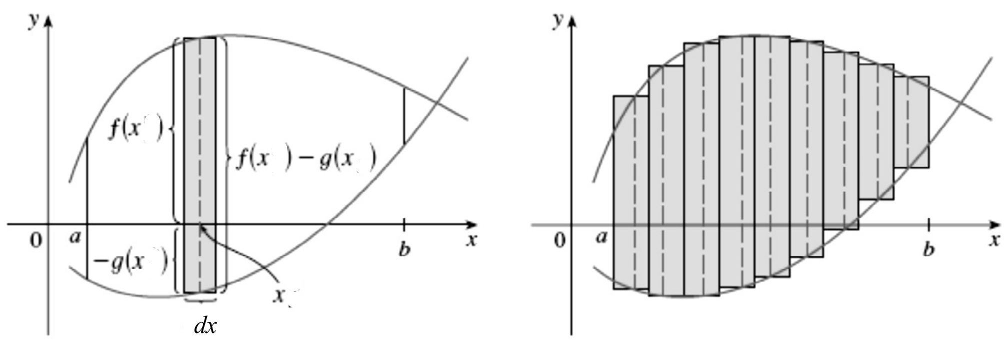
APPLICATIONS OF DEFINITE INTEGRAL

1. Area between Curves:

The area A of the region bounded by the curves $y=f(x)$, $y=g(x)$ and the lines $x=a$, $x=b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

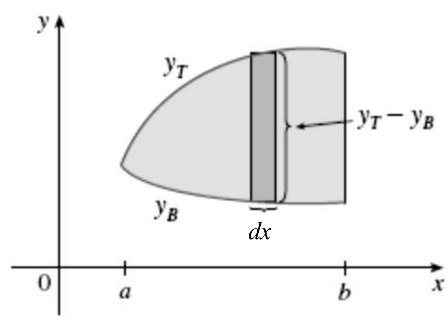
$$A = \int_a^b [f(x) - g(x)]dx$$



(a) Typical rectangle (b) Approximating rectangles

Steps to find area between two curves:

1. Sketch the graph of the curves together. This identify the up curve y_T and the bottom curve y_B
2. Find the limits of integration (if not given in the problem).
3. Write a formula of $[f(x) - g(x)]$ or $[y_T - y_B]$ and simplify it.
4. Integrate $[f(x) - g(x)]$ from a to b. The number you get it is the area.



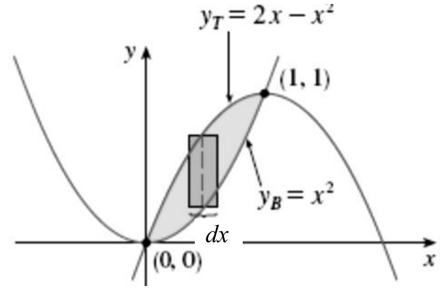
Example 1: Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$

Sol.: We first find the points of intersection of the parabolas by solving their equations simultaneously.

$$x^2 = 2x - x^2 \quad \Rightarrow \quad x^2 + x^2 - 2x = 0 \quad \Rightarrow \quad 2x^2 - 2x = 0 \quad \Rightarrow \quad 2x(x-1) = 0$$

either $2x = 0 \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad y = 0$

or $\quad x - 1 = 0 \quad \Rightarrow \quad x = 1 \quad \Rightarrow \quad y = 1$



The points of intersection are $(0,0)$ and $(1,1)$

We see from Figure that the top and bottom boundaries are

$$y_T = 2x - x^2 \text{ and } y_B = x^2$$

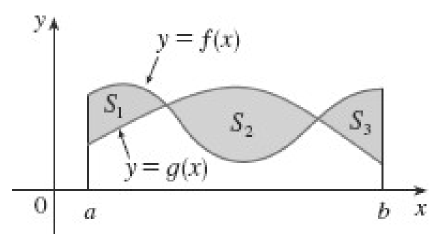
The area of a typical rectangle is

$$dA = y_T - y_B = (2x - x^2) - (x^2) = 2x - x^2 - x^2 = 2x - 2x^2$$

and the region lies between x=0 and x= 1. So the total area is

$$A = \int dA = \int_0^1 (2x - 2x^2)dx = \frac{2x^2}{2} - \frac{2x^3}{3}\Big|_0^1 = [(1)^2 - \frac{2(1)^3}{3}] - [0] = \frac{1}{3} \text{ square units}$$

If we are asked to find the area between the curves $y=f(x)$ and $y=g(x)$ where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for values of x, then we split the given region S into several regions $S_1, S_2, \ldots$ with areas A_1, $A_2, \ldots$ as shown in Figure. We then define the area of the region S to be the sum of the areas of the smaller regions $S_1, S_2, \ldots$ that is, $A=A_1+A_2+\ldots$. Since

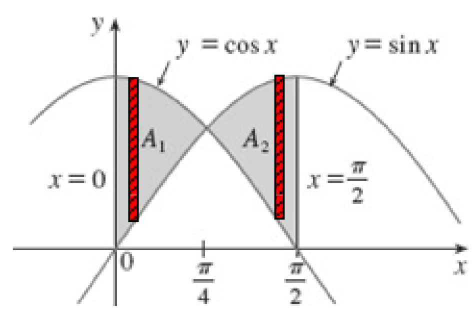


$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & when \quad f(x) \geq g(x) \\ g(x) - f(x) & when \quad g(x) \geq f(x) \end{cases}$$

Example 2: Find the area of the region bounded by the curves $y=\sin x$, $y=\cos x$, $x=0$, and $x=\pi/2$.

Sol.: The point of intersection occur when $\sin x = \cos x$, that is, when $x=\pi/4$.

Observe that $\cos x \geq \sin x$ when $0 \leq x \leq \pi/4$ but $\sin x \geq \cos x$ when $\pi/4 \leq x \leq \pi/2$. Therefore the required area is



$$A = \int_0^{\pi/2} |\cos x - \sin x| dx = A_1 + A_2$$

$$= \int_0^{\pi/4} [\cos x - \sin x]dx + \int_{\pi/4}^{\pi/2} [\sin x - \cos x]dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 \right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right)$$

$$= 2\sqrt{2} - 2$$

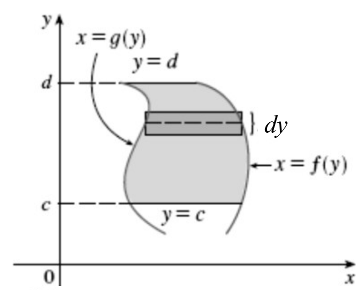
In this particular example we could have saved some work by noticing that the region is symmetric about $x=\pi/4$ and so,

$$A = 2A_1 = 2\int_0^{\pi/4} [\cos x - \sin x]dx$$

Integration with respect to *y* (horizontal strip)

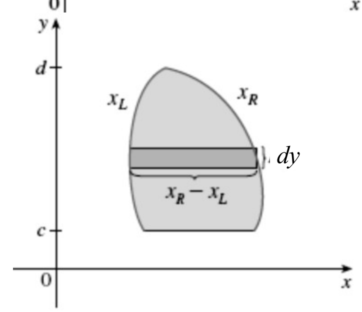
Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x=f(y)$, $x=g(y)$, $y=c$, and $y=d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ then its area is

$$A = \int_c^d [f(y) - g(y)]dy$$



If we write for the right boundary x_R and for the left boundary x_L, then we have

$$A = \int_c^d [x_R - x_L]dy$$



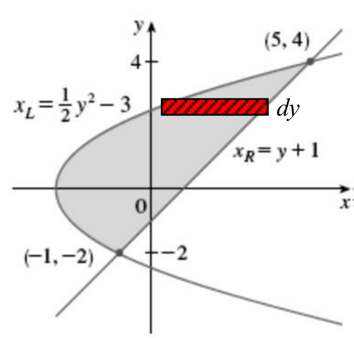
Example 3: Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$

Sol.: To find points of intersections put $x_{\text{line}} = x_{\text{curve}}$ so

$$y+1 = \frac{y^2 - 6}{2} \Rightarrow 2(y+1) = y^2 - 6 \Rightarrow y^2 - 2y - 8 = 0$$

$$\Rightarrow (y-4)(y+2) = 0 \text{ either } y=4 \Rightarrow x=5$$

$$\text{or} \quad y=-2 \Rightarrow x=-1$$



$\therefore$ (5,4) and (-1,-2) are the points of intersections of the two curves.

We can notice from Figure that the left and right boundary curves are

$$x_R = y+1 \qquad \text{and} \qquad x_L = \frac{1}{2}y^2 - 3$$

We must integrate between the appropriate y-values, y=-2 and y=4. Thus

$$A = \int_{-2}^{4} [x_R - x_L]dy$$

$$= \int_{-2}^{4} [(y+1) - (\frac{1}{2}y^2 - 3)]dy$$

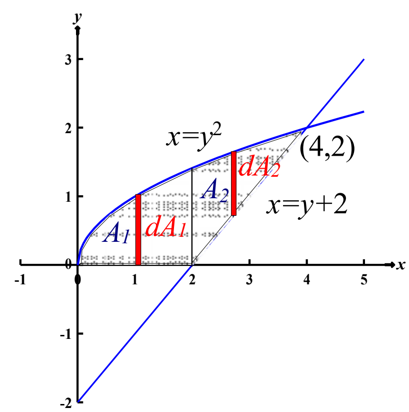
$$= \int_{-2}^{4} [-\frac{1}{2}y^2 + y + 4)]dy$$

$$= [-\frac{y^3}{2*3} + \frac{y^2}{2} + 4y]_{-2}^{4}$$

$$= \left(-\frac{4^3}{6} + \frac{4^2}{2} + 4*4\right) - \left(-\frac{(-2)^3}{6} + \frac{(-2)^2}{2} + 4*(-2)\right)$$

$$-\frac{64}{6} + 8 + 16 - \frac{8}{6} - 2 + 8 = 18 \text{ square units.}$$

Example 4: Find the area of the region between the curves $x=y^2$ and $x=y+2$ in the first quadrant.



Sol.: Graph the curves together

a. Using vertical strip: we should split the are into two areas by the line x=2

$$\therefore A = A_1 + A_2$$

The area of the first typical rectangle

$$dA_1 = (y_T - 0)dx = (\sqrt{x} - 0)dx = \sqrt{x}dx$$

$$\therefore A_1 = \int dA_1 = \int_0^2 \sqrt{x}dx = \left.\frac{x^{3/2}}{3/2}\right|_0^2 = \frac{2}{3}[2^{3/2} - 0] = 1.885618$$

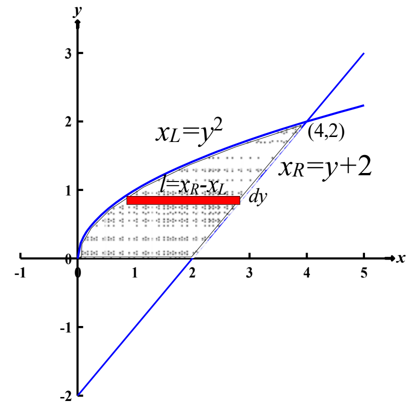
The area of the second typical rectangle

$$dA_2 = (y_T - y_B)dx = (\sqrt{x} - (x-2))dx = (\sqrt{x} - x + 2)dx$$

$$\therefore A_2 = \int dA_2 = \int_2^4 (\sqrt{x} - x + 2)dx = \frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x \Big|_2^4$$

$$= [\frac{4^{3/2}}{3/2} - \frac{4^2}{2} + 2*4] - [\frac{2^{3/2}}{3/2} - \frac{2^2}{2} + 2*2] = 1.447715$$

$$\therefore A = 1.885618 + 1.447715 = 3.333333 \text{ square units}$$



b. Using horizontal strip:

The area of the typical rectangle

$$dA = (x_R - x_l)dy = \{(y+2) - y^2\}dy$$

$$\therefore A = \int dA = \int_0^2 (y + 2 - y^2)dy = \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_0^2$$

$$= [\frac{2^2}{2} + 2*2 - \frac{2^3}{3}] - [0] = 3.33333 \text{ square units}$$

2. Volume of Solids of Revolution:

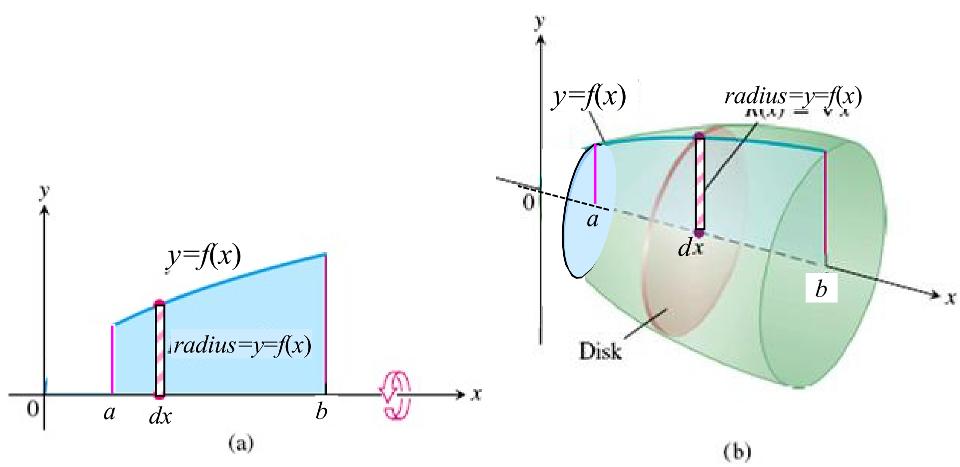
The Solid generated by rotating a plane region about an axis in its plane is called a **solid of revolution**. We will use the following methods to find this volume

a. The Disk Method (The strip is perpendicular to the axis of revolution):

i. Rotation about *x*-axis: The volume of the solid generated by revolving the region between the graph of continuous function $y=f(x)$ and the x-axis from $x=a$ to $x=b$ about the x-axis is

$$dV = \pi.(radius)^2(thickness) = \pi.y^2dx = \pi.\left(f(x)\right)^2 dx$$

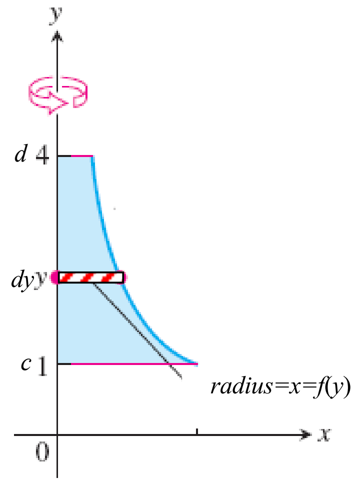
$$\text{Volume} = \int dV = \int_a^b \pi(radius)^2 dx = \int_a^b \pi\left(f(x)\right)^2 dx$$

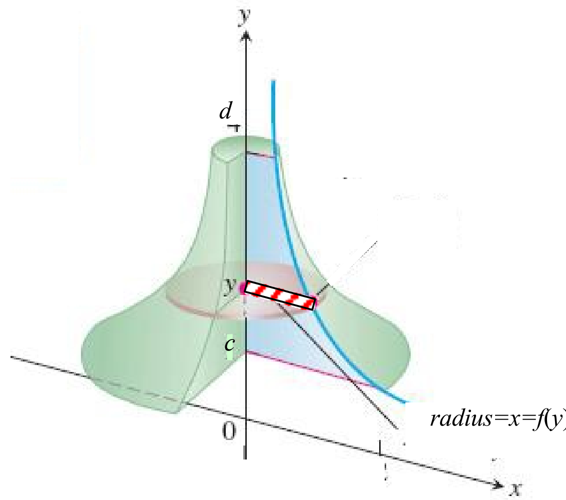


ii. Rotation about *y*-axis: If the region bounded between the continuous function $x=f(y)$ and y-axis is rotated about y-axis from $y=c$ to $y=d$ to generate a solid, then the volume of the solid is:

$$dV = \pi.(radius)^2(thickness) = \pi.x^2dy = \pi.\left(f(y)\right)^2 dy$$

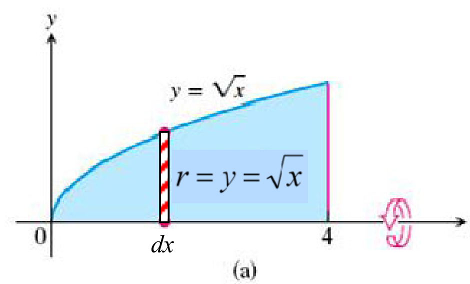
$$\text{Volume} = \int dV = \int_c^d \pi(radius)^2 dy = \int_c^d \pi\left(f(y)\right)^2 dy.$$





Example 1: The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Sol.: We draw figures showing the region, the typical radius and the generated solid. The volume of the disk is

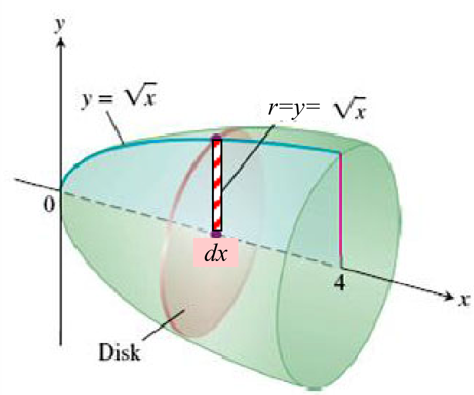


$$dV = \pi.(radius)^2(thickness) = \pi.r^2.t$$

Where $r = y = f(x) = \sqrt{x}$ and $t = dx$

$$\therefore dV = \pi(\sqrt{x})^2 dx = \pi.x.dx$$

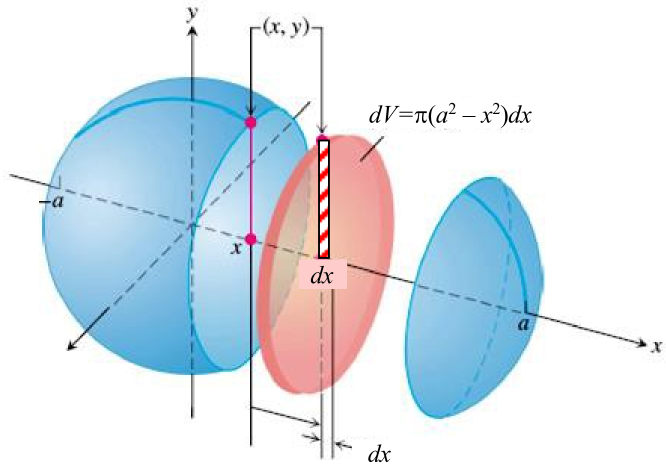
So the volume of the solid is



$$V = \int dV = \int_0^4 \pi.x.dx = \pi \left. \frac{x^2}{2} \right|_0^4$$

$$= \frac{\pi}{2}[4^2 - 0^2] = \frac{16\pi}{2} = 8\pi \text{ cubic units}$$

Example 2: The circle $x^2 + y^2 = a^2$ is rotated about the x-axis to generate a sphere. Find its volume.



Sol.: We imagine a sphere cut into thin slices by planes perpendicular to the x-axis. The volume of a typical slice at point x between a and $-a$ is

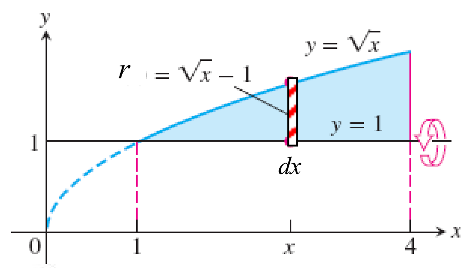
$$dV = \pi . r^2 . t = \pi . y^2 dx = \pi(a^2 - x^2)dx$$

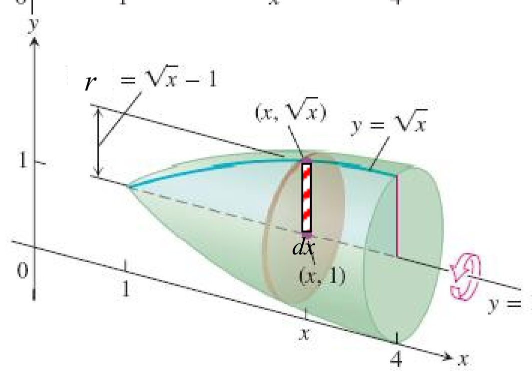
Therefore the volume is

$$V = \int dV = \int_{-a}^{a} \pi(a^2 - x^2)dx = 2\int_{0}^{a} \pi(a^2 - x^2)dx$$

$$= 2\pi(a^2 x - \frac{x^3}{3})\Big|_0^a = \frac{4}{3}\pi . a^3$$

Example 3: Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y=1$, $x=4$ about the line $y=1$.





Sol.: We draw figures showing the region, the typical radius and the generated solid. The volume of the disk is

$$dV = \pi . (radius)^2 (thickness) = \pi . r^2 . t$$

Where $r = y - 1 = \sqrt{x} - 1$ and $t = dx$

$$\therefore dV = \pi(\sqrt{x} - 1)^2 dx$$

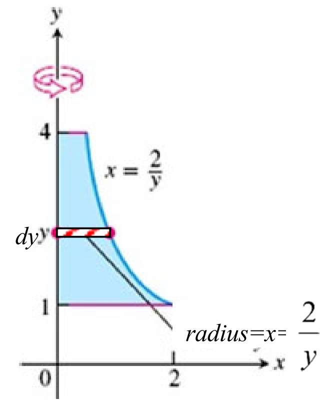
So the volume of the solid is

$$V=\int dV=\int_1^4 \pi(\sqrt{x}-1)^2\,dx=\int_1^4 \pi(x-2\sqrt{x}+1)dx \quad =\pi\left(\frac{x^2}{2}-\frac{2x^{3/2}}{3/2}+x\right)\Big|_1^4$$

$$=\pi\left[\left(\frac{4^2}{2}-\frac{4*4^{3/2}}{3}+4\right)-\left(\frac{1^2}{2}-\frac{4*1^{3/2}}{3}+1\right)\right]$$

$$=\frac{7\pi}{6}\text{ cubic units}$$

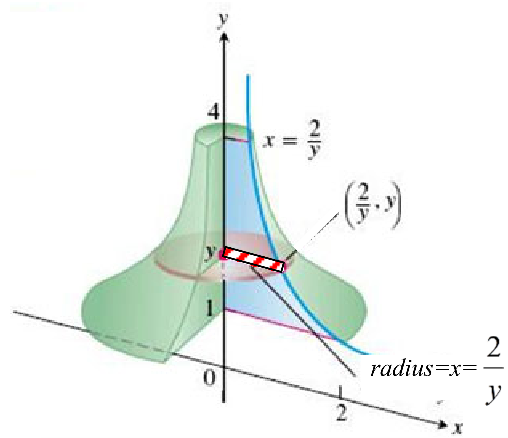
Example 4: Find the volume of the solid generated by revolving the region between the y-axis and the curve $x=2/y$, $1\le y\le 4$, about y-axis.



Sol.: We draw figures showing the region, the typical radius and the generated solid. The volume of the disk is

$$dV=\pi.(radius)^2(thickness)=\pi.r^2.t$$

Where $r=x=\dfrac{2}{y}$ and $t=dy$



$$\therefore dV=\pi\left(\frac{2}{y}\right)^2dy=\frac{4\pi}{y^2}dy$$

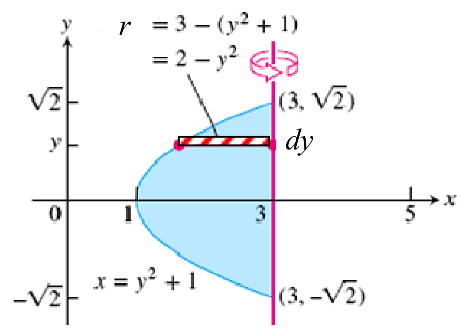
So the volume of the solid is

$$V=\int dV=\int_1^4\frac{4\pi}{y^2}dy=-\frac{4\pi}{y}\Big|_1^4$$

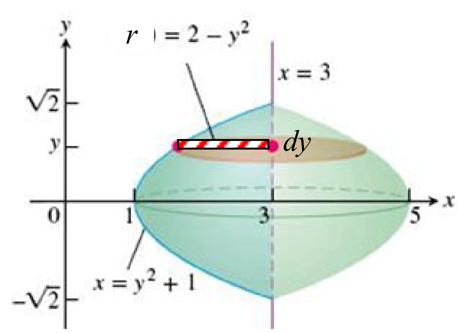
$$=4\pi\left[-\frac{1}{4}-\left(-\frac{1}{1}\right)\right]=4\pi*\frac{3}{4}$$

$$=3\pi\text{ cubic units}$$

Example 5: Find the volume of the solid generated by revolving the region between the parabola $x=y^2+1$ and the line $x=3$, about $x=3$.



Sol.: We draw figures showing the region, the typical radius and the generated solid. Note that the cross-sections are perpendicular to the line $x=3$. The volume of the disk is



$$dV = \pi.(radius)^2(thickness) = \pi.r^2.t$$

Where $r = 3 - x = 3 - (y^2 + 1) = 2 - y^2$ and $t = dy$

$$\therefore dV = \pi(2 - y^2)^2\, dy$$

So the volume of the solid is

$$V = \int dV = \int_{-\sqrt{2}}^{\sqrt{2}} \pi(2 - y^2)^2\, dy$$

$$= \int_{\sqrt{2}}^{\sqrt{2}} \pi(4 - 4y^2 + y^4)dy = 2\int_{0}^{\sqrt{2}} \pi(4 - 4y^2 + y^4)dy \qquad = 2\pi(4y - \frac{4}{3}y^3 + \frac{y^5}{5})\Big|_0^{\sqrt{2}}$$

$$= 2\pi[(4\sqrt{2} - \frac{4}{3}(\sqrt{2})^3 + \frac{(\sqrt{2})^5}{5}) - (0)] = \frac{64\pi\sqrt{2}}{15} \text{ cubic units}$$

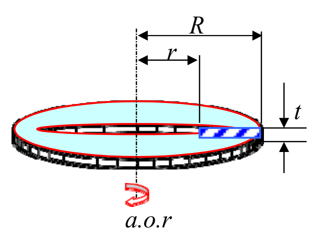
b. The Washer Method (The strip is perpendicular to the axis of revolution):

If the region we revolved to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it. The cross-sections perpendicular to the axis of revolution are washers instead of disks. The dimensions of a typical washer are

Outer radius: R

Inner radius: r

Thickness: t



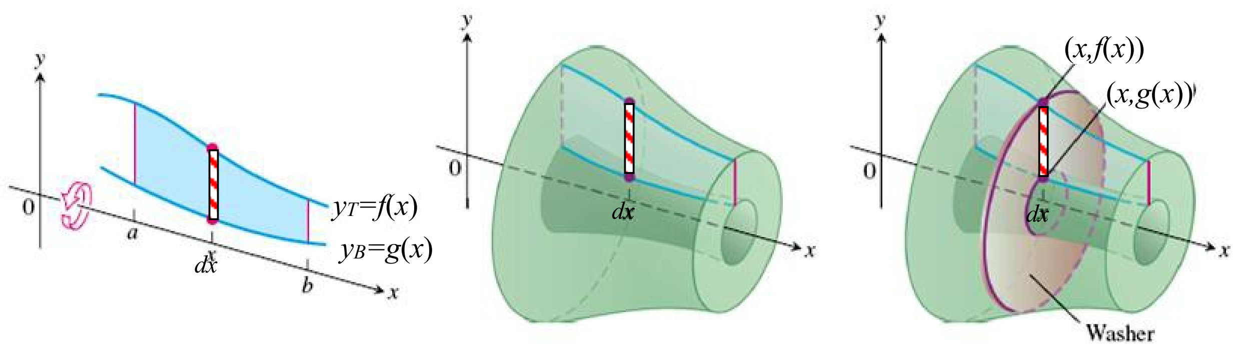
The washer's volume is: $dV = \pi\,[R^2 - r^2]\; t$

i. Rotation about *x*-axis:

If a region bounded by curves with equations $y=f(x)$, $y=g(x)$, $x=a$, and $x=b$, where f and g are continuous and $f(x) \geq g(x)$ for $a \leq y \leq b$ is rotated about x-axis then,

$R=y_T=f(x)$, $r=y_B=g(x)$ and $t=dx$

$$V = \int dV = \int_a^b \pi\{R^2 - r^2\}dx = \int_a^b \pi\{(y_T)^2 - (y_B)^2\}dx = \int_a^b \pi\{[f(x)]^2 - [g(x)]^2\}$$



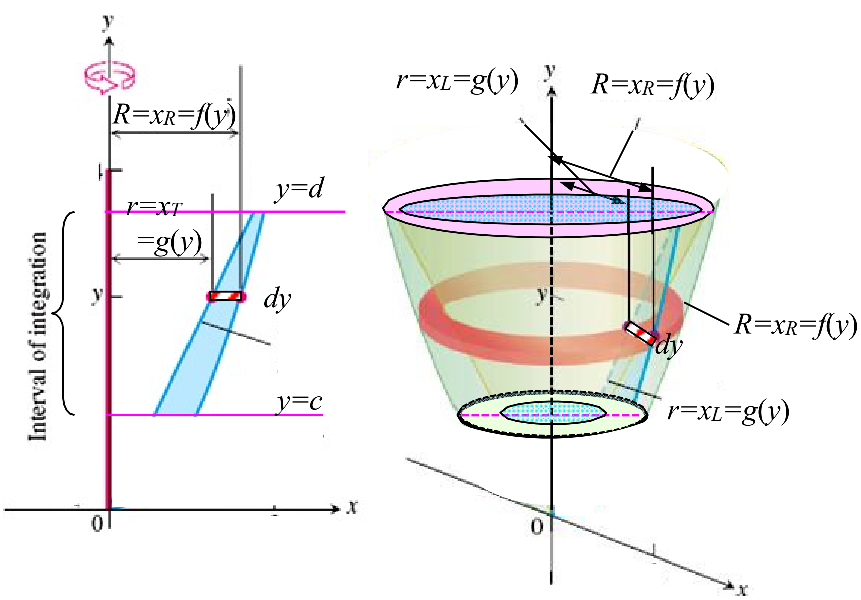
ii. Rotation about *y*-axis:

If a region bounded by curves with equations $x=f(y)$, $x=g(y)$, $y=c$, and $y=d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ is rotated about y-axis then,

$R=x_R=f(y)$, $r=x_L=g(y)$ *and* $t=dy$;

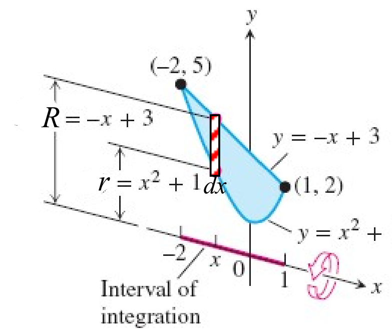
and the volume of the solid:

$$V=\int dV=\int_c^d \pi\{R^2-r^2\}dy=\int_c^d \pi\{(x_R)^2-(x_L)^2\}dy=\int_c^d \pi\{[f(y)]^2-[g(y)]^2\}dy$$



Example 6: The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find its volume.

Sol.:



1. Draw the region and sketch a strip across it perpendicular to the axis of revolution.
2. Find the outer and the inner radii of the washer that would be swept out by the strip if it were revolved about the x-axis along with the region. These radii are the distance of the ends of the strip from the axis of revolution.

   Outer radius: $R = y_T = -x + 3$

   Inner radius: $r = y_B = x^2 + 1$
3. Find the limits of integration by finding the x-coordinate of the intersection points of the curve and line (put $y_{\text{curve}} = y_{\text{line}}$).

   $x^2 + 1 = -x + 3$

   $x^2 + x - 2 = 0$

   $(x + 2)(x - 1) = 0$

   Either $x = -2 \Rightarrow y = 5$

   or $x = 1 \Rightarrow y = 2$
4. Find the washer's volume

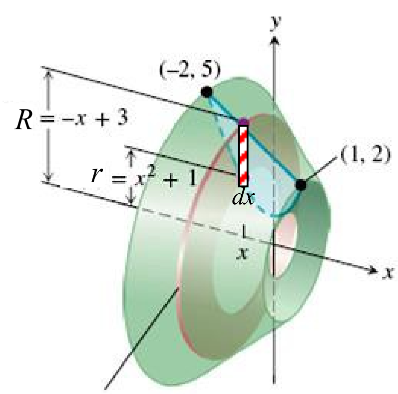
   $dV = \pi [R^2 - r^2]\, t$

   where $R = y_T = -x + 3$, $r = y_B = x^2 + 1$ and $t = dx$

   $\therefore dV = \pi [(-x + 3)^2 - (x^2 + 1)^2]\, dx$

   $= \pi [x^2 - 6x + 9 - (x^4 + 2x^2 + 1)]\, dx$

   $= \pi [\, 8 - 6x - x^2 - x^4 \,]\, dx$



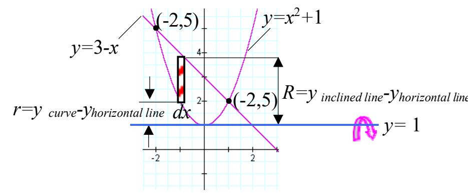
5. Evaluate the volume integral

$$V = \int dV = \int_{-2}^{1} \pi (8 - 6x - x^2 - x^4)\, dx$$

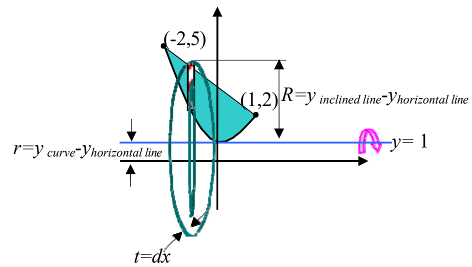
$$= \pi \left(8x - \frac{6x^2}{2} - \frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_{-2}^{1} = \frac{117\pi}{5} \text{ cubic units.}$$

Example 7: Repeat Example 6 but here rotate about the line $y=1$

Sol.:



1. Draw the region and sketch a strip across it perpendicular to the axis of revolution.
2. Find the outer and the inner radii of the washer that would be swept out by the strip. These radii are the distance of the ends of the strip from the axis of revolution.



Outer radius: $R=y_{\text{ inclined line}}-y_{\text{ horizontal line}}$

$=(-x + 3)-(1) = -x + 3-1=-x+2$

Inner radius: $r=y_{curve}-y_{\text{ horizontal line}}$

$=(x^2 + 1) -(1) = x^2$

3. Find the limits of integration: from previous example the limit of integration are from $x = -2$ to $x = 1$
4. Find the washer's volume

$dV = \pi\,[R^2 - r^2]\; t$

where $R= 2- x$, $r= x^2$ and $t=dx$

$\therefore dV= \pi\,[(2-x\,)^2 - (x^2\,)^2]\; dx$

$= \pi\,[4 - 4x - x^2- x^4\,)]\; dx$

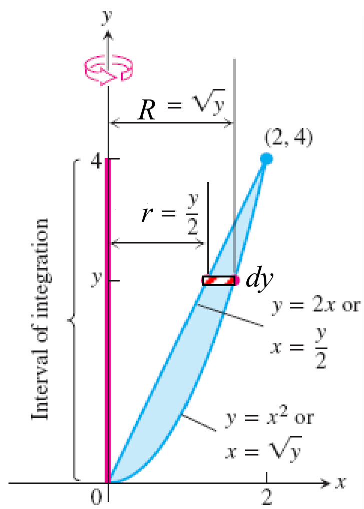
5. Evaluate the volume integral

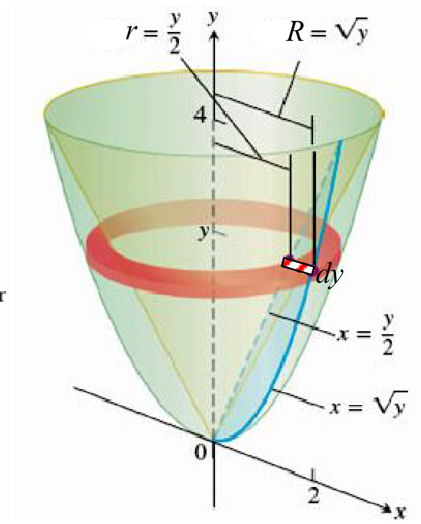
$$V = \int dV = \int_{-2}^{1} \pi(4-4x-x^2-x^4)dx$$

$$= \pi(4x - \frac{4x^2}{2} - \frac{x^3}{3} - \frac{x^5}{5})\Big|_{-2}^{1} = \frac{75\pi}{5} \text{ cubic units.}$$

Example 8: The region bounded by the parabola $y = x^2$ and the line $y = 2x$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Sol.: First we draw the region and draw a strip across it perpendicular to the axis of revolution (the y-axis). The radii of washer swept out by the strip are $R = x_R = \sqrt{y}$ and $r = x_L = y/2$ but its thickness is $t = dy$



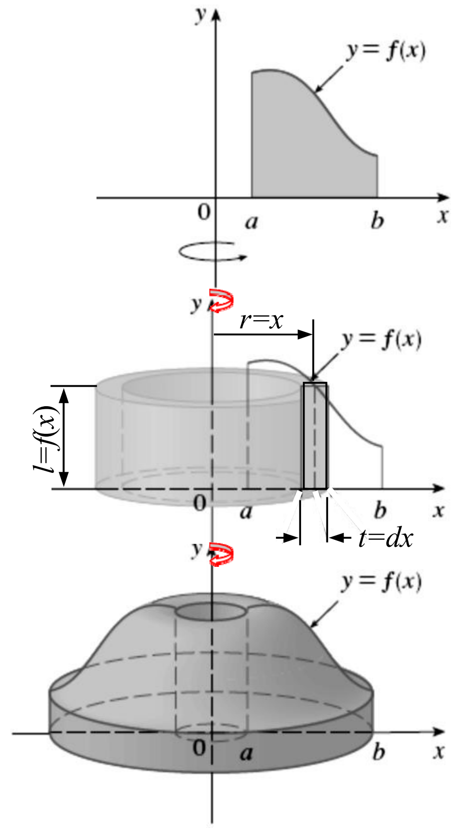


The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$dV = \pi[R^2 - r^2]t = \pi[(\sqrt{y})^2 - (y/2)^2]dy = \pi[y - \frac{y^2}{4}]dy$$

$$V = \int dV = \int_0^4 \pi[y - \frac{y^2}{4}]dy = \pi[\frac{y^2}{2} - \frac{y^3}{3*4}]\Big|_0^4 = \frac{8}{3}\pi \text{ cubic units}$$

c. Volumes by Cylindrical Shells (The strip is parallel to the axis of revolution)



The volume of the solid obtained by rotating the region bounded by $y=f(x)$ [where $f(x)>0$], $y=0$, $x=a$ and $x=b$ about the y-axis is obtained by:

Volume of typical cylindrical shell

$$dV=2\pi(r)(l)(t)$$

where r= radius of cylinder = x

l= cylindrical length (or height) = $f(x)$

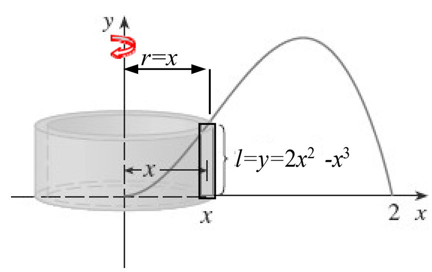
t= shell thickness = dx

$$\therefore\ dV=2\pi.x.[f(x)]dx$$

and volume of the solid

$$V=\int dV=\int_a^b 2\pi[f(x)]x.dx$$

Example 1: Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y=2x^2-x^3$ and $y=0$.



Sol.: From the sketch of the curve we see that a typical shell has radius x and height $y=f(x)$. So, by the shell method, the volume of typical shell is:

$$dV=2\pi r.l.t=2\pi x[f(x)]dx=2\pi x(2x^2-x^3)dx$$

To find the limits of integration put $y_{\text{curve}}=0$ so:

$$2x^2-x^3=0\ \Rightarrow\ x^2(2-x)=0$$

$\therefore$ either $x^2=0 \quad\Rightarrow\quad x=0$

or $(2-x)=0 \quad\Rightarrow\quad x=2$

so the volume of the solid:

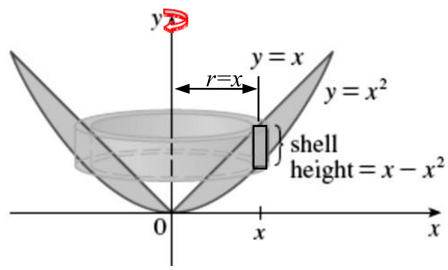
$$V = \int dV = \int_0^2 2\pi(2x^3 - x^4)dx = 2\pi[2*\frac{x^4}{4} - \frac{x^5}{5}]\Big|_0^2$$

$$= 2\pi[(2*\frac{2^4}{2} - \frac{2^5}{5}) - (0)] = 2\pi[16 - \frac{32}{5}] = \frac{96}{5}\pi \text{ cubic units}$$

Example 2: Find the volume of the solid obtained by rotating about the y-axis the region between $y=x$ and $y=x^2$.

Sol.: when we sketch the region we see that the shell has radius x, and height $x-x^2$. So the volume of typical cylindrical shell:

$$dV = 2\pi r.l.t = 2\pi x(x - x^2)dx$$



To find the limits of integration put $y_{curve} = y_{line}$ so

$$x = x^2 \quad \Rightarrow \quad x - x^2 = 0 \quad \Rightarrow \quad x(1-x) = 0$$

$\therefore$ either $x = 0 \quad \Rightarrow \quad y = 0$

or $(1-x) = 0 \quad \Rightarrow \quad x = 1$ and $y = 1$

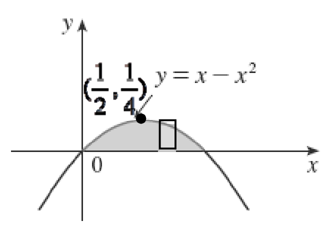
So the volume of the solid:

$$V = \int dV = \int_0^1 2\pi(x^2 - x^3)dx = 2\pi[\frac{x^3}{3} - \frac{x^4}{4}]\Big|_0^1$$

$$= 2\pi[(\frac{1^3}{3} - \frac{1^4}{4}) - (0)] = 2\pi[\frac{1}{3} - \frac{1}{4}] = \frac{\pi}{6} \text{ cubic units}$$

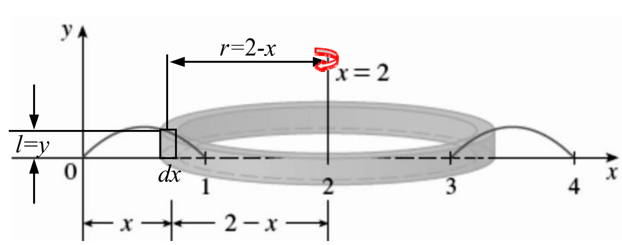
Example 3: Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Sol.: To graph the curve $y = x - x^2$, complete the square and compare the resulting equation with the curve $y = -x^2$



$$y = -(x^2 - x + \frac{1}{4}) + \frac{1}{4} = -(x - \frac{1}{2})^2 + \frac{1}{4}$$

Sketch the region by shifting the curve $y=-x^2$ by 1/2 units left and 1/4 units up.



The volume of typical cylindrical shell:

$$dV = 2\pi r.l.t$$

Where r=2-x, l=y=x-x^2 and t=dx

$$\therefore dV = 2\pi(2-x)(x-x^2).dx$$

$$= 2\pi(2x-2x^2-x^2+x^3)dx$$

$$= 2\pi(2x-3x^2+x^3)dx$$

To find the limits of integration put $y_{\text{curve}} = 0 \Rightarrow x - x^2 = 0 \quad \Rightarrow x(1-x) = 0$

$\therefore$ either $x = 0$

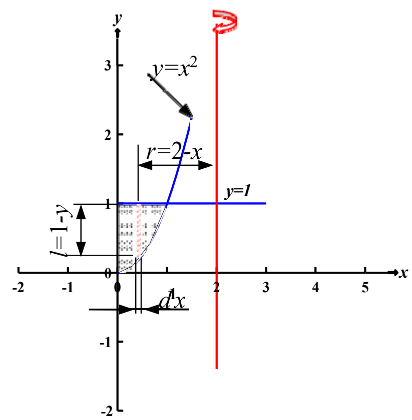
or $(1-x) = 0 \quad \Rightarrow \quad x = 1$

So the volume of the solid:

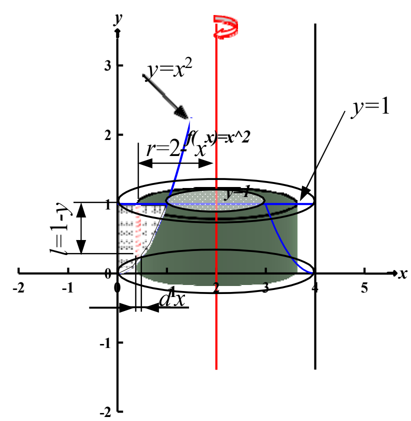
$$\therefore V = \int dV = \int_0^1 2\pi(2x-3x^2+x^3).dx$$

$$= 2\pi(x^2 - x^3 + \frac{x^4}{4})\Big|_0^1$$

$$= 2\pi[(1^2 - 1^3 + \frac{1^4}{4}) - (0)] = \frac{\pi}{2}$$



Example 4: The region bounded by the parabola y=x^2, the y-axis and the line y=1 in the first quadrant is revolved about the line x=2 to generate a solid. Find the volume of the solid.



Sol.: l=1-y, r=2-x and t=dx

$$dV = 2\pi.r.l.t = 2\pi\ (2-x)(1-y)dx$$

$$= 2\pi(2-x)(1-x^2)dx$$

The limits of integration from x=0 to x=1.

The volume of the solid:

$$\therefore V = \int dV = \int_0^1 2\pi(2-x)(1-x^2)dx$$

$$=\int_0^1 2\pi(2-2x^2-x+x^3)dx = 2\pi(2x-\frac{2x^3}{3}-\frac{x^2}{2}+\frac{x^4}{4})\Big|_0^1$$

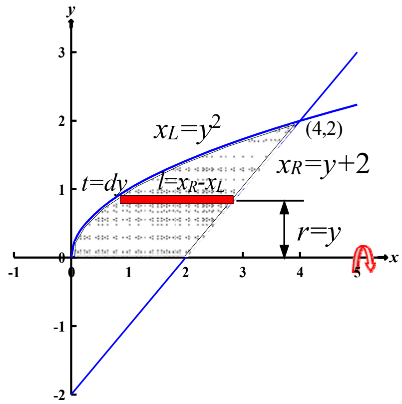
$$=2\pi[(2*1-\frac{2*1^3}{3}-\frac{1^2}{2}+\frac{1^4}{4})-(0)]$$

$$=\frac{13}{6}\pi \text{ cubic units}$$

Example 5: Find the volume of the solid which is generated by rotating the region bounded by $y=\sqrt{x}$, y=x-2 and x-axis about:

a. x-axis. b. y-axis.

Sol.: a. about x-axis (the strip is parallel to the axis of rotation so it will give cylindrical shell)



$dV=2\pi.r.l.t$

where $r=y$, $l=x_R$-x_L=(y+2)-(y^2)= y+2-y^2 and $t=dy$

$$\therefore dV = 2\pi.y(y+2-y^2)dy = 2\pi(2y+y^2-y^3)dy$$

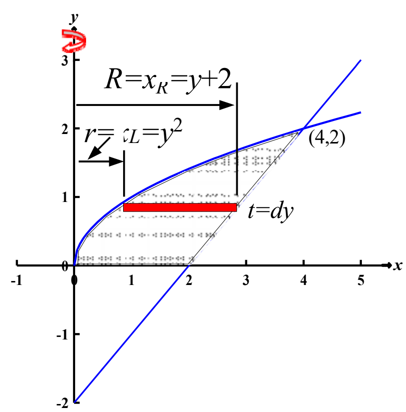
The limits of integration from y=0 to y=2

So the volume of the solid:

$$V=\int dV = \int_0^2 2\pi(2y+y^2-y^3)dy$$

$$=2\pi(y^2+\frac{y^3}{3}-\frac{y^4}{4})\Big|_0^2$$

$$=2\pi[(2^2+\frac{2^3}{3}-\frac{2^4}{4})-(0)]=\frac{16}{3}\pi \quad \text{cubic units}$$



b. about y-axis (the strip is perpendicular to the axis of rotation so it will give washer)

$dV=\pi(R^2-r^2).t$

where R= x_R=y+2, r=x_L=y^2 and $t=dy$

$$\therefore dV = \pi.[(y+2)^2-(y^2)^2]dy$$

$$=\pi[y^2+4y+4-y^4]dy$$

The limits of integration from y=0 to y=2

So the volume of the solid:

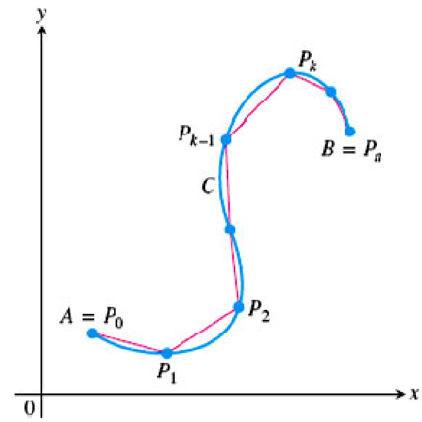
$$V = \int dV = \int_0^2 \pi(y^2 + 4y + 4 - y^4)dy$$

$$= \pi(\frac{y^3}{3} + 2y^2 + 4y - \frac{y^5}{5})\Big|_0^2$$

$$= \pi[(\frac{2^3}{3} + 2*2^2 + 4*2 - \frac{2^5}{5}) - (0)] = \frac{184}{15}\pi \text{ cubic units}$$

3. <u>Length of Plane Curves</u>:

i. Suppose that $y=f(x)$ is a smooth curve on the interval $[a, b]$, then:



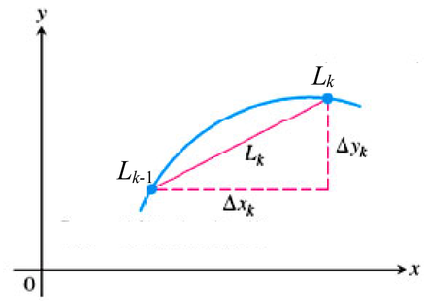
$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{(\Delta x_k)^2 [1 + \frac{(\Delta y_k)^2}{(\Delta x_k)^2}]}$$

$$= \sqrt{[1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2]}.(\Delta x_k)$$

$$\therefore L = \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{[1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2]}.(\Delta x_k)$$

When $n \to \infty \;\; \Rightarrow \;\; \Delta x \to 0$

So $\therefore L = \lim\limits_{\Delta x_k \to 0} \sum_{k=1}^{\infty} \sqrt{[1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2]}.(\Delta x_k)$



Remember that $\lim\limits_{\Delta x \to 0} \frac{\Delta y_k}{\Delta x_k} = f`(x)$

$$\boxed{\therefore L = \int_a^b \sqrt{1 + [f`(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx} \qquad \text{----(1)}$$

ii. Suppose that $x=f(y)$ is a continuous from $y=c$ to $y=d$, then the arc-length of the curve is:

$$\boxed{L = \int_c^d \sqrt{1 + [f`(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy} \qquad \text{----(2)}$$

iii. If the curve is represented by a parametric equations:

$x=x(t)$, $y=y(t)$ and $a \le t \le b$ and if $\frac{dx}{dt}$, $\frac{dy}{dt}$ are continuous functions on $a \le t \le b$, then the arc-length of the curve is:

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, .dt} \qquad \text{----(3)}$$

Example 1: Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1; \qquad 0 \le x \le 1.$$

Sol.: We use equation (1) with a=0 and b=1, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

$$\frac{dy}{dx} = \frac{3}{2} * \frac{4\sqrt{2}}{3}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.$$

The length of the curve from x=0 to x=1 is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}dx$$

$$= \frac{1}{8}\int_0^1 \sqrt{1 + 8x}8.dx = \frac{1}{8}.\frac{(1+8x)^{3/2}}{3/2}\Bigg|_0^1$$

$$= \frac{1}{12}.[(1+8*1)^{3/2} - (1+8*0)^{3/2}] = \frac{13}{6} \text{ unit length.}$$

Example 2: Find the length of the curve $y = \left(\frac{x}{2}\right)^{2/3}$ from x=0 to x=2.

Sol.: The derivative:

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3} * \frac{1}{2} = \frac{1}{3}\left(\frac{x}{2}\right)^{-1/3}$$

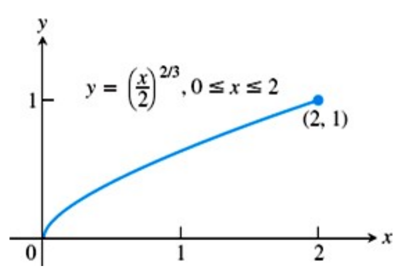
is not defined at x=0, so we can not find the curve's length with equation (1). We therefore rewrite the equation to express x in term of y (x=f(y)):

$$y = \left(\frac{x}{2}\right)^{2/3} \quad \Rightarrow \quad y^{3/2} = \frac{x}{2} \quad \Rightarrow \quad x = 2y^{3/2}$$

Note that when x=0 $\Rightarrow$ y=0

and x=1 $\Rightarrow$ y=1

from this we see that the curve whose length we want is also the graph $x = 2y^{3/2}$ from y=0 to y=1



The derivative

$$\frac{dx}{dy} = 2 * \frac{3}{2} y^{1/2} = 3y^{1/2}$$

is continuous from $y=0$ to $y=1$. We may therefore us equation (2) to find the curve's length:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 \sqrt{1 + \left(3y^{1/2}\right)^2} dy$$

$$= \int_0^1 \sqrt{1+9y} dy = \frac{1}{9} \frac{(1+9y)^{3/2}}{3/2}\Bigg|_0^1$$

$$= \frac{2}{27}[((1+9*1)^{3/2}) - ((1+9*0)^{3/2})]$$

$$= \frac{2}{27}(10\sqrt{10} - 1) \approx 2.27 \text{ unit length.}$$

Example 3: Find the length of the circle of radius r defined parametrically by

$x = r\cos t$ and $y = r\sin t$ $\quad 0 \le t \le 2\pi$.

Sol.: As the curve is defined by parametric equation, we use equation (3) to find the length of the curve

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} .dt$$

We find $\quad \frac{dx}{dt} = -r\sin t \quad \Rightarrow \quad \left(\frac{dx}{dt}\right)^2 = (-r\sin t)^2 = r^2 \sin^2 t$

$$\frac{dy}{dt} = r\cos t \quad \Rightarrow \quad \left(\frac{dy}{dt}\right)^2 = (r\cos t)^2 = r^2\cos^2 t$$

and $\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2\sin^2 t + r^2\cos^2 t$

$$= r^2(\sin^2 t + \cos^2 t) = r^2.$$

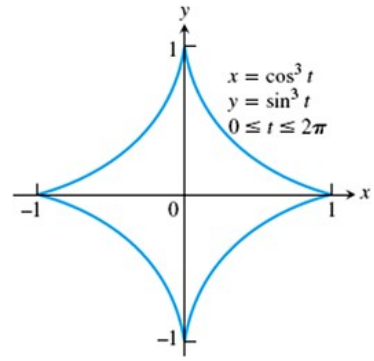
$$\therefore L = \int_0^{2\pi} \sqrt{r^2} .dt = \int_0^{2\pi} r.dt = r.t\Big|_0^{2\pi}$$

$$= r(2\pi - 0) = 2\pi.r \text{ unit length.}$$

Example 4: Find the length of the curve

$$x=\cos^3 t, \qquad y=\sin^3 t, \qquad 0 \le t \le 2\pi.$$

Sol.: Because the curve's symmetry with respect to coordinate axes, its length is four times the length of the first quadrant portion. We have



$$x=\cos^3 t, \qquad y=\sin^3 t$$

$$\left(\frac{dx}{dt}\right)^2 = [3\cos^2 t.(-\sin t)]^2 = 9\cos^4 t\sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3\sin^2 t.\cos t]^2 = 9\sin^4 t\cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)}$$

$$= \sqrt{9\sin^2 t\cos^2 t} = |3\sin t\cos t|$$

$$= 3\sin t\cos t \ \text{(because } \sin t.\cos t \ge 0 \text{ for } 0 \le t \le \pi/2)$$

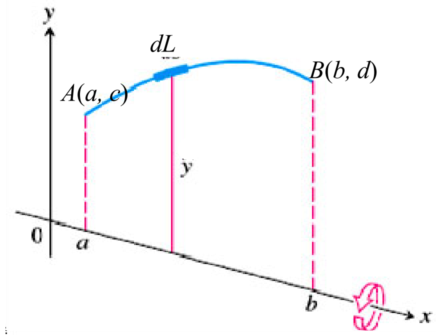
Therefore: The Length of the first quadrant portion= $\int_0^{\pi/2} 3\cos t\sin t.dt$

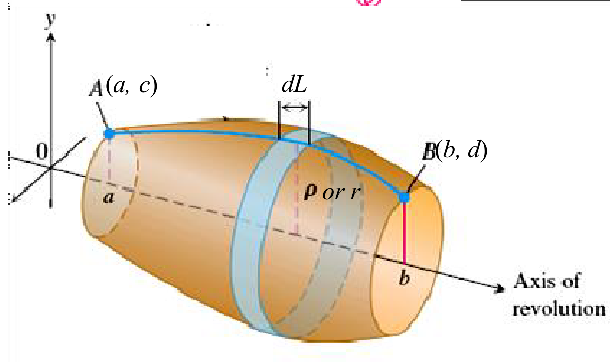
$$= \frac{3}{2}\int_0^{\pi/2} \sin 2t.dt = -\frac{3}{4}\cos 2t\Big|_0^{\pi/2} = \frac{3}{2}$$

.The length of the curve is four times this: 4(3/2)= 6 unit length

4. <u>Area of Surface of Revolution</u>:

If the function $y=f(x) > 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y=f(x)$ about the x-axis is calculated as following:





The surface area of typical cylinder is $dS=2\pi r.dL$

dL will be calculated from one of the following three relations:

i. $dL = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}.dx$

ii. $dL = \sqrt{1+\left(\dfrac{dx}{dy}\right)^2}.dy$

iii. $dL = \sqrt{\left(\dfrac{dy}{dt}\right)^2+\left(\dfrac{dx}{dt}\right)^2}.dt$

r or ρ is the radius of the typical cylinder: (As in this case when the curve is rotated about x-axis), then

$$r=y=f(x)$$

So the surface area: $S=\int dS=\int_a^b 2\pi.r.dL$

If we represent dL by the first equation, then:

$$S=\int_a^b 2\pi.y.\sqrt{1+\left(\frac{dy}{dx}\right)^2}.dx=\int_a^b 2\pi.f(x).\sqrt{1+[f`(x)]^2}.dx \qquad \text{----(1)}$$

When the same area is rotated about y-axis then:

$$r=x$$

The surface area is

$$S=\int_a^b 2\pi.x.\sqrt{1+[f`(x)]^2}.dx \qquad \text{----(2)}$$

Note: We can use this expression instead of equation (1) in case of the curve is expressed as $x=f(y)$

$$S=\int_c^d 2\pi.y.\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,.dy=\int_c^d 2\pi.y.\sqrt{1+[f`(y)]^2}\,.dy \qquad \text{----(3)}$$

and this expression instead of equation (2) in case of the curve is expressed as $x=f(y)$

$$S=\int_c^d 2\pi.x.\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,.dy=\int_c^d 2\pi.f(y).\sqrt{1+[f`(y)]^2}\,.dy \qquad \text{----(4)}$$

If the curve is expressed as parametric equation such:

$$x=x(t),\ y=y(t) \qquad a\le t\le b$$

and $\frac{dx}{dt}$, $\frac{dx}{dt}$ are both continuous in above interval then the area of surface area generated by revolving this curve

i. about x-axis is

$$S=\int_a^b 2\pi.y(t)\sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}\,.dt \qquad \text{----(5)}$$

ii. about y-axis is

$$S=\int_a^b 2\pi.x(t)\sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}\,.dt \qquad \text{----(6)}$$

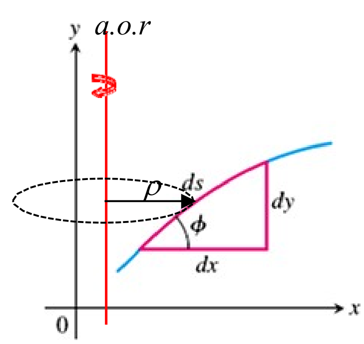
Or in general from short differential form

$$S=\int dS=\int 2\pi.\rho.dL$$

Where $dL=\sqrt{dx^2+dy^2}$

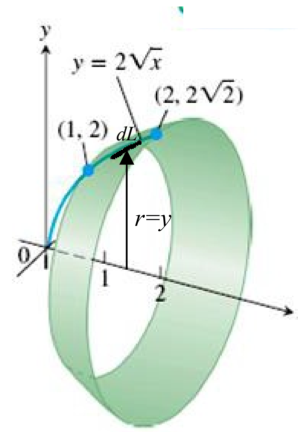
and ρ: is the radius from axis of revolution to an element of arc-length dL. If axis of rotation is

- $x=k$ then $\rho=x-k$
- $y=k$ then $\rho=y-k$



Example 1: Find the area of the surface generated by revolving the curve $y = 2\sqrt{x},\ 1 \le x \le 2$ about x-axis.

Sol.: $dS = 2\pi r.dL$



where $r = y = 2\sqrt{x}$

and $dL = \sqrt{1+[f`(x)]^2}.dx$

$$f`(x) = 2 * \frac{1}{2}x^{-1/2} = \frac{1}{\sqrt{x}}$$

$$dS = 2\pi(2\sqrt{x})\sqrt{1+[f`(x)]^2}.dx$$

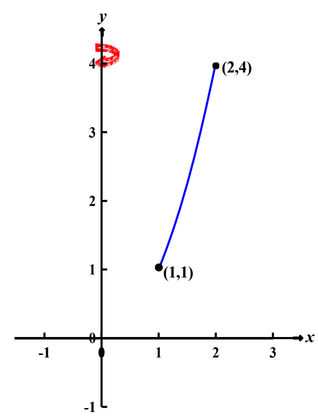
$$= 4\pi\sqrt{x}\sqrt{1+[\frac{1}{\sqrt{x}}]^2}.dx = 4\pi\sqrt{x}\sqrt{1+\frac{1}{x}}.dx$$

$$= 4\pi\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}.dx = 4\pi\sqrt{x+1}.dx \quad ➔ \quad \therefore S = \int dS = \int_1^2 4\pi\sqrt{x+1}.dx = 4\pi \left.\frac{(x+1)^{3/2}}{3/2}\right|_1^2$$

$$= \frac{8\pi}{3}[(2+1)^{3/2} - (1+1)^{3/2}] \approx 19.836 \text{ square units}$$

Example 2: Find the area of the surface generated by revolving the portion of the curve $y=x^2$ between $x=1$ and $x=2$ about y-axis.

Sol.: $dS = 2\pi r.dL$



where $r = x$

and $dL = \sqrt{1+[f`(x)]^2}.dx$

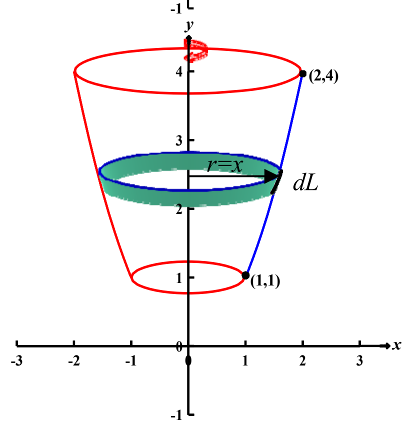
$y=x^2 \Rightarrow f`(x) = 2x$

$\therefore dL = \sqrt{1+(2x)^2}.dx = \sqrt{1+4x^2}.dx$

$$\therefore S = \int dS = \int_1^2 2\pi.x\sqrt{1+4x^2}.dx = 2\pi\left[\frac{1}{8}\frac{(1+4x^2)^{3/2}}{3/2}\right|_1^2$$

$$= \frac{\pi}{6}(1+4x^2)^{3/2}\Big|_1^2 = \frac{\pi}{6}[(1+4*2^2)^{3/2} - (1+4*1^2)^{3/2}]$$

$$= \frac{\pi}{6}[17^{3/2} - 5^{3/2}] \approx 30.85 \text{ square units.}$$



Another solution: Use $x=f(y)$

$$y=x^2 \Rightarrow \quad x=\sqrt{y} \quad \Rightarrow \quad \frac{dx}{dy}=\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \left(\frac{dx}{dy}\right)^2=\left(\frac{1}{2\sqrt{y}}\right)^2=\frac{1}{4y}$$

$$\therefore dL=\sqrt{1+\left(\frac{dx}{dy}\right)^2}.dy=\sqrt{1+\frac{1}{4y}}.dy=\sqrt{\frac{4y+1}{4y}}.dy=\frac{\sqrt{4y+1}}{2\sqrt{y}}.dy$$

The limits of integration:

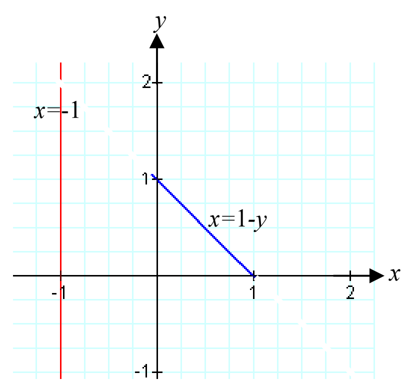
When $x=1 \Rightarrow y=(1)^2=1$ and when $x=2 \Rightarrow y=(2)^2=4$

$$\therefore S=\int dS=\int_1^4 2\pi x\frac{\sqrt{4y+1}}{2\sqrt{y}}.dy=\int_1^4 2\pi.\sqrt{y}\frac{\sqrt{4y+1}}{2\sqrt{y}}.dy$$

$$=\int_1^4 \pi.\sqrt{4y+1}dy=\pi*\frac{1}{4}\frac{(4y+1)^{3/2}}{3/2}\Big|_1^4=\frac{\pi}{3}[(4*4+1)^{3/2}-(4*1+1)^{3/2}]$$

$$=\frac{\pi}{6}[17^{3/2}-5^{3/2}]\approx 30.58 \text{ square units.}$$

Example 3: The line segment $x=1-y$, $0\le y\le 1$, is revolve about $x=-1$ to generate truncated cone. Find its lateral surface area (which excludes the top and base areas).



Sol.: $dS=2\pi r dL$

$r=x-k=x-(-1)=x+1$

$$dL=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.dx$$

and $\frac{dy}{dx}=\frac{d}{dx}(1-x)=-1$

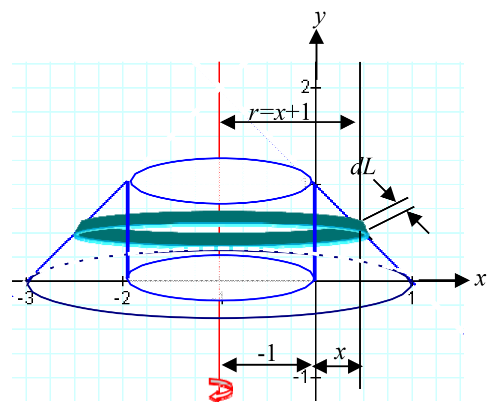
$$\Rightarrow \quad 1+\left(\frac{dy}{dx}\right)^2=1+(-1)^2=1+1=2$$

$$\therefore dL=\sqrt{2}.dx$$

So $dS=2\pi r dL=2\pi(x+1)\sqrt{2}dx$

When $y=0 \quad \Rightarrow \quad x=1-0=1$

$y=1 \quad \Rightarrow \quad x=1-1=0$

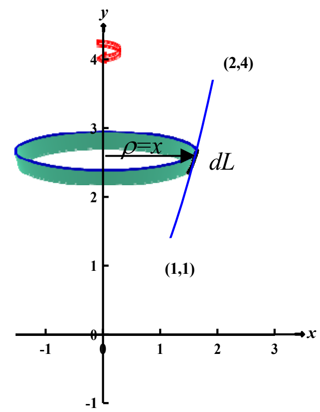


$$\therefore S=\int dS=2\sqrt{2}\pi\int_0^1 (x+1).dx=2\sqrt{2}\pi\left[\frac{x^2}{2}+x\right]_0^1$$

$$=2\sqrt{2}\pi\left[\left(\frac{1^2}{2}+1\right)-\left(\frac{0^2}{2}+0\right)\right]=2\sqrt{2}\pi\left(\frac{3}{2}\right)=3\sqrt{2}\pi \text{ square units}$$

Example 4: Find the area of the surface generated by revolving the parametric curve $x=\cos^2 t$, $y=\sin^2 t$, $0\le t\le \pi/2$ about y-axis.

Sol.: $dS=2\pi\rho.dL$



where $\rho=x=\cos^2 t$

and $dL=\sqrt{dx^2+dy^2}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}.dt$

$$x=\cos^2 t \;\Rightarrow\; \frac{dx}{dt}=-2\cos t\sin t \;\Rightarrow\; \left(\frac{dx}{dt}\right)^2=4\cos^2 t\sin^2 t$$

$$y=\sin^2 t \;\Rightarrow\; \frac{dy}{dt}=2\sin t\cos t \;\Rightarrow\; \left(\frac{dy}{dt}\right)^2=4\sin^2 t\cos^2 t$$

$$dL=\sqrt{8\sin^2\cos^2 t}.dt=2\sqrt{2}\sin t\cos t.dt$$

$$S=\int dS=\int_0^{\pi/2} 2\pi\cos^2 t(2\sqrt{2}\sin t\cos t)dt=\int_0^{\pi/2} 4\sqrt{2}\pi\cos^3 t\sin t.dt$$

$$=-4\sqrt{2}\pi\left.\frac{\cos^4 t}{4}\right|_0^{\pi/2}=-\sqrt{2}\pi[\cos^4\frac{\pi}{2}-\cos^4 0]=-\sqrt{2}\pi[(0)-1]=\sqrt{2}\pi \text{ square units.}$$