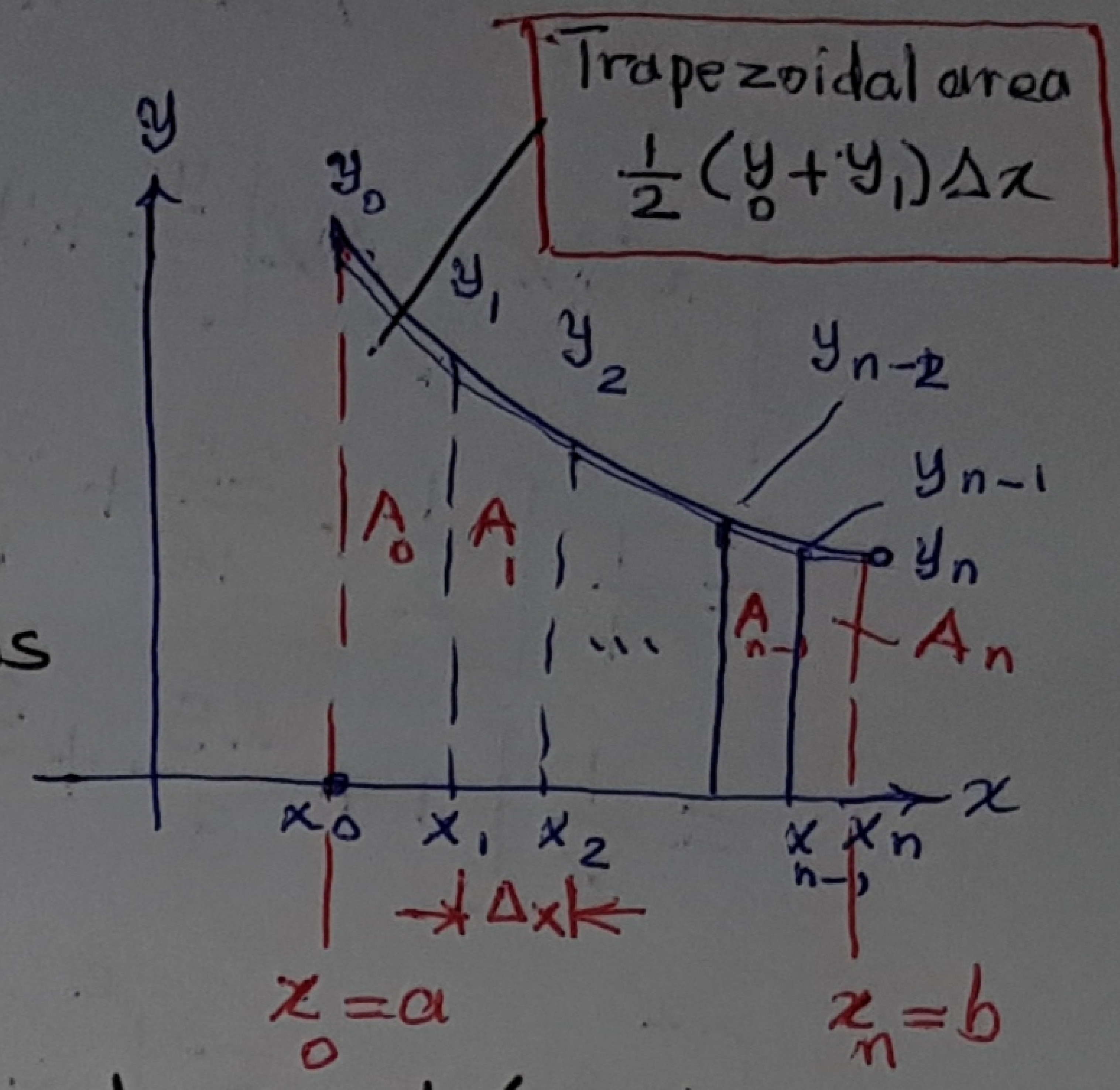
Numerical Integration

1. The Trapezoidal Rule

Let $y = f(x)$, continuous on (a,b). To calculate the area under the curve

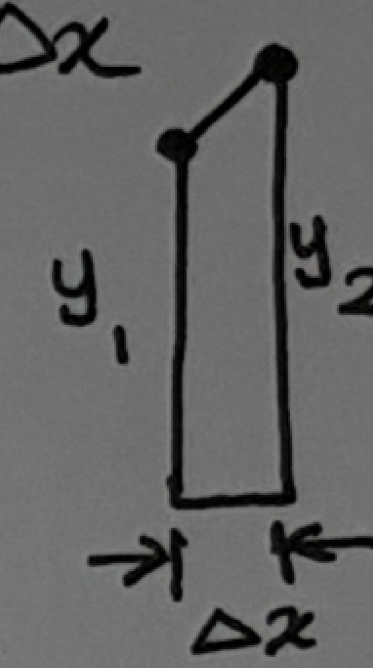
$$A = \int_a^b f(x)\,dx$$



1. Divide the area into (n) strips of uniform thickness (Δx),

$$\Delta x = \frac{b-a}{n}$$

2. Each strip will be a trapezoid, $A = \frac{1}{2}(y_1 + y_2)\Delta x$



$$\therefore A_0 = \frac{1}{2}(y_0 + y_1)\Delta x,$$

$$A_1 = \frac{1}{2}(y_1 + y_2)\Delta x$$

$$\vdots$$

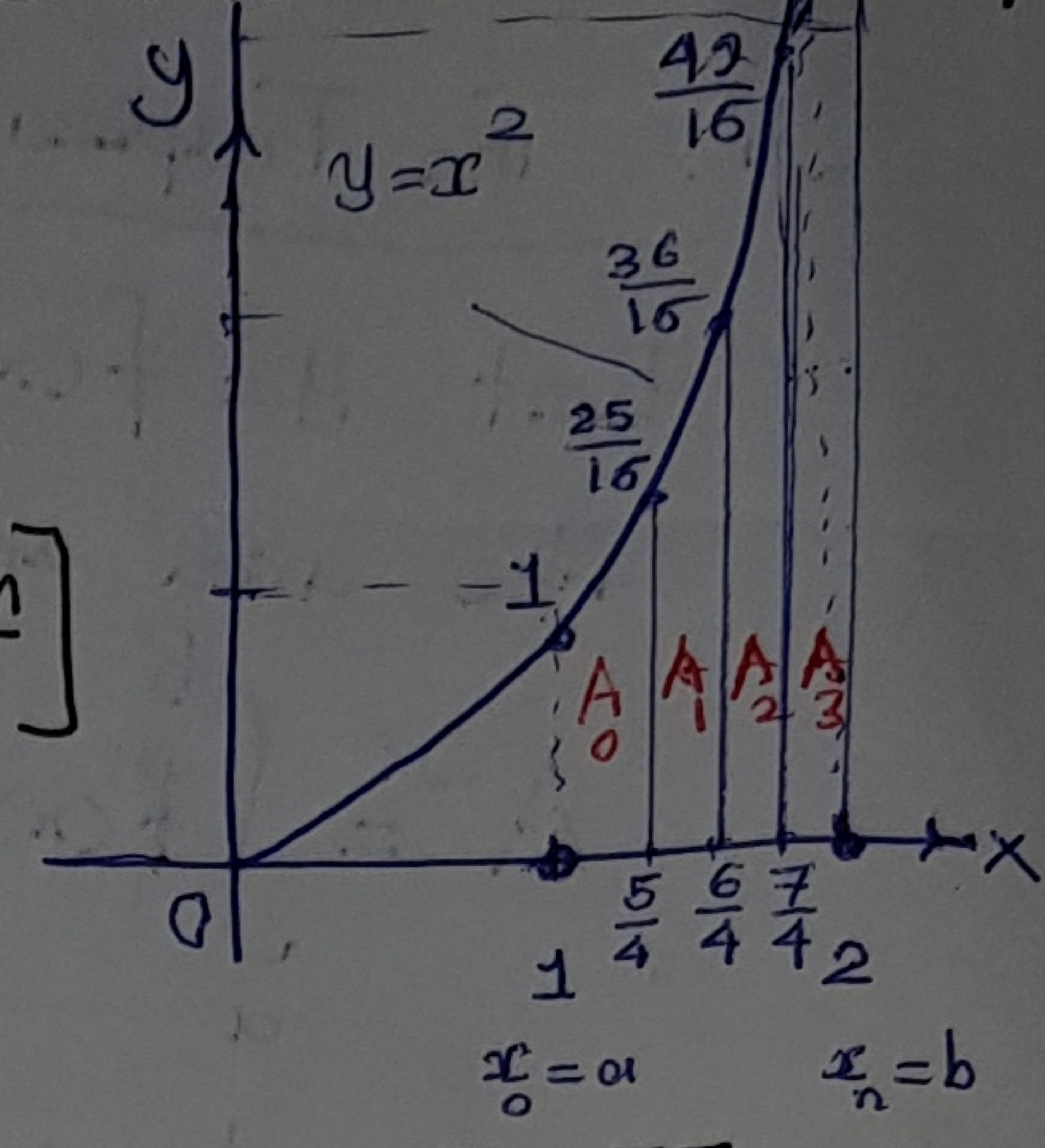
$$A_n = \frac{1}{2}(y_{n-1} + y_n)\Delta x$$

$$\therefore A = A_0 + A_1 + A_2 + \cdots + A_{n-1} + A_n$$

$$\therefore \int_a^b f(x)\,dx = \Delta x\left[\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2}\right]$$

$y_0 = f(x_0) = a$, $y_1 = f(x_1)$, …, $y_{n-1} = f(x_{n-1})$, $y_n = f(x_n) = b$.

Ex. use the Trapezoidal Rule with $n=4$ to estimate $\int_1^2 x^2 dx$. Compare the estimate with the exact value.



Solution: $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$

$$\int_1^2 x^2 dx = \Delta x\left[\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2}\right]$$

$$= \frac{1}{4}\left[\frac{1}{2} + \frac{25}{16} + \frac{36}{16} + \frac{49}{16} + \frac{4}{2}\right]$$

$$= \frac{1}{8}\left[1 + \frac{25}{8} + \frac{36}{8} + \frac{49}{8} + 4\right] = \frac{1}{8}\left[5 + \frac{110}{8}\right] = \frac{75}{32} = 2.3438$$

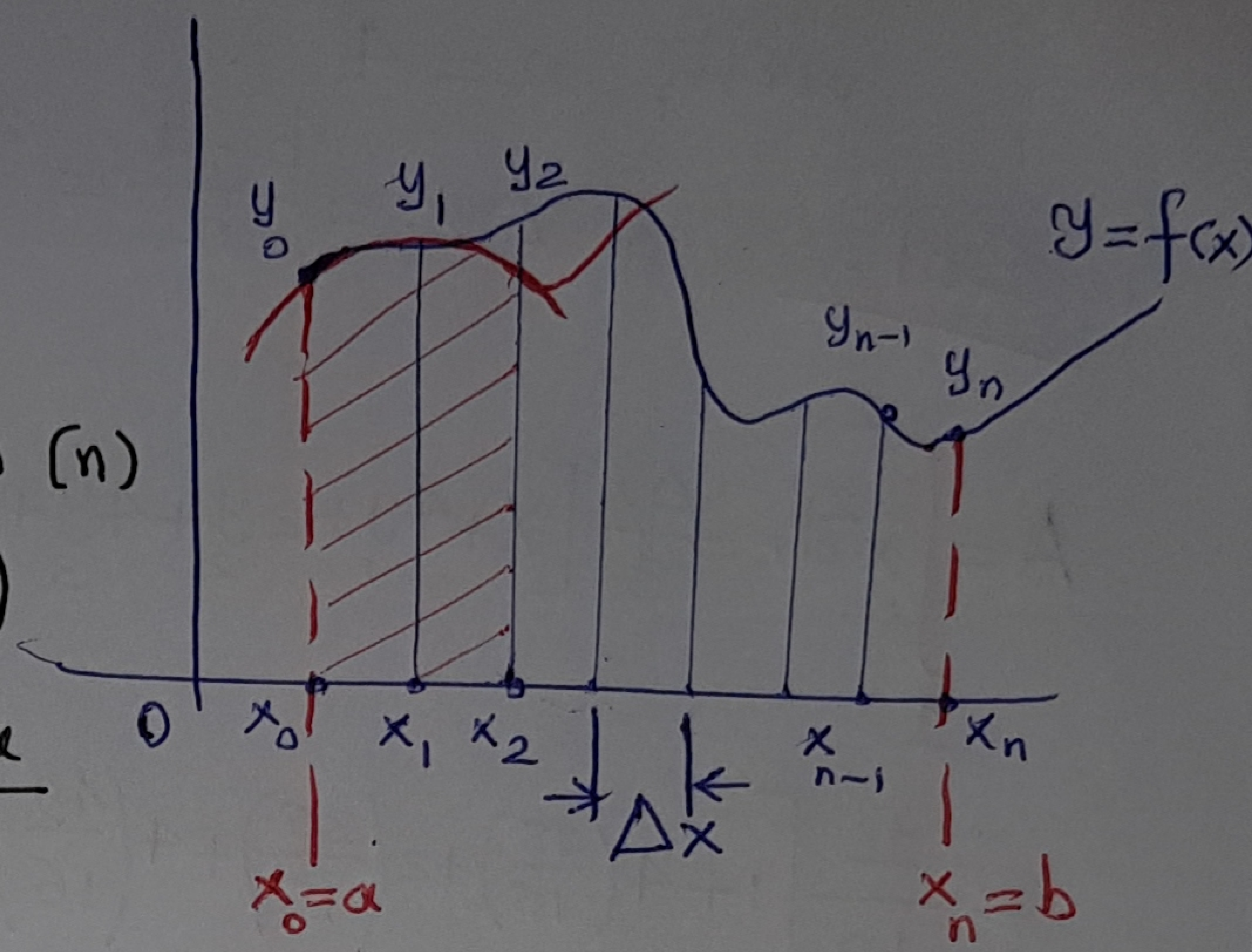
$$\int_1^2 x^2 dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{1}{3}[8-1] = \frac{7}{3}$$

$$\frac{\left(\frac{75}{32} - \frac{7}{3}\right)}{\left(\frac{7}{3}\right)} \simeq 0.00446 \simeq 0.446\%$$

	x	$y = x^2$
x_0	1	1
x_1	$1.25 = \frac{5}{4}$	$\frac{25}{16}$
x_2	$1.5 = \frac{6}{4}$	$\frac{36}{16}$
x_3	$1.75 = \frac{7}{4}$	$49/16$
x_4	2	4

2. Simpson's Rule: Approximations using Parabolas

$$A = \int_a^b f(x)\,dx$$



1. Divide the area into (n) strips (must be even) of thickness $\Delta x = \frac{b-a}{n}$

2. The area ~~of~~ under each Parabola

$$A_0^2 = A_p = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

$$A_2^4 = \frac{\Delta x}{3}(y_2 + 4y_3 + y_4)$$

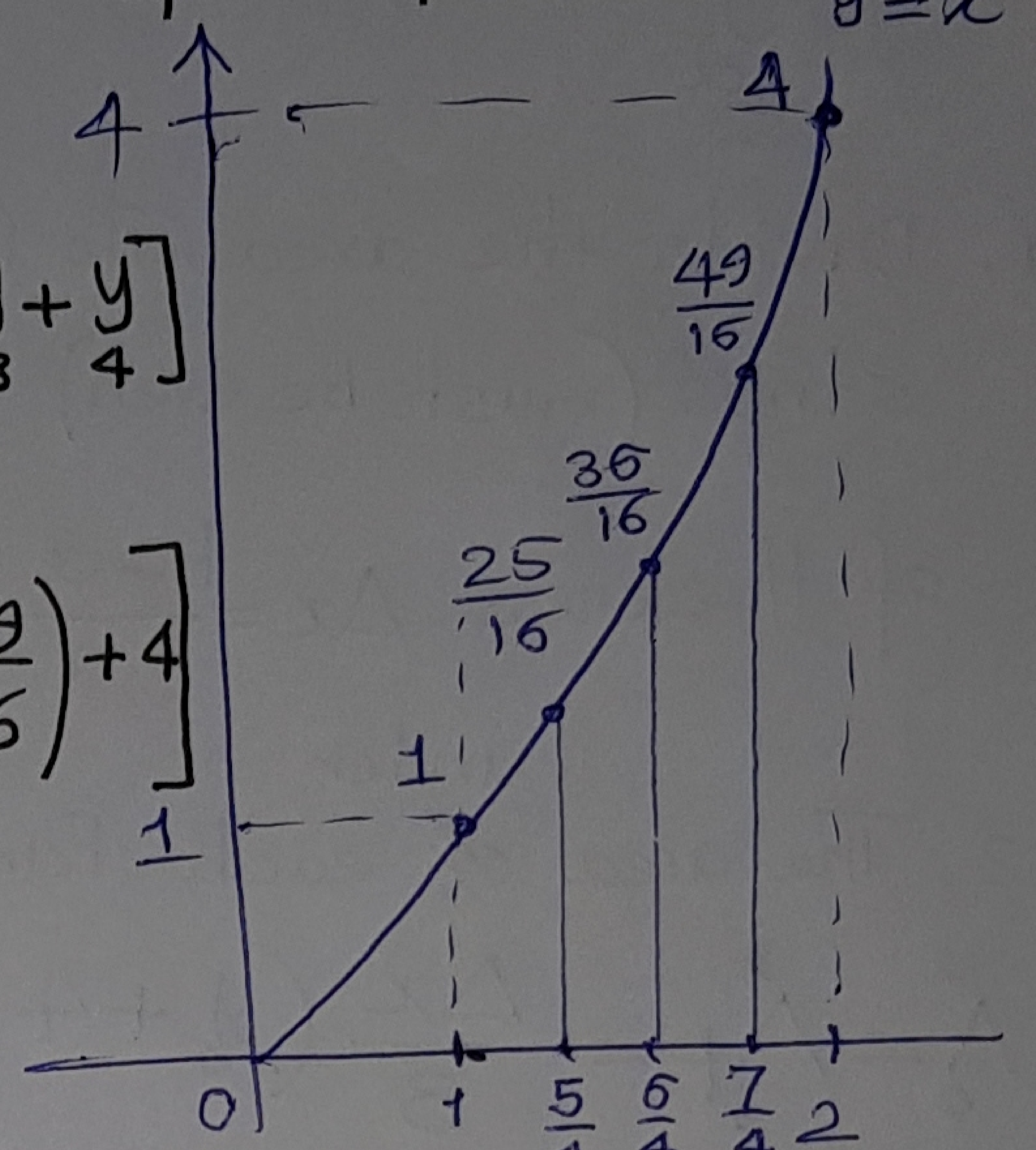
$$\vdots$$

$$A_{n-2}^n = \frac{\Delta x}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$\therefore A = \int_a^b f(x)\,dx = \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + y_n\right].$$

Ex. Estimate $\int_1^2 x^2\,dx$ using Simpson's Rule with $n = 4$.

Sol. $\Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{4} = \dfrac{1}{4}$.



$$A = \int_1^2 x^2\,dx = \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right]$$

$$= \frac{1}{12}\left[1 + 4\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 4\left(\frac{49}{16}\right) + 4\right]$$

$$= \frac{215}{96} = 2.2396.$$

Comparing with $\int_1^2 x^2\,dx = \dfrac{7}{3}$

$$\frac{\frac{215}{96} - \frac{7}{3}}{\frac{7}{3}} = -0.04018 = -4.018$$

Home Work: Use (a) the trapezoidal rule (b) Simpson's rule to estimate: ① $\int_1^2 x\,dx$ ② $\int_{-1}^{1} (x^2+1)\,dx$

③ $\int_0^2 (t^3+t)\,dt$ ④ $\pi = 4\int_0^1 \dfrac{1}{1+x^2}\,dx.$