\frac{d5e^{-x}}{dx} = 5e^{-x}(-1) = -5e^{-x}.$

(c) $\int e^{x^3} x^2 \, dx = \frac{1}{3} \int e^{x^3} . \ 3x^2 dx = \frac{1}{3} e^{x^3} + C.$

(d) $\int e^{-2x} \, dx = -\frac{1}{2} \int e^{-2x} . (-2) dx = -\frac{1}{2} e^{-2x} + C.$

(e) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx = 2e^{\sqrt{x}} + C.$ $u = \sqrt{x} \rightarrow du = \frac{1}{2} x^{-\frac{1}{2}} \, dx = \frac{1}{2\sqrt{x}} dx.$

(f) $\int_0^{\pi/2} e^{\sin x} \cos x \, dx = e^{\sin x} \Big|_0^{\pi/2} = e^1 - e^0 = e - 1.$

(g) $\int_0^{\ln 2} e^{3x} \, dx = \frac{1}{3} \int_0^{\ln 2} e^{3x} \, 3 \, dx = \frac{1}{3} e^{3x} \Big|_0^{\ln 2} = \frac{1}{3} \left(e^{3 \ln 2} - e^0 \right)$

$$= \frac{1}{3} \left(e^{\ln 2^3} - 1 \right) = \frac{1}{3} (8 - 1) = 7/3.$$

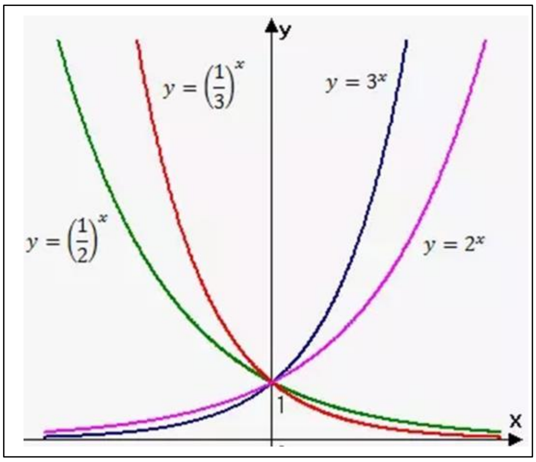
Home Work: Exercises 7.3 Page 493.

7.2 The General Exponential and Logarithmic Functions (a^x and $\log_a x$)

NOTE: $\ln x$ is a special case of the general logarithmic ($\log_a x$) Functions *and* e^x is a special case of the general exponential (a^x) .

The general exponential function

$y = f(x) = a^x$, (a = constant, a > 0 and a ≠ 1).



The Domain: $(-\infty, \infty)$

The Range: $y > 0$

The Derivative of a^x

Let $y = a^u$, $u = f(x)$,

Taking ln → $\ln y = \ln a^u = u \ln a$

d.w.r.t.x → $\frac{1}{y}\frac{dy}{dx} = \ln a \frac{du}{dx} \rightarrow \frac{dy}{dx} = y \ln a \, \frac{du}{dx}$

If a > 0 and *u* is a differentiable function of *x*,

$$\frac{da^u}{dx} = a^u \ln a \frac{du}{dx}$$

$$\int a^u \, du = \frac{a^u}{\ln a} + C$$

Examples:

(a) $\frac{d3^x}{dx} = 3^x \ln 3.$

(b) $\frac{d}{dx} 7^{x+1} = 7^{x+1} \ln 7.$

(c) $\frac{d2^{-x}}{dx} = 2^{-x} \ln 2 \ (-1) = -2^{-x} \ln 2 \, .$

(d) $\frac{d3^{\sin x}}{dx} = 3^{\sin x} \, (\ln 3) \ \cos x.$

(e) $\int 5^x dx = \frac{5^x}{\ln 5} + C.$

Logarithms with Base a ($y = f(x) = \log_a x$, (a > 0 and a ≠ 1)

Definition: $\log_a x$

For any positive number a ≠ 1, $\log_a x$ is the inverse function of a^x.

Domain: x > 0. Range: $(-\infty, \infty)$.

$$a^{\log_a x} = x \quad (x > 0), \quad and \quad \log_a(a^x) = x \quad (for\ all\ x)$$

Examples:

(a) $10^2 = 100 \rightarrow \log_{10} 100 = 2$

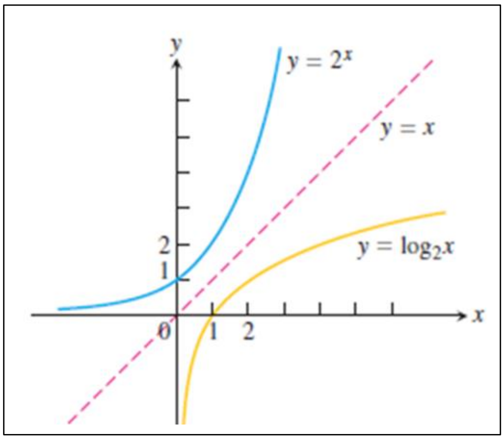
$(\log_a a^x = x \rightarrow \log_{10} 100 = \log_{10} 10^2 = 2)$

(b) $10^{-2} = \frac{1}{100} \rightarrow \log_{10}(\frac{1}{100}) = -2$

(c) $2^5 = 32 \rightarrow \log_2 32 = 5$

(d) $a^0 = 1 \rightarrow \log_a 1 = 0$

(e) $a^1 = a \rightarrow \log_a a = 1$



NOTE: $\log_a x = \frac{\ln x}{\ln a}$

NOTE: The properties of $\log_a x$ is the same as the properties of lnx.

The Derivative and Integral of $\log_a x$

If $y = \log_a u, \quad u = f(x) \rightarrow u = a^y$

Taking ln $\rightarrow \ln u = \ln a^y = y \ln a$

$\therefore \ln u = y \ln a$.

D. w. r. t. x $\rightarrow \frac{1}{u}\frac{du}{dx} = \ln a \frac{dy}{dx} \rightarrow \therefore \frac{dy}{dx} = \frac{1}{u} \frac{1}{\ln a} \frac{du}{dx}$

$$\frac{d\log_a u}{dx} = \frac{1}{\ln a}\,\frac{1}{u}\,\frac{du}{dx}$$

$$\int \log_a u\,du = \int \frac{\ln u}{\ln a}du$$

Examples: (a) $\frac{d}{dx}\log_5(x^2+1) = \frac{1}{\ln 5}\,\frac{1}{x^2+1}(2x).$

(b) $\int \frac{\log_2 x}{x}dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}dx = \frac{(\ln x)^2}{2\ln 2} + C$

NOTE: Find dy/dx if $y = x^x$.

Taking ln → $lny = \ln x^x = x\ln x$

D. w. r. t. x. → $\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \ln x\,.(1) \rightarrow \frac{dy}{dx} = y(1+\ln x) = x^x(1+\ln x).$

Example: Find dy/dx if $y^{\frac{2}{3}} = \frac{(x^2+1)\sqrt{3x+4}}{\sqrt[5]{(2x-3)(x^2-4)}}$

Solution: taking ln → $\ln y^{\frac{2}{3}} = \ln \frac{(x^2+1)\sqrt{3x+4}}{\sqrt[5]{(2x-3)(x^2-4)}}$

$\rightarrow \frac{2}{3}\ln y = \ln(x^2+1) + \frac{1}{2}\ln(3x+4) - \frac{1}{5}\{(\ln(2x-3) + \ln(x^2-4)\}$

D. w. r. t. x. →

$$\frac{2}{3}\,\frac{1}{y}\,\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{3}{2(3x+4)} - \frac{1}{5}\,\frac{2}{(2x-3)} - \frac{1}{5}\,\frac{2x}{(x^2-4)}$$

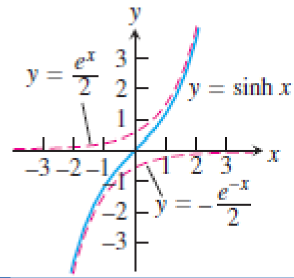
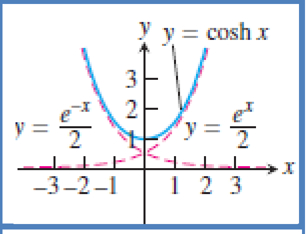
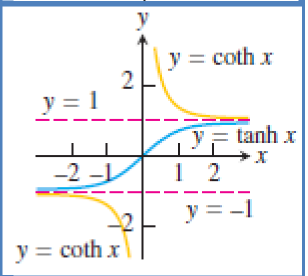
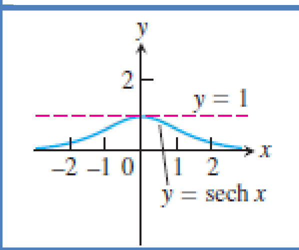
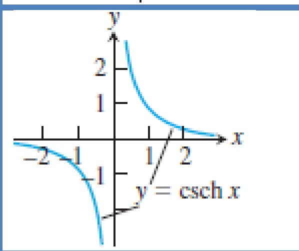
$$\therefore \frac{dy}{dx} = \frac{3}{2}y\left(\frac{2x}{x^2+1} + \frac{3}{2(3x+4)} - \frac{1}{5}\,\frac{2}{(2x-3)} - \frac{1}{5}\,\frac{2x}{(x^2-4)}\right)$$

Home Work: Exercises 7.4 Page 500.

Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}.

Definitions and Identities

$\sinh x = \dfrac{e^x - e^{-x}}{2}$	
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	
$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	
$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$	
$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$	
$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$	

Identities

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

$\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

$\coth^2 x = 1 + \operatorname{csch}^2 x$

Derivatives and Integrals

Derivatives	**Integral Formulas**
$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$	$\int \sinh u\, du = \cosh u + C$
$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$	$\int \cosh u\, du = \sinh u + C$
$\frac{d}{dx}(\tanh u) = \text{sech}^2 u \frac{du}{dx}$	$\int \text{sech}^2 u\, du = \tanh u + C$
$\frac{d}{dx}(\coth u) = -\text{csch}^2 u \frac{du}{dx}$	$\int \text{csch}^2 u\, du = -\coth u + C$
$\frac{d}{dx}(\text{sech}\, u) = -\text{sech}\, u \tanh u \frac{du}{dx}$	$\int \text{sech}\, u \tanh u\, du = -\text{sech}\, u + C$
$\frac{d}{dx}(\text{csch}\, u) = -\text{csch}\, u \coth u \frac{du}{dx}$	$\int \text{csch}\, u \coth u\, du = -\text{csch}\, u + C$

Examples: Find the following derivatives and integrals

(a) $\frac{d}{dx}(\tanh\sqrt{1+x^2}) = \text{sech}^2\sqrt{1+x^2}\ \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)$

$$= \frac{x}{\sqrt{1+x^2}}\text{sech}^2\sqrt{1+x^2}$$

(b) $\int \coth 3t\, dt = \int \frac{\cosh 3t}{\sinh 3t} dt = \frac{1}{3}\int \frac{3\cosh 3t dt}{\sinh 3t} = \frac{1}{3}\ln|\sinh 3t| + C.$

(c) $\int_0^1 \cosh^2 x\, dx = \int_0^1 \frac{\cosh 2x+1}{2} dx = \frac{1}{2}\int_0^1 (\cosh 2x + 1)dx$

$$= \frac{1}{2}(\frac{1}{2}\sinh 2x + x)\Big|_0^1 = \frac{1}{2}\{(\frac{1}{2}\sinh 2 - \sinh 0) + (1-0)\}$$

$$= \frac{\sinh 2}{4} + \frac{1}{2} \approx 1.4067$$

Home Work: Page 534 (Derivatives and Integrals).

Inverse Hyperbolic Functions

The inverses of the six basic hyperbolic functions are very useful in integration.

Derivatives and Integrals of Inverse Hyperbolic Functions

Derivatives	Integrals Formulas
$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$	**1.** $\int \frac{du}{\sqrt{1+u^2}} \sinh^{-1} u + C$
$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx},$	**2.** $\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$
$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx},$	**3.** $\int \frac{du}{\sqrt{1-u^2}} = -\operatorname{sech}^{-1} u + C$
$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx},$	**4.** $\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1} u + C$
$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}},$	**5.** $\int \frac{du}{1-u^2} = \begin{cases}\tanh^{-1} u + C, & u < 1 \\ \coth^{-1} u + C, & u > 1\end{cases}$
$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{\lvert u\rvert\sqrt{1+u^2}},$	

Home Work: Page 542 (Derivatives and Integrals).