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مشتقات مقلوب الدوال المثلثية  
Derivatives of the Inverse Trigonometric  
Functions:

$$1 - \frac{d}{dx} (\sin^{-1}(u)) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$2 - \frac{d}{dx} (\cos^{-1}(u)) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$3 - \frac{d}{dx} (\tan^{-1}(u)) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$4 - \frac{d}{dx} (\cot^{-1}(u)) = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$$

$$5 - \frac{d}{dx} (\sec^{-1}(u)) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$6 - \frac{d}{dx} (\csc^{-1}(u)) = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

مثال // برهان المبرنة (1)

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\text{let } y = \sin^{-1}(u) \Rightarrow \sin(y) = u$$

$$\Rightarrow \cos(y) \cdot \frac{dy}{dx} = \frac{du}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} \cdot \frac{du}{dx}$$

$$\therefore \sin(y) = u \Rightarrow \sin^2(y) = u^2 \Rightarrow 1 - \sin^2(y) = 1 - u^2$$

$$\Rightarrow \cos^2(y) = 1 - u^2 \Rightarrow \sqrt{\cos^2(y)} = \sqrt{1 - u^2}$$

$$\Rightarrow \cos(y) = \sqrt{1 - u^2} \quad \therefore \frac{dy}{dx} = \frac{d}{dx} \sin^{-1}(u) = \frac{1}{\sqrt{1 - u^2}} \cdot \frac{du}{dx}$$

1)  $\frac{dy}{dx}$

$$1 - y = \sin^{-1}(3x^2)$$

$$y' = \frac{1}{\sqrt{1-(3x^2)^2}} \cdot 6x = \frac{6x}{\sqrt{1-9x^4}}$$

$$2 - y = \tan^{-1}(3 \tan(x))$$

$$y' = \frac{1}{1+(3 \tan(x))^2} \cdot 3 \sec^2(x) = \frac{3 \sec^2(x)}{1+9 \tan^2(x)}$$

$$3 - y = x \sin^{-1} x + \sqrt{1-x^2}$$

$$y' = \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x + \frac{-2x}{2\sqrt{1-x^2}} = \sin^{-1} x$$

$$4 - \sqrt{x^2-1} - \sec^{-1} x = y$$

$$\begin{aligned} \therefore y' &= \frac{2x}{2\sqrt{x^2-1}} - \frac{1}{x\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}} - \frac{1}{x\sqrt{x^2-1}} \\ &= \frac{x^2-1}{x\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x} \end{aligned}$$

$$5 - x \cos^{-1} 2x - \frac{1}{2} \sqrt{1-4x^2} = y$$

$$y' = \frac{-2x}{\sqrt{1-4x^2}} + \cos^{-1} 2x - \frac{-8x}{4\sqrt{1-4x^2}} = \cos^{-1} 2x$$

أسئلة ثقافت الروال الأ... واللوعار تقييد

$$1 - \frac{d}{dx} (\ln(u)) = \frac{1}{u} \cdot \frac{du}{dx}$$

$$2 - \frac{d}{dx} (\log_a(u)) = \frac{1}{u \ln(a)} \cdot \frac{du}{dx}$$

$$3 - \frac{d}{dx} (a^u) = a^u \ln(a) \cdot \frac{du}{dx}$$

$$4 - \frac{d}{dx} (e^u) = e^u \cdot \frac{du}{dx}$$

أسئلة ثقافت الروال الأ... واللوعار تقييد

$$1 - y = \ln(x^3)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x^3} \cdot 3x^2 = \frac{3}{x}$$

$$2 - y = \ln(\sin^{-1}(2x))$$

$$\Rightarrow y' = \frac{1}{\sin^{-1}(2x)} \cdot \frac{2}{\sqrt{1-4x^2}} = \frac{2}{\sin^{-1}(2x)\sqrt{1-4x^2}}$$

$$3 - y = (100)^{x^2+2x}$$

$$\Rightarrow \frac{dy}{dx} = (100)^{x^2+2x} \cdot \ln(100) \cdot (2x+2)$$

$$4 - y = e^{\sin x}$$

$$\Rightarrow y' = e^{\sin x} \cos(x)$$

2.

$$5 - y = X \log_3 X$$

$$\Rightarrow y' = \cancel{X} \cdot \left( \frac{1}{\cancel{X} \ln(3)} \right) + \log_3 X$$
$$= \frac{1}{\ln(3)} + \log_3 X$$

$$6 - y = e^{\ln(x) + X}$$

$$\Rightarrow \frac{dy}{dx} = e^{\ln(x) + X} \cdot \left( \frac{1}{x} + 1 \right) = e^{\ln(x) + X} \left( \frac{1 + X}{x} \right)$$

$$\Rightarrow e^{\ln(x)} \cancel{e^X} \left( \frac{1 + X}{x} \right) = e^{\cancel{X}} \cancel{X} \left( \frac{1 + X}{x} \right)$$
$$= e^X (1 + X)$$

$$7 - y = \ln(x^2 + 4) - x \tan^{-1} \frac{X}{2}$$

$$y' = \frac{2X}{X^2 + 4} - \left( \frac{X \cdot \frac{1}{2}}{1 + \left(\frac{X}{2}\right)^2} + \tan^{-1} \frac{X}{2} \right)$$

$$= \frac{\cancel{2X}}{X^2 + 4} - \frac{\cancel{2X}}{4 + X^2} - \tan^{-1} \frac{X}{2} = -\tan^{-1} \frac{X}{2}$$

$$8 - y = \log_5 (x^2 + 5X)$$

$$y' = \frac{2X + 5}{(x^2 + 5X) \ln 5}$$

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derivatives hyperbolic functions مشتقات الدوال الزائدية

$$1 - \frac{d}{dx} (\sinh(u)) = \cosh(u) \cdot \frac{du}{dx}$$

$$2 - \frac{d}{dx} (\cosh(u)) = \sinh(u) \cdot \frac{du}{dx}$$

$$3 - \frac{d}{dx} (\tanh(u)) = \operatorname{sech}^2(u) \cdot \frac{du}{dx}$$

$$4 - \frac{d}{dx} (\coth(u)) = -\operatorname{csch}^2(u) \cdot \frac{du}{dx}$$

$$5 - \frac{d}{dx} (\operatorname{sech}(u)) = -\operatorname{sech}(u) \tanh(u) \cdot \frac{du}{dx}$$

$$6 - \frac{d}{dx} (\operatorname{csch}(u)) = -\operatorname{csch}(u) \coth(u) \cdot \frac{du}{dx}$$

\* برهنه الطريقة 1

$$y = \sinh(u) = \frac{e^u - e^{-u}}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left( e^u \cdot \frac{du}{dx} + e^{-u} \frac{du}{dx} \right)$$

$$= \frac{1}{2} (e^u + e^{-u}) \frac{du}{dx} = \cosh(u) \cdot \frac{du}{dx}$$

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دالة  $y$  كدالة لـ  $x$   $\frac{dy}{dx}$   $\rightarrow$   $\frac{dy}{dx}$

$$1 - y = \sinh^2(5x)$$

$$\begin{aligned}\Rightarrow \frac{dy}{dx} &= 2 \sinh(5x) \cdot \cosh(5x) \cdot 5 \\ &= 10 \sinh(5x) \cosh(5x)\end{aligned}$$

$$2 - y = \tanh(x^3) \coth(x^2)$$

$$\begin{aligned}\Rightarrow \frac{dy}{dx} &= \tanh(x^3) (-\operatorname{csch}^2(x^2)) \cdot (2x) + \\ &\quad \coth(x^2) (\operatorname{sech}^2(x^3)) \cdot (3x^2)\end{aligned}$$

$$3 - y = \cosh(e^{2x})$$

$$\Rightarrow \frac{dy}{dx} = \sinh(e^{2x}) \cdot e^{2x} \cdot 2 = 2e^{2x} \sinh(e^{2x})$$

$$4 - y = \ln(\sinh(2x))$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sinh(2x)} \cdot \cosh(2x) \cdot (2)$$

$$= \frac{2 \cosh(2x)}{\sinh(2x)} = 2 \coth(2x)$$

مشتقة الدوال الزائدية العكسية  
Inverse hyperbolic function

$$1 - \frac{d}{dx} (\sinh^{-1}(u)) = \frac{1}{\sqrt{u^2+1}} \cdot \frac{du}{dx}$$

$$2 - \frac{d}{dx} (\cosh^{-1}(u)) = \frac{1}{\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$3 - \frac{d}{dx} (\tanh^{-1}(u)) = \frac{1}{1-u^2} \cdot \frac{du}{dx}$$

$$4 - \frac{d}{dx} (\coth^{-1}(u)) = \frac{1}{1-u^2} \cdot \frac{du}{dx} \quad |u| > 1$$

$$5 - \frac{d}{dx} (\operatorname{sech}^{-1}(u)) = \frac{-1}{u\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$6 - \frac{d}{dx} (\operatorname{csch}^{-1}(u)) = \frac{-1}{|u|\sqrt{1+u^2}} \cdot \frac{du}{dx}$$

\* برهان العبارة (1)

$$\text{let } y = \sinh^{-1}(u) = \ln(u + \sqrt{u^2+1})$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{u + \sqrt{u^2+1}} \cdot \left(1 + \frac{2u}{2\sqrt{u^2+1}}\right) \cdot \frac{du}{dx}$$

$$= \frac{1}{u + \sqrt{u^2+1}} \cdot \left(\frac{u + \sqrt{u^2+1}}{\sqrt{u^2+1}}\right) \cdot \frac{du}{dx} = \frac{1}{\sqrt{u^2+1}} \cdot \frac{du}{dx}$$

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دائری لکھو کہ  $\frac{dy}{dx}$  کا // اسی

$$1 - y = \tanh^{-1}(\cos(x))$$

$$\begin{aligned}\Rightarrow y' &= \frac{1}{1 - \cos^2(x)} \cdot (-\sin(x)) = \frac{-\sin(x)}{1 - \cos^2(x)} \\ &= \frac{-\sin(x)}{\sin^2(x)} = \frac{-1}{\sin(x)}\end{aligned}$$

$$2 - y = \operatorname{sech}^{-1}(\sin(2x))$$

$$\begin{aligned}\Rightarrow \frac{dy}{dx} &= \frac{-1}{\sin(2x)\sqrt{1 - \sin^2(2x)}} \cdot (2\cos(2x)) \\ &= \frac{-2\cos(2x)}{\sin(2x)\sqrt{\cos^2(2x)}} = \frac{-2\cos(2x)}{\sin(2x)\cos(2x)} \\ &= \frac{-2}{\sin(2x)}\end{aligned}$$

$$3 - y = \operatorname{cosh}^{-1}(e^x)$$

$$\Rightarrow y' = \frac{1}{\sqrt{(e^x)^2 - 1}} \cdot e^x = \frac{e^x}{\sqrt{e^{2x} - 1}}$$

$$4 - y = \operatorname{sech}^{-1}(\cos(x))$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\cos(x)\sqrt{1 - \cos^2(x)}} \cdot (-\sin(x)) = \sec(x)$$



أمثلة ١- أوجد المشتقة الثانية للدالة

$$y = \sin x$$

$$y' = \frac{d}{dx} (\sin x) = \cos x$$

$$y'' = \frac{d}{dx} (\cos x) = -\sin x$$

٢- أوجد المشتقة الثالثة إذا كانت

$$y = 6x^5$$

$$y' = \frac{d}{dx} (6x^5) = 5 \times 6x^4 = 30x^4$$

$$y'' = \frac{d}{dx} (30x^4) = 30 \times 4x^3 = 120x^3$$

$$y''' = \frac{d}{dx} (120x^3) = 120 \times 3x^2 = 360x^2$$

٣- إذا كانت  $y = e^{-x} \ln x$  فأوجد  $y''$

$$\begin{aligned} y' &= e^{-x} \frac{d}{dx} \ln(x) + \ln(x) \frac{d}{dx} e^{-x} \\ &= \frac{e^{-x}}{x} - e^{-x} \ln x = \frac{e^{-x}}{x} - y \end{aligned}$$

٤- إذا كانت  $y = e^{-x} \ln x^2$  فأوجد  $y''$

$$y = e^{-x} \ln x^2 = 2e^{-x} \ln x$$

ومن هنا نجد المشتقة الأولى فـ

$$y' = -2e^{-x} \left[ \frac{2}{x} + \frac{1}{x^2} - \ln x \right]$$

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$$y = \sin 3x + \cos 2x \quad \text{إذا كانت } = 0$$

من المتكافئة إذا

$$y' = \cos(3x) * (3) - \sin(2x) * (2)$$

$$y'' = -3 \sin(3x) * 3 - 2 \cos(2x) * (2)$$

$$y''' = -9 \cos(3x) * (3) + 4 \sin(2x) * (2)$$

$$y^{(4)} = 27 \sin(3x) * 3 + 8 \cos(2x) * (2)$$

$$y^{(5)} = 81 \cos(3x) * (3) - 16 \sin(2x) * 2$$

## قاعده لوبيتال L Hopitals rule

تتعلق هذه القاعدة بالاستقاقات بهدف إيجاد النهايات  
لصيغ غير محددة. يتم هذه القاعدة في الرياضيات الفزيائية  
غنيوم وي لوبيتال.

ستتعلق هذه القاعدة في حالة  $\frac{0}{0}$  مثل

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x^2 + 3x} = \frac{0}{0+0} = \frac{0}{0}$$

بمناسبة البسطة والمقام وتصل على

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x^2 + 3x} = \lim_{x \rightarrow 0} \frac{\cos(x)}{2x + 3} = \frac{\cos(0)}{0 + 3} = \frac{1}{3}$$

وستتعلق هذه القاعدة في حالة  $\frac{\infty}{\infty}$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln(x)} = \frac{\infty}{\infty}$$

بمناسبة البسطة والمقام

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln(x)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{\sqrt{x}\sqrt{x}}}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{2} = +\infty$$

أمثلة تستعمل قاعدة لوبيتال عدة مرات للوصول إلى النتيجة

$$\lim_{x \rightarrow 0} \frac{\cos(2x) - 1}{x^3 + 5x^2} = \lim_{x \rightarrow 0} \frac{-2\sin(2x)}{3x^2 + 10x}$$

$$= \lim_{x \rightarrow 0} \frac{-4\cos(2x)}{6x + 10} = \frac{-2}{5}$$

وقد يمكن إيجاد بعض النهايات التي لا تقبل بشكل مباشر

كسور باستخدام هذه القاعدة

$$\lim_{x \rightarrow \infty} x - \sqrt{x^2 - x} = \lim_{x \rightarrow \infty} \frac{1 - \sqrt{1 - 1/x}}{1/x}$$

$$= \lim_{x \rightarrow \infty} \frac{1 - \sqrt{1 - h}}{h} = \lim_{x \rightarrow \infty} \frac{1}{2\sqrt{1 - h}}$$

$$= \frac{1}{2}$$

$$\infty - \infty \quad \lim_{x \rightarrow \infty} x - \sqrt{x^2 - x} = \frac{1}{2}$$

أمثلة :-

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\pi - 2x}$$

عند النقطة بقيه  $x = \frac{\pi}{2}$  فان النتيجة  $\frac{0}{0}$

نستخدم قاعدة لوبيتال

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\pi - 2x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\sin x}{-2} = \frac{\sin \frac{\pi}{2}}{2} = \frac{1}{2}$$

$$\lim_{x \rightarrow 0} \frac{x}{\sin \sqrt{x}}$$

اوفاً نتبيح القويه  $\frac{0}{0}$  لذلك نستخدم القاعده

$$\lim_{x \rightarrow 0} \frac{x}{\sin \sqrt{x}} = \lim_{x \rightarrow 0} \frac{1}{\cos \sqrt{x} * \frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow 0} \frac{2\sqrt{x}}{\cos \sqrt{x}}$$

$$= \frac{0}{1} = 0$$

$$\lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2}$$

$$\lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2} = \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} = \frac{0}{0}$$

بما ان النتيجة باقيه  $\frac{0}{0}$  نطبقه افتر

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{2x} = \lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$$

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$$\lim_{x \rightarrow \infty} \frac{\log X}{X}$$

-٤ - حد

عند الحدود تكون النتيجة  $\frac{\infty}{\infty}$  لذلك سنستخدم قاعدة لوبيتال

$$\therefore \lim_{x \rightarrow \infty} \frac{\log X}{X} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x}$$

$$= \frac{1}{\infty} = 0$$

٥ - حد أنت  $\lim_{x \rightarrow 0} \log X^x = 0$

$$\therefore \lim_{x \rightarrow 0} \log X^x = \lim_{x \rightarrow 0} x \log X = \lim_{x \rightarrow 0} \frac{\log X}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0} (-x) = 0$$

$$\lim_{x \rightarrow 0} \frac{2+x+\sin x}{x^3+x-\cos x}$$

-٦ - حد

$$\lim_{x \rightarrow 0} \frac{2+x+\sin x}{x^3+x-\cos x} = \frac{2+0+\sin(0)}{0+0-\cos(0)}$$

$$= \frac{2}{-1} = -2$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} \quad \text{ـ } \Delta \text{ ـ } \nabla$$

عند القبول بقيه  $x=0$  فان الناتج  $\frac{0}{0}$

$$\lim_{x \rightarrow 0} \frac{(1+x)^{1/2} - 1}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}(1+x)^{-1/2}}{1} = \frac{1}{2}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x + x^2} \quad \text{ـ } \Delta \text{ ـ } \nabla$$

الناتج  $\left(\frac{0}{0}\right)$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0$$

الناتج  $\left(\frac{0}{0}\right)$

$$\lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3} \quad \text{ـ } \Delta \text{ ـ } \nabla$$

$$\lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3} = \lim_{x \rightarrow 3} \frac{4x^3}{1} = 108$$

الناتج  $\frac{\infty}{\infty}$

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} \quad \text{ـ } \Delta \text{ ـ } \nabla$$

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x}$$

$$= \frac{2}{\infty} = 0$$

الناتج  $0 \cdot \infty$

$$\lim_{x \rightarrow \infty} x^2 e^{-x} \quad \text{ـ } \Delta \text{ ـ } \nabla$$

$$\lim_{x \rightarrow \infty} x^2 e^{-x} = \lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x}$$

$$= \frac{2}{\infty} = 0$$

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$\infty - \infty$     السع     $\lim_{x \rightarrow 0} \left( \csc(x) - \frac{1}{x} \right) \rightarrow \Delta - \infty$

$$\therefore \lim_{x \rightarrow 0} \left( \csc(x) - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left( \frac{x - \sin(x)}{x \sin(x)} \right) = \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x \cos(x) + \sin(x)} = \lim_{x \rightarrow 0} \frac{\sin(x)}{\cos(x) + \cos(x) - x \sin(x)}$$

$$= \frac{0}{2} = 0$$

\* أيضا ساعد قاعدة لوبيتال للمنفرد  $(0^0, \infty^0, 1^\infty)$

$1^\infty$  السع     $\lim_{x \rightarrow 0} (\cos(x))^{\frac{1}{x^2}} \rightarrow \Delta - \infty$

Let  $y = (\cos(x))^{\frac{1}{x^2}} \Rightarrow \ln y = \frac{1}{x^2} \ln \cos(x)$

$$\Rightarrow \lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{1}{x^2} \ln(\cos(x))$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{-\sin(x)}{2x \cos(x)} = \lim_{x \rightarrow 0} \frac{-\cos(x)}{2(\cos(x) - x \sin(x))} = -\frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow 0} \ln(y) = -\frac{1}{2} \Rightarrow \ln(\lim_{x \rightarrow 0} y) = -\frac{1}{2}$$

$$\Rightarrow e^{\ln(\lim_{x \rightarrow 0} y)} = e^{-\frac{1}{2}} \Rightarrow \lim_{x \rightarrow 0} y = e^{-\frac{1}{2}}$$

$$\therefore \lim_{x \rightarrow 0} (\cos(x))^{\frac{1}{x^2}} = e^{-\frac{1}{2}}$$