Properties of Logarithmic Functions :- خواص الدوال اللوغاريتمية

اذا كانت $b>0$ ، $b \neq 1$ ، $a>0$ ، $c>0$
و r اي عدد حقيقي فان

① $\log_b(ac) = \log_b(a) + \log_b(c)$

② $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$

③ $\log_b a^r = r \log_b a$

④ $\log_b(1) = 0$

⑤ $\log_b x$ هنا يكون غير معرف عندما $x<0$

⑥ $\log_b b = 1$

⑦ $\log_b\left(\frac{1}{c}\right) = -\log_b(c)$

⑧ $\ln(e^x) = x$ لكل قيم x

⑨ $e^{\ln(x)} = x$

⑩ $\log_b(x) = \frac{\ln(x)}{\ln(b)}$

⑪ $\log_b b^x = x$ لكل قيم x

مثال : جد $\text{Log} \dfrac{x y^5}{\sqrt{z}}$

ملاحظة : عندما لا يذكر الاساس (b) هذا يعني انه اللوغارتم عشري اي ان $b=10$

$$\text{Log} \frac{x y^5}{\sqrt{z}} = \text{Log}(x y^5) - \text{Log}(\sqrt{z})$$

$$= \text{Log}\, x + \text{Log}\, y^5 - \text{Log}(z^{\frac{1}{2}})$$

$$= \text{Log}\, x + 5\, \text{Log}\, y - \frac{1}{2} \text{Log}\, z$$

مثال :- جد $\frac{1}{3} \ln(x) - \ln(x^2-1) + 2 \ln(x+3)$

الحل

$$\frac{1}{3} \ln(x) - \ln(x^2-1) + 2 \ln(x+3)$$

$$= \ln(x)^{\frac{1}{3}} - \ln(x^2-1) + \ln(x+3)^2$$

$$= \ln(x)^{\frac{1}{3}} + \ln(x+3)^2 - \ln(x^2-1)$$

$$= \ln\left((x)^{\frac{1}{3}} (x+3)^2\right) - \ln(x^2-1)$$

$$= \ln\left(\frac{\sqrt[3]{x}\,(x+3)^2}{x^2-1}\right)$$

مثال : جد قيمة x لكل مما يأتي :

① $\text{Log}\, x = 2$

الحل

$$\text{Log}\, x = 2 \Rightarrow \text{Log}\, x = \frac{\ln(x)}{\ln(10)} = 2 \Rightarrow$$

$$\Rightarrow \ln(x) = 2\ln(10) \Rightarrow \ln(x) = \ln(10)^2 \Rightarrow$$

$$\Rightarrow \ln(x) = \ln(100)$$ بأخذ exp للطرفين

$$\Rightarrow e^{\ln(x)} = e^{\ln(100)}$$

$$= \boxed{x = 100}$$

② $\ln(x+1) = 5$

بأخذ exp للطرفين

$$= e^{\ln(x+1)} = e^{5}$$

$$= x+1 = e^5 \Rightarrow \boxed{x = e^5 - 1}$$

③ $5^x = 7$

بأخذ ln للطرفين

$$\Rightarrow \ln(5^x) = \ln(7)$$

$$\Rightarrow x\ln(5) = \ln(7)$$

$$x = \frac{\ln(7)}{\ln(5)}$$

④ $\dfrac{e^{x} - e^{-x}}{2} = 1$

الحل $= e^{x} - e^{-x} = 2$

$= e^{x} - \dfrac{1}{e^{x}} = 2$

$= \dfrac{e^{2x} - 1}{e^{x}} = 2$

$\Rightarrow e^{2x} - 1 = 2e^{x} \Rightarrow e^{2x} - 2e^{x} - 1 = 0$ $\boxed{\pm 1}$

$\Rightarrow e^{2x} - 2e^{x} - 1 - 1 + 1 = 0$

$e^{2x} - 2e^{x} + 1 - 2 = 0$

$(e^{x} - 1)(e^{x} - 1) - 2 = 0$

$\therefore (e^{x} - 1)^{2} = 2$ بالجذر

$e^{x} - 1 = \pm\sqrt{2} \Rightarrow e^{x} = \pm\sqrt{2} + 1$ بأخذ $\ln$

$\ln e^{x} = \ln(\sqrt{2} + 1)$ + فقط

$\boxed{\therefore x = \ln(1 + \sqrt{2})}$

⑤ $x^{\log x} = 100x$ H.W

Hyperbolic Functions الدوال الزائدية :-

① $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$ where $D_f = R$, $R_f = R$

② $\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$ where $D_f = R$, $R_f = [1, \infty)$

③ $\tanh(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \dfrac{\sinh(x)}{\cosh(x)}$ wher $D_f = R$, $R_f = (-1, 1)$

④ $\coth(x) = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}} = \dfrac{1}{\tanh(x)} = \dfrac{\cosh(x)}{\sinh(x)}$

where $D_f = R / \{0\}$

$R_f = R / (-1, 1)$

⑤ $\operatorname{sech}(x) = \dfrac{2}{e^{x} + e^{-x}} = \dfrac{1}{\cosh(x)}$ where $D_f = R$

$R_f = (0, 1]$

⑥ $\operatorname{csch}(x) = \dfrac{2}{e^{x} - e^{-x}} = \dfrac{1}{\sinh(x)}$ where $D_f = R / \{0\}$

$R_f = R / \{0\}$

⑦ $\cosh^2(x) - \sinh^{\boxed{2}}(x) = 1$ ⑦ مبرهنة

$$\left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2 = 1$$

$$\frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1$$

$$\frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4} = 1$$

$$= \frac{4}{4} = 1$$

⑧ $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

⑨ $\coth^2(x) - 1 = \operatorname{csch}^2(x)$

ملاحظات حول الدوال الزائدية :-

دالة فردية 1) $\sinh(-x) = \dfrac{e^{-x} - e^{-(-x)}}{2} = \dfrac{e^{-x} - e^{x}}{2} = \dfrac{-(e^{x} - e^{-x})}{2} = -\sinh(x)$

دالة زوجية 2) $\cosh(-x) = \dfrac{e^{-x} + e^{-(-x)}}{2} = \dfrac{e^{-x} + e^{x}}{2} = \dfrac{e^{x} + e^{-x}}{2} = \cosh(x)$

③ $\tanh(-x) = -\tanh(x)$ دالة فردية

④ $\coth(-x) = -\coth(x)$ دالة فردية

⑤ $\operatorname{sech}(-x) = \operatorname{sech}(x)$ دالة زوجية

⑥ $\operatorname{csch}(-x) = -\operatorname{csch}(x)$ دالة فردية

Properties of Hyperbolic Functions خواص الدوال الزائدية

① $\sinh(x \mp y) = \sinh(x)\cosh(y) \mp \sinh(y)\cosh(x)$

② $\cosh(x \mp y) = \cosh(x)\cosh(y) \mp \sinh(x)\sinh(y)$

③ $\tanh(x \mp y) = \dfrac{\tanh(x) \mp \tanh(y)}{1 \mp \tanh(x)\tanh(y)}$

④ $\sinh(2x) = 2\sinh(x)\cosh(x)$

⑤ $\cosh(2x) = \sinh^2(x) + \cosh^2(x)$

وباستفادة من المتطابقة $\cosh^2(x) - \sinh^2(x) = 1$ نجد

$\cosh(2x) = 2\sinh^2(x) + 1$

or $= 2\cosh^2(x) - 1$

⑥ $\sinh^2(x) = \dfrac{\cosh(2x) - 1}{2}$

⑦ $\cosh^2(x) = \dfrac{\cosh(2x) + 1}{2}$

مثال : اذا كانت $\cosh(x) = 5$, أن $x > 0$, جد
$\sinh(x)$, $\tanh(x)$, $\coth(x)$, $\operatorname{sech}(x)$ and $\operatorname{csch}(x)$

الحل :

$\because \cosh^2(x) - \sinh^2(x) = 1$

$(5)^2 - \sinh^2(x) = 1$

$\therefore \sinh^2(x) = (5)^2 - 1$

$\sinh^2(x) = 25 - 1$

$\sinh^2(x) = 24$ بالجذر

$\therefore \boxed{\sinh(x) = \sqrt{24}}$

$\because \tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{\sqrt{24}}{5}$

$\because \coth(x) = \dfrac{\cosh(x)}{\sinh(x)} = \dfrac{5}{\sqrt{24}}$

$\because \operatorname{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{1}{5}$

$\because \operatorname{csch}(x) = \dfrac{1}{\sinh(x)} = \dfrac{1}{\sqrt{24}}$

مثال : اثبت أن

$\cosh(x) + \sinh(x) = e^{x}$

الحل :

$\dfrac{e^{x} + e^{-x}}{2} + \dfrac{e^{x} - e^{-x}}{2} = \dfrac{e^{x} + \cancel{e^{-x}} + e^{x} - \cancel{e^{-x}}}{2}$

$\Rightarrow \dfrac{\cancel{2}e^{x}}{\cancel{2}} = e^{x}$

مثال :- اثبت ان $\cosh(x) - \sinh(x) = e^{-x}$ H.W

مثال : اثبت أن $\tanh\left(\frac{1}{2}\ln(x)\right) = \frac{x-1}{x+1}$

الحل :

$$\tanh\left(\tfrac{1}{2}\ln(x)\right) = \frac{e^{\frac{1}{2}\ln(x)} - e^{-\frac{1}{2}\ln(x)}}{e^{\frac{1}{2}\ln(x)} + e^{-\frac{1}{2}\ln(x)}}$$

$$= \frac{e^{\ln(x)^{1/2}} - e^{\ln(x)^{-1/2}}}{e^{\ln(x)^{1/2}} + e^{\ln(x)^{-1/2}}}$$

$$= \frac{e^{\ln(\sqrt{x})} - e^{\ln\left(\frac{1}{\sqrt{x}}\right)}}{e^{\ln(\sqrt{x})} + e^{\ln\left(\frac{1}{\sqrt{x}}\right)}}$$

$$= \frac{\sqrt{x} - \frac{1}{\sqrt{x}}}{\sqrt{x} + \frac{1}{\sqrt{x}}}$$

$$= \frac{\frac{x-1}{\cancel{\sqrt{x}}}}{\frac{x+1}{\cancel{\sqrt{x}}}}$$

$$= \frac{x-1}{x+1}$$

مثال : اثبت أن $\tanh(x+y) = \dfrac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}$

الحل :- نأخذ الطرف الايمن

$$\frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)} = \frac{\dfrac{e^x - e^{-x}}{e^x + e^{-x}} + \dfrac{e^y - e^{-y}}{e^y + e^{-y}}}{1 + \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \dfrac{e^y - e^{-y}}{e^y + e^{-y}}}$$

$$\Rightarrow \frac{\dfrac{(e^x - e^{-x})(e^y + e^{-y}) + (e^y - e^{-y})(e^x + e^{-x})}{\cancel{(e^x + e^{-x})(e^y + e^{-y})}}}{\dfrac{(e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})}{\cancel{(e^x + e^{-x})(e^y + e^{-y})}}}$$

$$\frac{e^{(x+y)} + \cancel{e^{(x-y)}} - \cancel{e^{-y-x}} - e^{-(x+y)} + e^{(x+y)} + \cancel{e^{(y-x)}} - \cancel{e^{(x-y)}} - e^{-(x+y)}}{e^{(x+y)} + \cancel{e^{x-y}} + \cancel{e^{y-x}} + e^{-(x+y)} + e^{x+y} - \cancel{e^{x-y}} - \cancel{e^{y-x}} + e^{-(x+y)}}$$

$$= \frac{2e^{(x+y)} - 2e^{-(x+y)}}{2e^{(x+y)} + 2e^{-(x+y)}}$$

$$= \frac{e^{(x+y)} - e^{-(x+y)}}{e^{(x+y)} + e^{-(x+y)}} = \tanh(x+y)$$

مثال :- اثبت أن

$$\tanh(2x) = \frac{2\tanh(x)}{1 + \tanh^2(x)}$$

H.W.

Inverse of Hyperbolic Functions: معكوس الدوال الزائدية :

① If $y=\sinh(x) \Rightarrow x=\sinh^{-1}(y)$ حيث $D_f=R$, $R_f=R$

② If $y=\cosh(x) \Rightarrow x=\cosh^{-1}(y)$ حيث $D_f=[1,\infty]$, $R_f=[0,\infty)$

③ If $y=\tanh(x) \Rightarrow x=\tanh^{-1}(y)$ حيث $D_f=[-1,1)$, $R_f=R$

④ If $y=\coth(x) \Rightarrow x=\coth^{-1}(y)$ حيث $D_f=R/[-1,2]$, $R_f=R/\{0\}$

⑤ If $y=\text{sech}(x) \Rightarrow x=\text{sech}^{-1}(y)$ حيث $D_f=(0,1]$, $R_f=R$

⑥ If $y=\text{csch}(x) \Rightarrow x=\text{csch}^{-1}(y)$ حيث $D_f=R/\{0\}$, $R_f=R/\{0\}$

Relations Between Functions : العلاقة بين الدوال *

① $\sinh^{-1}(x)=\ln(x+\sqrt{x^2+1})$

② $\cosh^{-1}(x)=\ln(x+\sqrt{x^2-1})$

③ $\tanh^{-1}(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $|x|<1$

④ $\coth^{-1}(x)=\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$, $|x|>1$

⑤ $\operatorname{sech}^{-1}(x) = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \leq 1$

⑥ $\operatorname{csch}^{-1}(x) = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right), \quad \forall x \in \mathbb{R}/\{0\}$

* برهن أنّ $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$

الحل:

نفرض أن $y = \sinh^{-1}(x)$

$\therefore x = \sinh(y) \Rightarrow x = \dfrac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - e^{-y}$

$\Rightarrow e^y - 2x - e^{-y} = 0 \qquad * e^y$

$\therefore e^{2y} - 2xe^y - 1 = 0$ هذه المعادلة يمكن ان تحل بالدستور

$e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2} \Rightarrow e^y = \dfrac{\cancel{2}(x \pm \sqrt{x^2+1})}{\cancel{2}}$

$\therefore e^y = x \pm \sqrt{x^2+1}$

$\because e^y > 0 \Rightarrow e^y = x + \sqrt{x^2+1})$ بأخذ ln للطرفين

$\ln e^y = \ln(x + \sqrt{x^2+1})$

$\therefore y = \ln(x + \sqrt{x^2+1})$

$\therefore \sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$

* برهن أن $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

<u>الحل</u>

نفرض أن

$$y = \tanh^{-1}(x) \Rightarrow x = \tan(y) \Rightarrow x = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$$

$$\Rightarrow e^{y} - e^{-y} = x e^{y} + x e^{-y}$$

$$\Rightarrow e^{y} - e^{-y} - x e^{y} - x e^{-y} = 0$$

$$\Rightarrow (1-x) e^{y} - (1+x) e^{-y} = 0 \qquad * \; e^{y}$$

$$\Rightarrow (1-x) e^{2y} - (1+x) = 0$$

$$\therefore e^{2y} = \frac{1+x}{1-x}$$

بأخذ $\ln$ للطرفين

$$\ln e^{2y} = \ln\left(\frac{1+x}{1-x}\right)$$

$$\therefore 2y = \ln\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\therefore \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

برهن أن

$$\coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$\boxed{\text{H.W.}}$