

University of Basrah
College of Engineering
Department of Mechanical Engineering

Advanced Vibration Lecture-5 and 6

Dr. Jaafrar Khalaf

Eigenvalue problems and Matlab

For N-DOF system represented by:

$$[M]\{\ddot{x}\} + [K]\{x\} = 0 \quad \dots(35)$$

When the response is harmonic $\{x\} = \{X\} \sin \omega t$, then:

$$[K]\{X\} = \lambda [M]\{X\} \quad \dots(36)$$

Equation (36) is called generalized eigenvalue problem. Pre-multiplying both sides of eq. (36) by $[M]^{-1}$ results in:

$$[M]^{-1}[K]\{X\} = \lambda [I]\{X\} \quad \dots(37)$$

Equation (37) can be re-written as:

$$(\mathbf{D} - \lambda \mathbf{I}) \mathbf{X} = 0 \quad \dots(38)$$

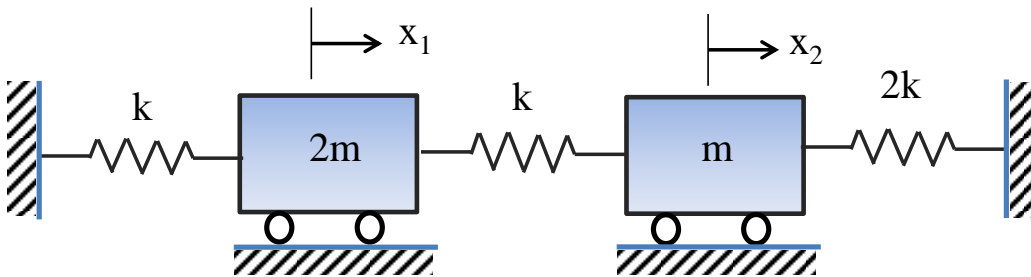
Where \mathbf{D} is the system dynamic matrix $\mathbf{D} = \mathbf{M}^{-1}\mathbf{K}$. Equation (38) is standard Eigenvalue problem. For N-DOF system, eq. (38) can be solved by LR or QR methods which provide stability of the calculations are perhaps the best known methods for Eigensolution.

Matlab ready function `eig(K,M)` or `eig(D)` can be used to obtain Eigenvalues and Eigenvectors from known mass and stiffness matrices. Where `eig(K,M)` can be used to directly evaluate Eigensolution for eq. (36) and `eig(D)` can be used for standard Eigenvalue problem as given in eq. (38).

Note: Equation (38) sometimes named as characteristics equation of matrix **D**. In general, if we have two square matrices **A** and **B** then both **AB** and **BA** have the same characteristic equation.

Example-7

For the system shown in Figure below, find the Eigenvalues and Eigenvectors using Matlab:



Solution:

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} = m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix} = k \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

Assuming harmonic motion:

$$k \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \{X\} = \lambda m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \{X\} \quad \text{where } \lambda = \omega^2$$

Using $\mu = \lambda \frac{m}{k}$, the above equation can be written as:

$$\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \{X\} = \mu \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \{X\}$$

The above equation can be cast into Eigenvalue problem. Using Matlab code:

```
K = [2 -1; -1 3];
M = [2 0; 0 1];
mu = eig(K,M); %calculate eigenvalues only
[X,EV] = eig(K,M); %calculate eigenvectors and eigenvalues
```

Note: Eigenvectors provided by Matlab are mass-normalized mode shapes.

Ans: $\mu = \{0.7753; 3.2247\}$

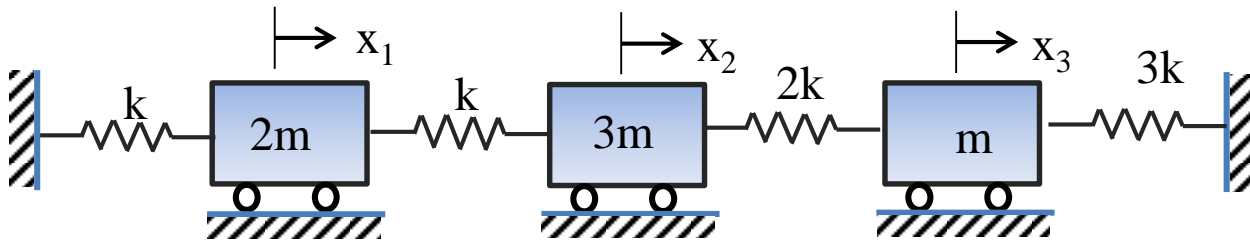
X =

$$\begin{bmatrix} -0.9121 & 0.2193 \\ -0.4100 & -0.9757 \end{bmatrix}$$

EV =

$$\begin{bmatrix} 0.7753 & 0 \\ 0 & 3.2247 \end{bmatrix}$$

Example-8:



Solution:

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 3k & -2k \\ 0 & -2k & 5k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 2m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & m \end{bmatrix} = m \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2k & -k & 0 \\ -k & 3k & -2k \\ 0 & -2k & 5k \end{bmatrix} = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

The above equation can be cast into Eigenvalue problem. Using Matlab code:

```
K = [2 -1 0; -1 3 -2; 0 -2 5];
M = [2 0 0; 0 3 0; 0 0 1];
mu = eig(K,M); %calculate eigenvalues only
[X,EV] = eig(K,M); %calculate eigenvectors and eigenvalues
```

Ans:

mu =
0.4210
1.2670
5.3120

X =
-0.3991 -0.5835 0.0175
-0.4621 0.3116 -0.1506
-0.2019 0.1669 0.9651

EV =
0.4210 0 0
0 1.2670 0
0 0 5.3120

Damped MDOF System

When damping is negligible the above treatment can be used. However, when damping exists, special treatment must be used.

In the undamped MDOF systems, we can apply eq. (9) $\Psi^T \mathbf{M} \Psi = [m_r]$ to uncouple mass matrix, or we can use eq. (10) $\Psi^T \mathbf{K} \Psi = [k_r]$ to uncouple the coupled stiffness matrix. Hence, the equation of motion can be uncoupled as a whole.

In damped system, we cannot uncouple equation of motion generally. However, there is an exception for that problem. When the damping is *proportional* or so called *Classical*, then equation of motion can be uncoupled using the mode shape matrix extracted from the same system assuming zero damping.

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0 \quad \dots(39)$$

If damping is proportional:

$$[C] = \alpha[M] + \beta[K] \quad \dots(40)$$

Or when the damping matrix generally satisfies the following identity (classical damping):

$$\begin{aligned}
[K][M]^{-1}[C] &= [C][M]^{-1}[K] \\
\text{or: } [M][K]^{-1}[C] &= [C][K]^{-1}[M] \quad \dots(41) \\
\text{or: } [M][C]^{-1}[K] &= [K][C]^{-1}[M]
\end{aligned}$$

Then the undamped mode shapes obtained by assuming zero damping, can be used to diagonalize damping matrix also:

$$\Psi^T [M] \Psi \{\ddot{x}\} + \Psi^T [C] \Psi \{\dot{x}\} + \Psi^T [K] \Psi \{x\} = \mathbf{0} \quad \dots(42)$$

$$[\cdot m_r.] \{\ddot{x}\} + [\cdot c_r.] \{\dot{x}\} + [\cdot k_r.] \{x\} \quad \dots(43)$$

Where $[\cdot c_r.]$ is the modal damping matrix or generalized damping matrix. In fact, when damping is classical, then Ψ is a planar matrix and can diagonalize damping as well as mass and stiffness matrices. Consequently:

$$\begin{aligned}
\omega_{d,r} &= \omega_r \sqrt{1 - \zeta_r^2} \\
\zeta_r &= \frac{c_r}{2m_r \omega_r} = \frac{\alpha}{2\omega_r} + \frac{\beta \omega_r}{2} \quad \dots(44)
\end{aligned}$$

Where ζ_r is the modal damping coefficient of the r^{th} mode. To evaluate the response of the system to force, the following equation can be used to evaluate the elements of the receptance matrix (to replace eq. (28)):

$$\alpha_{ij}(\omega) = \sum_{r=1}^N \frac{\phi_{ir} \phi_{jr}}{\omega_r^2 - \omega^2 + 2j\zeta_r \omega \omega_r} \quad \dots(45)$$

The same procedure used to derive eq. (28) can be used to derive eq. (45) provided that $\Phi^T [\cdot c_r.] \Phi = [2\omega_r \zeta_r.]$.

The uncoupled equation of motion in case of classical damping is given by:

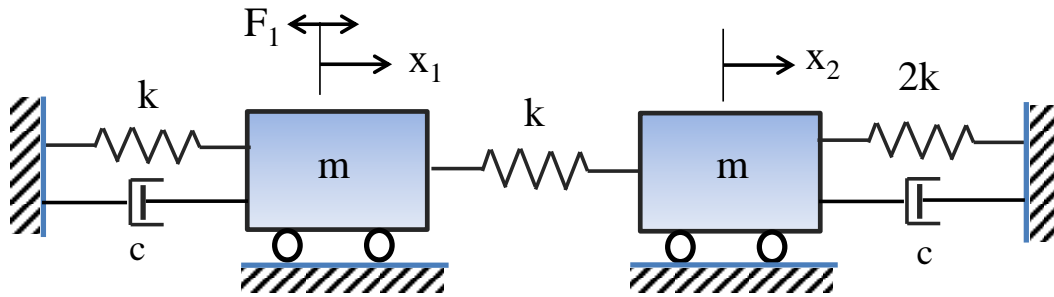
$$m_n \ddot{q}_n(t) + c_n \dot{q}_n(t) + k_n q_n(t) = \boldsymbol{\psi}_n^T \mathbf{f}(t) \quad (46)$$

For $n = 1, 2 \dots N$.

Once eq. (46) is solved for each mode, then eq. (30) can be used to find the response.

Example-9:

For the system shown below, determine modal damping matrix and find the response to the force shown:



Solution:

$$m\ddot{x}_1 + c\dot{x}_1 + kx_1 + k(x_1 - x_2) = 0$$

$$m\ddot{x}_2 + c\dot{x}_2 + 2kx_2 + k(x_2 - x_1) = 0$$

In matrix form

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is the same system of Example-2 but with added damping

It is clear that damping is proportional and the modal damping matrix is:

$$\begin{aligned} \boldsymbol{\psi}^T \mathbf{C} \boldsymbol{\psi} &= \begin{bmatrix} -0.618 & 1 \\ 1.618 & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} -0.618 & 1.618 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1.382c & 0 \\ 0 & 3.618c \end{bmatrix} \end{aligned}$$

$$\zeta_r = \frac{c_r}{2m_r\omega_r}$$

From Example-2: $\omega_1 = \sqrt{3.618\left(\frac{k}{m}\right)}$ and $\omega_2 = \sqrt{1.382\left(\frac{k}{m}\right)}$

so far: $\zeta_1 = \frac{1.382c}{2 \times 1.382m \sqrt{3.618\frac{k}{m}}} = \frac{c}{2\sqrt{3.618km}}$

and $\zeta_2 = \frac{3.618c}{2 \times 3.618m \sqrt{1.382\frac{k}{m}}} = \frac{c}{2\sqrt{1.382km}}$

Since $F_2 = 0$, then we have:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} \alpha_{11}(\omega) & \alpha_{12}(\omega) \\ \alpha_{21}(\omega) & \alpha_{22}(\omega) \end{bmatrix} \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

We need to find only $\alpha_{11}(\omega)$ and $\alpha_{21}(\omega)$

Since damping is proportional, the mode shapes are same as undamped systems which are evaluated in example-4 as:

$$\{\phi\}_1 = \frac{1}{\sqrt{1.382m}} \begin{Bmatrix} -0.618 \\ 1 \end{Bmatrix} \text{ and } \{\phi\}_2 = \frac{1}{\sqrt{3.618m}} \begin{Bmatrix} 1.618 \\ 1 \end{Bmatrix}$$

Using eq. (37):

$$\alpha_{11}(\omega) = \frac{\phi_{11}\phi_{11}}{\omega_1^2 - \omega^2 + 2j\omega\zeta_1\omega_1} + \frac{\phi_{12}\phi_{12}}{\omega_2^2 - \omega^2 + 2j\omega\zeta_2\omega_2}$$

$$= \frac{1}{1.382m} \frac{(-0.618)^2}{(\omega_1^2 - \omega^2 + 2j\omega\zeta_1\omega_1)} + \frac{1}{3.618m} \frac{(1.618)^2}{(\omega_2^2 - \omega^2 + 2j\omega\zeta_2\omega_2)}$$

$$\alpha_{21}(\omega) = \frac{\phi_{21}\phi_{11}}{\omega_1^2 - \omega^2 + 2j\omega\zeta_1\omega_1} + \frac{\phi_{22}\phi_{12}}{\omega_2^2 - \omega^2 + 2j\omega\zeta_2\omega_2}$$

$$= \frac{1}{1.382m} \frac{(1)(-0.618)}{(\omega_1^2 - \omega^2 + 2j\omega\zeta_1\omega_1)} + \frac{1}{3.618m} \frac{(1)(1.618)}{(\omega_2^2 - \omega^2 + 2j\omega\zeta_2\omega_2)}$$

General Damping Case

Generally the solution of eq. (39) (damped MDOS) is given by:

$$\{x\} = \{X\} e^{st} \quad \dots(47)$$

Where s is Laplace operator. Assuming zero initial conditions, eq. (39) becomes:

$$(s^2[M] + s[C] + [K])\{X\} = 0 \quad \dots(48)$$

The Eigenvalues and Eigenvectors of eq. (48) can be found by using state-space

$$\{y\} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \quad \dots(49)$$

Hence, equation of motion can be written in terms of state vector y as follows:

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \{y\} = \{0\}$$

$$\text{or : } [A]\{\dot{y}\} + [B]\{y\} = \{0\} \quad \dots(50)$$

Where both A and B are square matrix with dimension $2n \times 2n$. Equation (50) is a standard Eigenvalue problem. Since $\{\dot{y}\} = \lambda\{y\}$:

$$[B]\{Y\} = -\lambda[A]\{Y\} \quad \dots(51)$$

Here we must remember that for damped SDOF, the characteristics equation has a complex conjugate root $\lambda_{1,2} = -\zeta\omega_n \mp j\omega_n\sqrt{1-\zeta^2} = \sigma \mp j\omega_d$

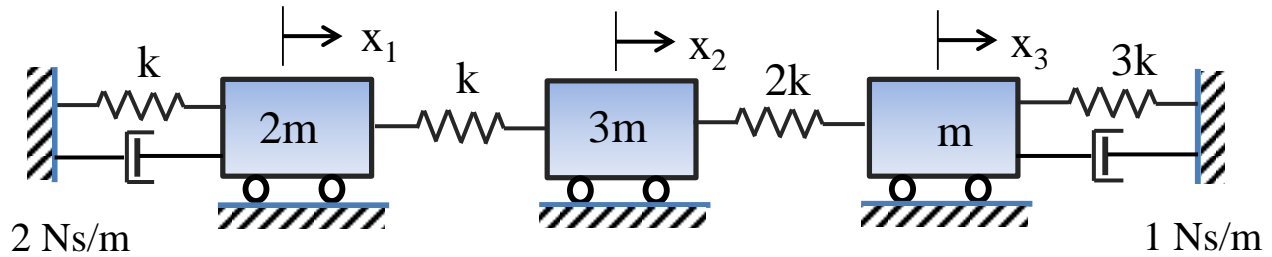
Therefore, the characteristics equation of (51) has $2N$ Eigenvalues in complex conjugate form. Also, it has $2N$ corresponding Eigenvectors that are complex conjugate.

Notes:

1. The eigenvalues and eigenvectors can be found using Matlab function $\text{eig}(B,A)$
2. The last n rows of eigenvectors are normally discarded since they are associated with \dot{x}
3. When damping matrix is proportional, eigenvectors are either in-phase or out of phase. Hence, when they are normalized by a certain column element, the result will be real eigenvectors. In this case, equation of motion can be decoupled using eigenvectors. Also, every point of the system undergoes harmonic motion and passes through the equilibrium position simultaneously.
4. When damping matrix is not proportional, eigenvectors will have phase angles from 0 to 180° between them. In this case, the points do not pass through the equilibrium position at the same time and there will be phase shift between them depending on the complexity of the system.

Example-10:

For the system shown below, determine eigenvalues, natural frequencies and mode shapes:



Given that: $m = 1 \text{ kg}$, $k = 100 \text{ N/m}$

Solution:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 200 & -100 & 0 \\ -100 & 300 & -200 \\ 0 & -200 & 500 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 200 & -100 & 0 \\ -100 & 300 & -200 \\ 0 & -200 & 500 \end{bmatrix}$$

The above equation can be cast into Eigenvalue problem. Using Matlab code:

```
K = [200 -100 0; -100 300 -200; 0 -200 500];
M = [2 0 0; 0 3 0; 0 0 1];
C = [2 0 0; 0 0 0; 0 0 1];
Z=[0 0 0; 0 0 0; 0 0 0];

A=[C M; M Z];
B=[K Z; Z -M];
lambda=-eig(B,A)
[Yd,EV] = eig(B,A); %calculate eigenvectors and eigenvalues
Xd = 100*Yd(1:3,:)
```

Ans:

```
lambda =
-0.4660 -23.0415i
-0.4660 +23.0415i
-0.3541 -11.2384i
-0.3541 +11.2384i
-0.1799 - 6.4932i
-0.1799 + 6.4932i
```

Xd =

Columns 1 through 4

$$\begin{array}{cccc} -0.0071 + 0.0762i & -0.0071 - 0.0762i & 0.9600 + 7.7249i & 0.9600 - 7.7249i \\ 0.0639 - 0.6565i & 0.0639 + 0.6565i & -0.0029 - 4.1619i & -0.0029 + 4.1619i \\ -0.1975 + 4.2240i & -0.1975 - 4.2240i & 0.0180 - 2.2286i & 0.0180 + 2.2286i \end{array}$$

Columns 5 through 6

$$\begin{array}{cc} 0.4508 + 11.7900i & 0.4508 - 11.7900i \\ 1.5005 + 13.5659i & 1.5005 - 13.5659i \\ 0.6018 + 5.9335i & 0.6018 - 5.9335i \end{array}$$

So far,

$$-\zeta_1 \omega_1 + j \omega_1 \sqrt{1 - \zeta_1^2} = -0.1799 + j6.4932$$

$$\zeta_1 \omega_1 = 0.1799$$

$$\omega_1 \sqrt{1 - \zeta_1^2} = 6.4932$$

$$\therefore \omega_1 = \sqrt{(6.4932)^2 + (0.1799)^2} = 6.496$$

$$\therefore \zeta_1 = \frac{0.1799}{6.496} = 0.02769$$

Similarly, we can show that:

$$\omega_2 = 11.244, \zeta_2 = 0.03149$$

$$\omega_3 = 23.0462, \zeta_3 = 0.02022$$

Response of Generally Damped MDOF System

We have previously discussed how to calculate the response of harmonic excitation for proportional damping case. Now we will discuss how to find the response for any damping case.

For free Vibration:

$$x_i = \sum_{r=1}^{2n} c_r \psi_{ir} e^{\lambda_r t} \quad \dots(52)$$

Here, ψ_{ir} represents the mode shape i -th element of r -th column and c_r depends on initial conditions.

For forced vibration, Laplace transform method can be used:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \quad \dots(53)$$

Using Laplace transform with zero initial conditions:

$$(s^2[M] + s[C] + [K])\{X\} = \{F\} \quad \dots(54)$$

Hence:

$$\begin{aligned} \{X(s)\} &= (s^2[M] + s[C] + [K])^{-1} \{F(s)\} \\ &= \{\alpha(s)\} \{F(s)\} \end{aligned} \quad \dots(55)$$

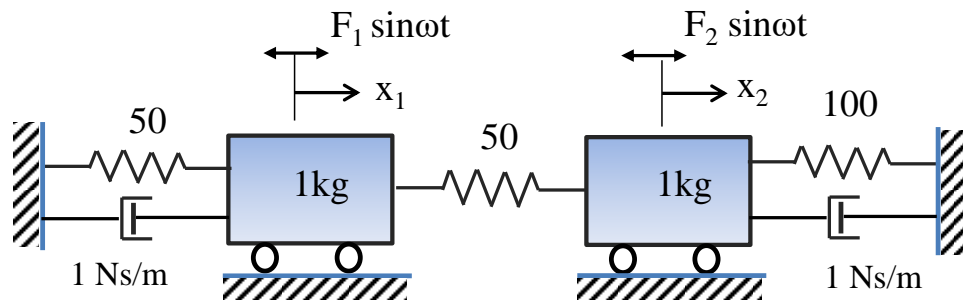
With:

$$\{\alpha(s)\} = (s^2[M] + s[C] + [K])^{-1} = \frac{adj(s^2[M] + s[C] + [K])}{\det(s^2[M] + s[C] + [K])} \quad \dots(56)$$

Once $\{X(s)\}$ are evaluated, the inverse Laplace transform can be used to find the response $\{x(t)\}$.

H.W:

Q1: for the system shown below with proportional damping,

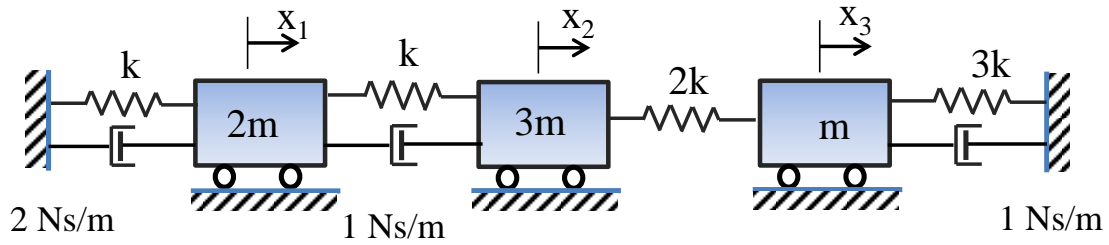


Find:

- (a) Natural frequencies and mode shapes
- (b) Mass-normalized mode shapes

- (c) Modal damping matrix
- (d) Responses in terms of the applied forces

Q2: for the following system,



$$m=1\text{kg}, k=60\text{ N/m}$$

Determine:

- (a) Equation of motion, mass, stiffness and damping matrices
- (b) Eigenvalues and natural frequencies (using Matlab)
- (c) Mode shapes