University of Basrah

College of Engineering

Department of Mechanical Engineering

Advanced Vibration Lecture-8

Dr. Jaafrar Khalaf

Vibration in Rotating Machines

1. Introduction and Definitions

Rotordynamics is the field of science that studies the response of rotating parts subjected to dynamic forces. This field is more than 160 year old and it is acquiring more attention day by day. Rotating parts are playing important role in our modern life starting from small domestic equipment up to the spacecrafts and submersible ships. A rotor is a body suspended through a set of cylindrical hinges or bearings that allow it to rotate freely about an axis fixed in space. Engineering components concerned with the subject of rotor dynamics are rotors in machines, especially of turbines, generators, motors, compressors, blowers and the like. The parts of the machine that do not rotate are referred to with general definition of stator. Rotors of machines have, while in operation, a great rotational energy and small vibrational energy. Rotational energy can be very large such that a relatively small turbine propels a huge aircraft a high speeds. The Rotordynamics as a subject aims to provide the following benefits:

1. Reduce vibrational energy as much as possible
2. Prevent rapid failure due to whirling at critical speeds
3. Obtain smooth operation without noise
4. Extend machine parts life by proper design
5. Reduce experimental works and prototyping costs

The above goals can be achieved through:

1. Evaluating the critical speeds and mode shapes of the rotor

2. Obtaining Frequency Response Function (FRF) and other parameters for the system
3. Study the effect of dynamic loads such as unbalance, misalignment, eccentricity and gyroscopic effect on the response
4. Study the effect of bearings stiffness, viscous and structural damping on the overall performance
5. Interpreting the effect of power disks such as blades, impellers, fans, pulleys and gears on the torsional and transverse vibration

Rotordynamics is not limited to design stage; it can provide useful tools for testing and diagnosis of machine fault during actual operation, thus, motivating preventive and predictive maintenance.

Whirling can be defined as the rotation of the plane generated by the bent shaft and the centerline connecting the bearings. Rotors tend to bow out due to various effects such as unbalance, hysteresis damping in the shaft, gyroscopic effects and fluids in contact with the rotor. At certain spin speeds, known as Critical Speeds, the deflection become very large if there is no sufficient damping and may cause catastrophic failure. Whirling is the main cause of machinery breakdown especially if the machine is left to rotate at a critical speed for long period.

2. Historical Introduction

The industrial revolution began with reciprocating steam engines as devised by James Watt in 1780, and the 19th century witnessed a rapid expansion in various industrial sectors. Unfortunately, the reciprocating steam engine had several problems because of external combustion and excessive alternating load due to reciprocating masses that limited speeds and capacities. The industry was looking for non-reciprocating systems, purely rotating systems that could usher in an era of so called “Vibration Free” engines. The dynamics of rotating structures are different from those of stationary structures. Basically, all the vibration phenomena will be valid, however, there are several differences and we have to set up new procedures for handling rotors and their vibratory phenomena.

There were practically no known attempts to understand vibrations of a rotating structure or a rotor for over a century after beam theory was well understood and expanded to other structures, e.g., Plates. Rankine (1820–1872) made significant contributions to Thermodynamics, particularly Steam Engines and his publication in 1859 was the first attempt at a practical approach to steam-engine theory. The

Rankine cycle is a thermodynamic sequence of events and is still used as a standard for rating steam power plant performance. Ten years later in 1869, the first attempts were made to understand Rotor Dynamics when Rankine performed the first analysis of a spinning shaft. He studied centrifugal whirling of shafts and critical speeds. He chose an unfortunate model and predicted that; beyond a certain spin speed ". . . the shaft is considerably bent and whirls around in this bent form." He defined this certain speed as the "whirling speed" of the shaft. In fact, it can be shown that beyond this whirling speed the radial deflection of Rankine's model increases without limit. Rankine did add the term "whirling" to the rotor dynamics vocabulary. Some believe now that he may have been responsible for setting back the science of rotor dynamics by nearly 50 years.

Though the basics of rotor dynamics were not yet fully understood, Laval built the first impulse turbine in 1883 which ran successfully at 40000 RPM! From simple equilibrium conditions, he derived a correct relation for the whirl radius based on single-degree of freedom model. Rayleigh, using the energy principle, provided an approximate (upper bound) method to determine the first critical speed of a rotor considering it as a stationary beam. Dunkerley derived an empirical relation for estimating the lower bound value of critical speed. The demonstration that a shaft can have several critical speeds is more than a century behind Lagrange's work for stationary systems. Stodola presented a graphical method to determine the critical speeds of practical rotors. This method continued to be widely used for over five decades until transfer matrix methods and digital computers became available.

Some of the earlier rotor failures belong to propeller shafts in torsion of steam driven war ships during I world war. The story goes thus: When a propeller shaft failed, it was felt that designers did not provide sufficient diameter of the shaft to take care of the transmitted torque, therefore its diameter was increased by 10%. The modified shaft however failed in less than half time of the previous shaft failure. Then the designers began taking rotor failures seriously to adopt dynamic design. It was found that by increasing the diameter of the shaft the natural frequency became closer to the excitation harmonic resulting in an earlier failure. Holzer presented a tabular method to determine the torsional natural frequencies of systems, which can be discretized in the form of several rigid inertias, connected by massless torsional springs. This is a simple method in which the inertia torque and torsional amplitude of each disk are calculated sequentially beginning from one end with amplitude equal to unity and arbitrarily chosen frequency. After completing the calculations till the end of the train, the boundary condition was checked – in this case the total inertia of the system in free vibration to be equal to zero.

3. Rotordynamics

Whirling is one of the most frequent causes of rotating machinery damage. Improper design and/or variation of system parameters may lead to rapid failures when machines are operated close to a critical frequency. For example, a dredger machine, shown in Fig. 1, suffered from repeated fracture of the shaft of the jet pump due to bending fatigue as shown as determined by fracto-graphical examination. Design changes to the shaft, relating to material strength, radii and thread geometry to fit the impeller did not restore the desired reliability. The metallographic examination did not reveal any initial material or manufacturing defect, so the cause of the failure must be related to stress in excess of the fatigue limit of the material. These stresses had to be related to specific operating conditions, as the stress according the engineering calculations were low and should not result in fatigue

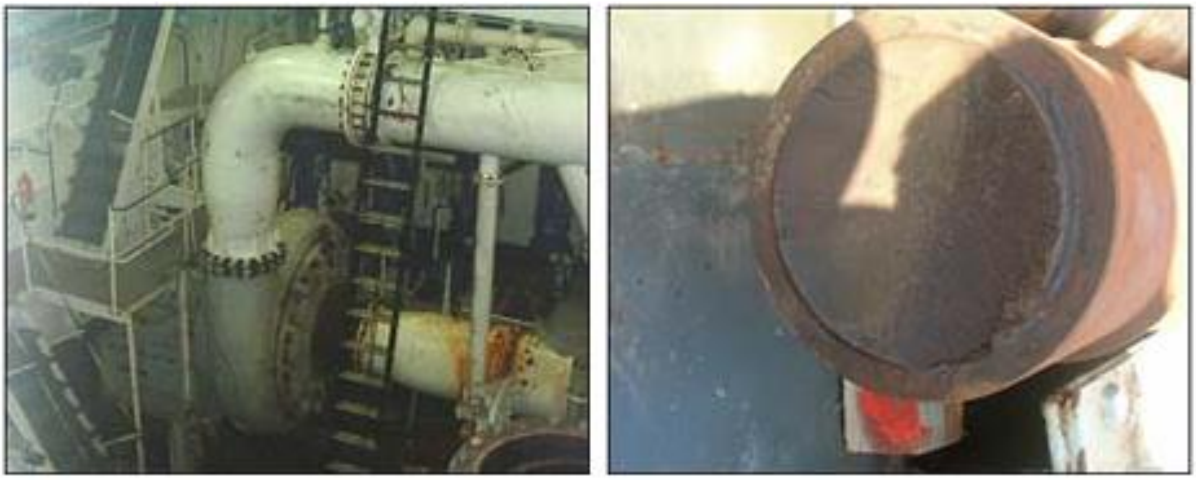


Figure 1 Dredger Pump (left) and its broken power shaft (right)

Three types of movements are considered in rotor dynamics; transverse, torsional and axial. They are depicted in Fig. 2.

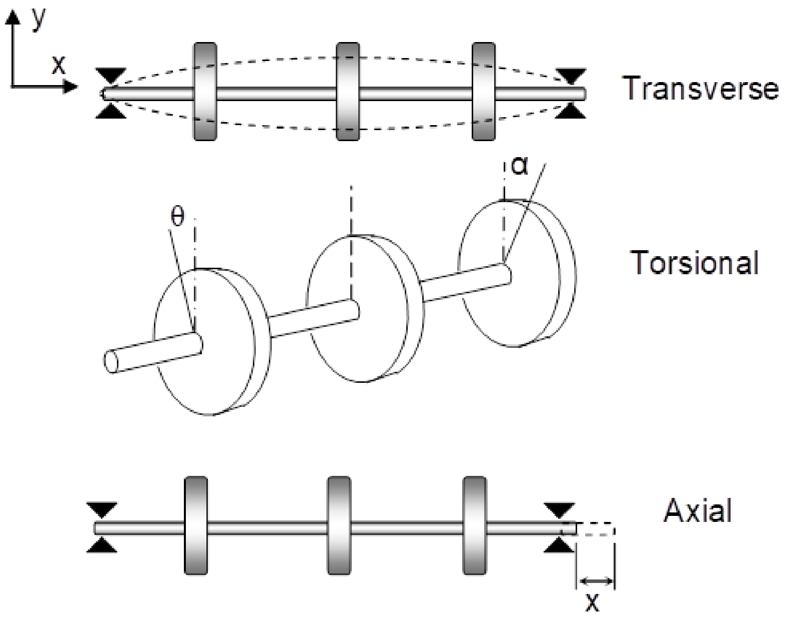


Figure 2 Types of Motions in Rotordynamics

4. Transverse Motion of Rotors

4.1 Simple Model (no damping)

This motion is the most considered in rotor dynamics. Whirling of rotors is mostly related to this type of motion. One of the earliest models used to represent this motion was that proposed by De Laval given by:

$$y = \frac{\omega^2 \delta}{\frac{k}{M} - \omega^2} \qquad (1)$$

Where *y* is whirling radius, δ is the eccentricity, *k* is the effective stiffness, *M* is rotor mass (kg) and ω is angular speed of rotation. In this equation, y approaches infinity as the speed approaches natural frequency (critical speed) of the system, i.e. when the denominator approaches zero. When the rotation speed becomes higher than critical speed, *y* decreases and it approaches δ when the rotation speed is much higher than critical speed. Fig. 3 illustrate a shaft undergoes whirling; in (a), a whirling shaft is shown in its deflected configuration, and in (b), the situation occurring in x-y plane is sketched. As the spin speed is equal to the whirl speed, the zone of the cross section of the shaft subjected to tensile stresses (shaded part close to point B) remains always under tensile loading, whereas that subjected to compression is always compressed.

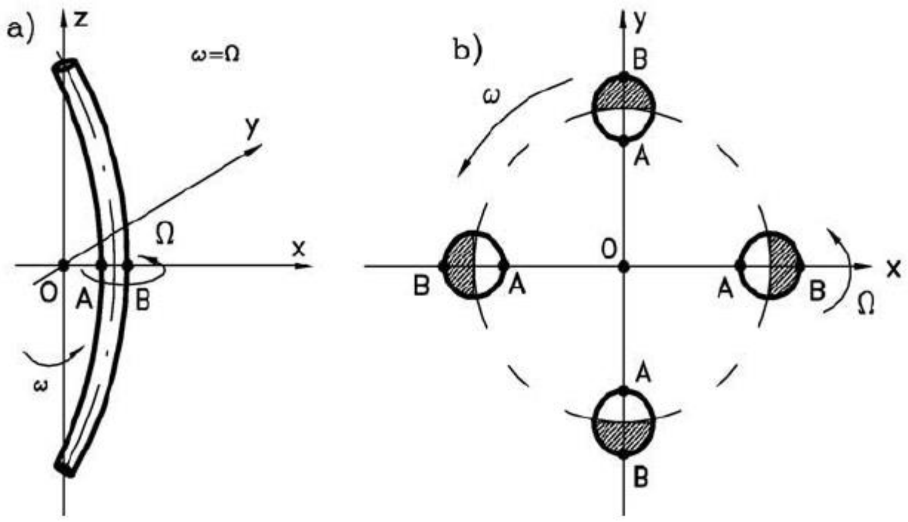


Figure 3 Illustration of Whirling Phenomenon

4.2 Damped Motion

Equation (1) neglects damping effect which can reduce amplitude of whirling even when spin speed approaches critical speed. If damping is considered, the equation would be

$$y = \frac{\delta\omega^2}{\sqrt{\left(\frac{k}{M} - \omega^2\right)^2 + \left(\frac{c\omega}{M}\right)^2}} \sin(\omega t - \phi)$$
$$where\ \phi = \tan^{-1}\frac{\frac{c\omega}{M}}{\frac{k}{M} - \omega^2} \qquad (2)$$

All the above equations assume the mass of shaft is concentrated at its geometrical centerline. Or alternatively, a lumped weight of mass M attached to a massless shaft as shown in Fig. 4.

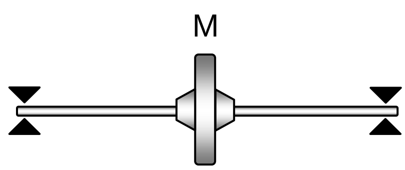


Figure 4 Simple Rotor Model

4.3 Continuous System

Assuming a shaft of mass per unit length of *m*, modulus of elasticity *E*, second moment of area about centerline of *I* and length *L* without any weight attached, the following equations give the critical speeds in *Hz*;

(a) Simply Supported Shaft: like shaft supported by self-aligned bearings

$$f = n^2 \frac{\pi}{2} \sqrt{\frac{EI}{mL^4}} \qquad (3)$$

Where n is the mode shapes = 1, 2, 3, …

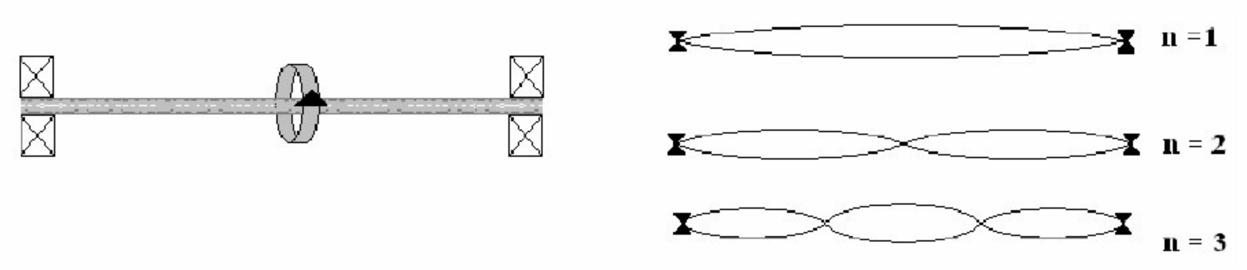


Figure (5) Simply supported shaft

(b) Fixed Ends: example, shaft supported by roller bearing or sleeve bearing

$$f = \left(n + \frac{1}{2}\right)^2 \frac{\pi}{2} \sqrt{\frac{EI}{mL^4}} \qquad (4)$$

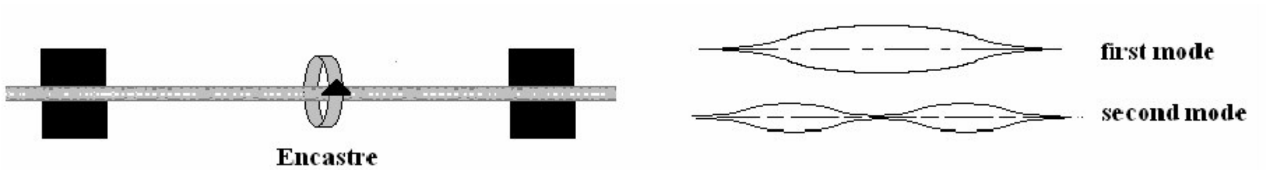


Figure (6) Fixed Ends supported shaft

(c) Cantilever

$$f = \left(n - \frac{1}{2}\right)^2 \frac{\pi}{2} \sqrt{\frac{EI}{mL^4}} \qquad (5)$$

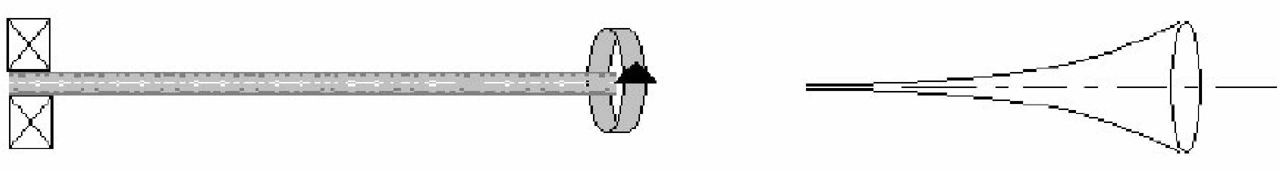


Figure (7) One End Fixed Shaft (Cantilever)

4.4 Dunkerley's Equation

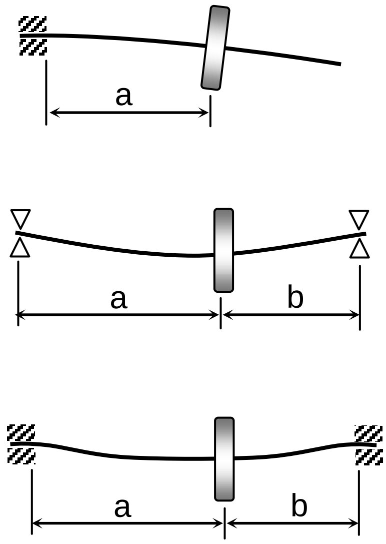
When weights (such as power disks) are attached to the shaft, the critical frequency of the complete system can be evaluated using Dunkerley's Equation:

$$\frac{1}{f_c^2} = \frac{1}{f_s^2} + \frac{1}{f_1^2} + \frac{1}{f_2^2} + \frac{1}{f_3^2} + \dots \quad (6)$$

Where f_c is the frequency of the complete systems, f_s: the frequency of shaft alone (from Eq. 3, 4 and 5 above), f_1, f_2, f_3, etc. are the frequencies corresponding to weights 1, 2, 3, etc. individually. For example, f_1 can be calculated as follows:

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_{eff,1}}{M_1}} \quad (7)$$

Where M_1 is mass of weight No. 1 and $k_{eff,1}$ is the effective stiffness at the location of it. Fig. 8 shows the effective stiffness at a point for cantilever, simply supported and fixed shafts.



$$k = \frac{3EI}{a^3}$$

$$k = \frac{3LEI}{a^2b^2}$$

$$k = \frac{3L^3EI}{a^3b^3}$$

Figure 8 Effective Stiffness for Different Types of Support

5. Gyroscopic Effect

To understand the gyroscopic effect, we will start with simple example consisting of a rotating disk subjected to angular precession. Let's also define our right-handed coordinate system, shown in Fig. 9, as follows: thumb point to *z*-axis as curl of fingers represents motion from *Ox* to *Oy*, thumb point to *y*-axis as fingers are curled from *Oz* to *Ox*, and thumb points to *x*-axis as fingers are curled from *Oy* to *Oz*.

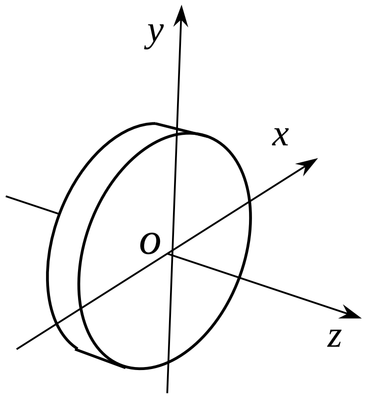


Figure 9

It is important to consider gyroscopic effects in the analysis of rotordynamics. These effects are introduced by the principle of conversation of angular momentum. Consider a disk of polar moment of inertia of I_p rotating about *z*-axis with angular speed Ω in the positive direction that comply with the above convention as shown in Fig. 10. If we view the disk from *z* towards *O*, such that *x*-axis is to the right, the disk is rotating *counterclockwise*.

The direction of the angular momentum $I_p \Omega$ is determined by the right hand rule which is toward *z*-axis in this case. When the disk is precessing around *y*-axis with angular speed $\dot{\psi}$ in the positive direction, the direction of the angular momentum is continuously changing producing gyroscopic moment M_x:

$$M_x = I_p \Omega \dot{\psi} \qquad (8)$$

Likewise, when the disk is precessing around *x*-axis with speed $\dot{\theta}$ in the positive direction, a moment M_y is introduced, but in the negative direction:

$$M_y = -I_p \Omega \dot{\theta} \qquad (9)$$

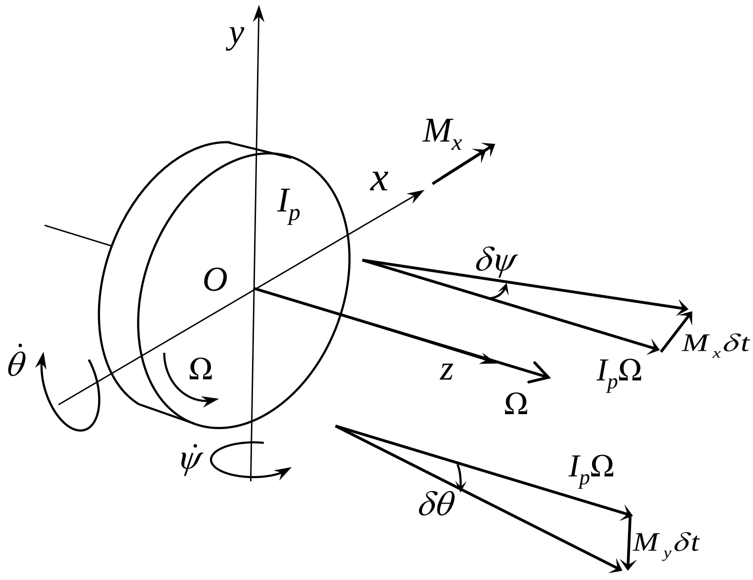


Figure 10

6. Simple Rigid Rotor

To illustrate the effect of gyroscopic couples on the dynamics of rotating structures, we will start with undamped simple rigid rotor on flexible support shown in Fig. 11. The supports have no angular stiffness, i.e. *short bearings*. The rotor has four degree of freedom, translational in *x* and *y* directions and rotation about the same axes. The translational motion is normally called *bounce* and rotational one is called *tilt*.

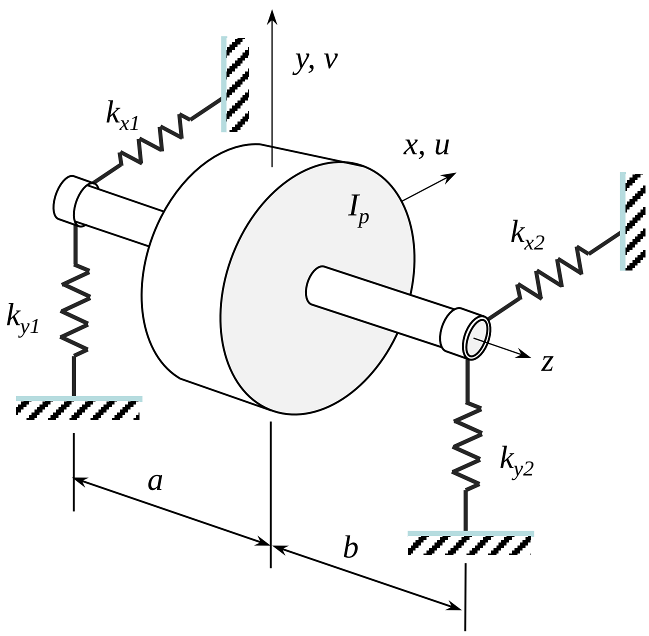


Figure 11

To develop equations of motion, Newton's second low can be applied. We will choose the center of the mass to describe the motion of the rotor.

$$
\begin{aligned}
&\text{Forces acting in } x\text{-direction}: && m\ddot{u} = -f_{x1} - f_{x2} \\
&\text{Forces acting in } y\text{-direction}: && m\ddot{v} = -f_{y1} - f_{y2} \\
&\text{Moments around } x\text{-axis}: && I_d\ddot{\theta} + I_p\Omega\dot{\psi} = -af_{y1} + bf_{y2} \\
&\text{Moments around } y\text{-axis}: && I_d\ddot{\psi} - I_p\Omega\dot{\theta} = af_{x1} - bf_{x2}
\end{aligned} \qquad (10)
$$

Where I_d is the diametral moment of inertia (moment of inertia about Ox or Oy axis).

Assuming the system is subjected to positive displacement and rotation angles and assuming small angles of rotation, forces in the springs can be written as:

$$
\begin{aligned}
f_{x1} &= k_{x1}(u - a\psi) \\
f_{x2} &= k_{x2}(u + b\psi) \\
f_{y1} &= k_{y1}(v + a\theta) \\
f_{y2} &= k_{y2}(v - b\theta)
\end{aligned} \qquad (11)
$$

Substituting eqs. (11) into (10) and re-arranging results in:

$$
\begin{aligned}
&m\ddot{u} + (k_{x1} + k_{x2})u + (-ak_{x1} + bk_{x2})\psi = 0 \\
&m\ddot{v} + (k_{y1} + k_{y2})u + (ak_{y1} - bk_{y2})\theta = 0 \\
&I_d\ddot{\theta} + I_p\Omega\dot{\psi} + (ak_{y1} - bk_{y2})v + (a^2k_{y1} + b^2k_{y2})\theta = 0 \\
&I_d\ddot{\psi} - I_p\Omega\dot{\theta} + (-ak_{x1} + bk_{x2})u + (a^2k_{x1} + b^2k_{x2})\psi = 0
\end{aligned} \qquad (12)
$$

Using:

$$
\begin{aligned}
&k_{Tx} = k_{x1} + k_{x2}, && k_{Ty} = k_{y1} + k_{y2} \\
&k_{Cx} = -ak_{x1} + bk_{x2}, && k_{Cy} = ak_{y1} - bk_{y2} \\
&k_{Rx} = a^2k_{x1} + b^2k_{x2}, && k_{Ry} = a^2k_{y1} + b^2k_{y2}
\end{aligned} \qquad (13)
$$

Equation (12) can be re-written as:

$$
\begin{aligned}
&m\ddot{u} + k_{Tx}u + k_{Cx}\psi = 0 \\
&m\ddot{v} + k_{Ty}v + k_{Cy}\theta = 0 \\
&I_d\ddot{\theta} + I_p\Omega\dot{\psi} + k_{Cy}v + k_{Ry}\theta = 0 \\
&I_d\ddot{\psi} - I_p\Omega\dot{\theta} + k_{Cx}u + k_{Rx}\psi = 0
\end{aligned} \qquad (14a)
$$

In matrix form:

$$\mathbf{M}\ddot{\mathbf{q}} + \Omega\mathbf{G}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0 \qquad (14b)$$

with:

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & 0 & 0 & I_d \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_p \\ 0 & 0 & -I_p & 0 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} k_{Tx} & 0 & 0 & k_{Cx} \\ 0 & k_{Ty} & k_{Cy} & 0 \\ 0 & k_{Cy} & k_{Ry} & 0 \\ k_{Cx} & 0 & 0 & k_{Rx} \end{bmatrix}$$

Where

$$\mathbf{q} = \begin{bmatrix} u \\ v \\ \theta \\ \psi \end{bmatrix}$$

It is clear that there is elastic coupling between first and fourth equation, as well as second and third equations. Also, there is gyroscopic coupling between third and fourth equations.

To further simplify the problem, we will assume isotropic bearings ($k_x = k_y$) in both bearings which results in ($k_{Tx} = k_{Ty} = k_T$) and ($k_{Rx} = k_{Ry} = k_R$), also we will assume symmetric rotor ($a = b$) which results in $k_{Cx} = k_{Cy} = 0$, hence:

$$\begin{aligned} & m\ddot{u} + k_T u = 0 \\ & m\ddot{v} + k_T v = 0 \\ & I_d\ddot{\theta} + I_p\Omega\dot{\psi} + k_R\theta = 0 \\ & I_d\ddot{\psi} - I_p\Omega\dot{\theta} + k_R\psi = 0 \end{aligned} \qquad (15)$$

Assuming harmonic responses:

$$u = u_0 e^{st},\ v = v_0 e^{st},\ \theta = \theta_0 e^{st},\ \psi = \psi_0 e^{st}$$

Equations (15) become:

$$\begin{aligned} & (ms^2 + k_T)u_0 = 0 \\ & (ms^2 + k_T)v_0 = 0 \\ & (I_d s^2 + k_R)\theta_0 + I_p\Omega\, s\,\psi_0 = 0 \\ & -I_p\Omega\, s\,\theta_0 + (I_d s^2 + k_R)\psi_0 = 0 \end{aligned} \qquad (16)$$

The first and second equations are uncoupled and can be solved directly to obtain:

$s^2 = -\dfrac{k_T}{m}$ twice from the first and second equations. So far:

$$s_1 = s_2 = j\sqrt{\frac{k_T}{m}} \quad and \quad s_5 = s_6 = -j\sqrt{\frac{k_T}{m}}$$

If the gyroscopic effect is neglected, such as when the spin speed is zero, the third and fourth equations become uncoupled and consequently the following roots are obtained (static structure):

$$s_3 = s_4 = j\sqrt{\frac{k_R}{I_d}} \quad and \quad s_7 = s_8 = -j\sqrt{\frac{k_R}{I_d}}$$

However, when the gyroscopic effect is included, the last two equations are coupled and can be solved by substituting one equation into another to obtain:

$$(I_d s^2 + k_R)^2 + (I_p \Omega s)^2 = 0 \qquad (17)$$

Moving the second term of eq. (17) to the right side:

$$(I_d s^2 + k_R)^2 = -(I_p \Omega s)^2 \Rightarrow I_d s^2 + k_R = \mp j I_p \Omega s$$

So

$$I_d s^2 \pm j I_p \Omega s + k_R = 0 \qquad (18)$$

Considering the negative sign of the second term and solving for *s* gives:

$$s = j\frac{I_p\Omega}{2I_d} \mp \sqrt{-\left(\frac{I_p\Omega}{2I_d}\right)^2 - \frac{k_R}{I_d}} = j\left(\frac{I_p\Omega}{2I_d} \mp \sqrt{\left(\frac{I_p\Omega}{2I_d}\right)^2 + \frac{k_R}{I_d}}\right)$$

Similarly, considering the positive sign will produce:

$$s = j\frac{I_p\Omega}{2I_d} \mp \sqrt{-\left(\frac{I_p\Omega}{2I_d}\right)^2 - \frac{k_R}{I_d}} = j\left(-\frac{I_p\Omega}{2I_d} \mp \sqrt{\left(\frac{I_p\Omega}{2I_d}\right)^2 + \frac{k_R}{I_d}}\right)$$

To comply with the results of neglected gyroscopic effect, and since the value of the square root is always larger than the first term, we will choose:

$$s_3 = j\left(-\frac{I_p\Omega}{2I_d}+\sqrt{\left(\frac{I_p\Omega}{2I_d}\right)^2+\frac{k_R}{I_d}}\right) \text{ and } s_4 = j\left(\frac{I_p\Omega}{2I_d}+\sqrt{\left(\frac{I_p\Omega}{2I_d}\right)^2+\frac{k_R}{I_d}}\right)$$

Therefore, $s_7 = -s_3$ and $s_8 = -s_4$

The natural frequencies are evaluated from the first two roots:

$$\omega_3 = -\frac{I_p\Omega}{2I_d}+\sqrt{\left(\frac{I_p\Omega}{2I_d}\right)^2+\frac{k_R}{I_d}} \text{ and } \omega_4 = \frac{I_p\Omega}{2I_d}+\sqrt{\left(\frac{I_p\Omega}{2I_d}\right)^2+\frac{k_R}{I_d}}$$

The natural frequencies are dependent on the spin speed Ω. As the spin speed tends to zero, the natural frequencies become identical and equal to that of static structure. As the spin speed increases, ω_3 decreases while ω_4 increases. The whirl speed map or Campbell diagram can be used to illustrate the dependence of natural frequency on spin speed as shown in Fig. 12. For the forward whirl, the gyroscopic effect contributes negative kinetic energy and tends to raise the stiffness apparently. While when the gyroscopic effect contributes positive energy, it tends to lower the natural frequency.

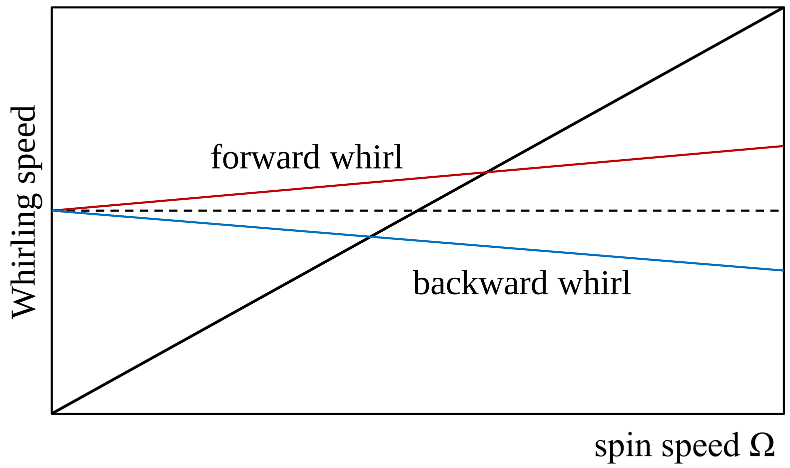


Figure 12 Whirl speed map

By rearranging the last two equations of (16);

$$\left(\frac{\theta_0}{\psi_0}\right)_i = -\frac{I_p\Omega s_i}{I_d s_i^2 + k_R} \quad \text{from third equation}$$

$$\left(\frac{\theta_0}{\psi_0}\right)_i = \frac{I_d s_i^2 + k_R}{I_p\Omega s_i} \quad \text{from fourth equation}$$

Multiplying the above two equations results in:

$$\left(\frac{\theta_0}{\psi_0}\right)_i^2 = -1 \;\; or \;\; \left(\frac{\theta_0}{\psi_0}\right)_i = \mp j \qquad (19)$$

Since the roots are complex conjugate $s_i = \mp j\omega_i$, then:

$$\left(\frac{\theta_0}{\psi_0}\right)_{i,i+4} = -\frac{\mp j\omega_i I_p \Omega}{k_R - \omega_i^2 I_d} = -j\,\text{sign}\left(\frac{\mp\omega_i I_p \Omega}{k_R - \omega_i^2 I_d}\right)$$

$$where: \left|\frac{\omega_i I_p \Omega}{k_R - \omega_i^2 I_d}\right| = 1$$

If $k_R > \omega_i^2 I_d$ (or $\omega_i^2 < k_R / I_d$) then:

$$\left(\frac{\theta_0}{\psi_0}\right)_i = \begin{cases} -j & s_i = j\omega_i \\ j & s_i = -j\omega_i \end{cases} \qquad (20)$$

If $k_R < \omega_i^2 I_d$ (or $\omega_i^2 > k_R / I_d$) then:

$$\left(\frac{\theta_0}{\psi_0}\right)_i = \begin{cases} j & s_i = j\omega_i \\ -j & s_i = -j\omega_i \end{cases} \qquad (21)$$

Considering ψ_0 is always 1, the responses can be evaluated from the mode shapes as follows:

$$\begin{Bmatrix} \theta(t) \\ \psi(t) \end{Bmatrix} = \begin{Bmatrix} -j \\ 1 \end{Bmatrix} e^{j\omega_i t} + \begin{Bmatrix} j \\ 1 \end{Bmatrix} e^{-j\omega_i t} = 2\begin{Bmatrix} \sin\omega_i t \\ \cos\omega_i t \end{Bmatrix} \text{ for } \omega_i^2 < k_R / I_d \qquad (22)$$

And

$$\begin{Bmatrix} \theta(t) \\ \psi(t) \end{Bmatrix} = \begin{Bmatrix} j \\ 1 \end{Bmatrix} e^{j\omega_i t} + \begin{Bmatrix} -j \\ 1 \end{Bmatrix} e^{-j\omega_i t} = 2\begin{Bmatrix} -\sin\omega_i t \\ \cos\omega_i t \end{Bmatrix} \text{ for } \omega_i^2 > k_R / I_d \qquad (23)$$

Equation (22) states that the orbit in the θ-ψ plane is circular and rotating clockwise as shown in Fig. 13(a), while the rotor is by convention rotating counterclockwise. This is *backward* mode. On the other hand, eq. (23) states that the orbit is rotating counterclockwise as shown in Fig. 13(b), so it is forward mode. In fact, the dashed line in Fig. 12 represent the static structure frequency line where $\omega_i^2 = k_R / I_d$, so when the natural frequency is less than static structure frequency, a backward mode is obtained, while when it is higher, a forward mode is obtained.

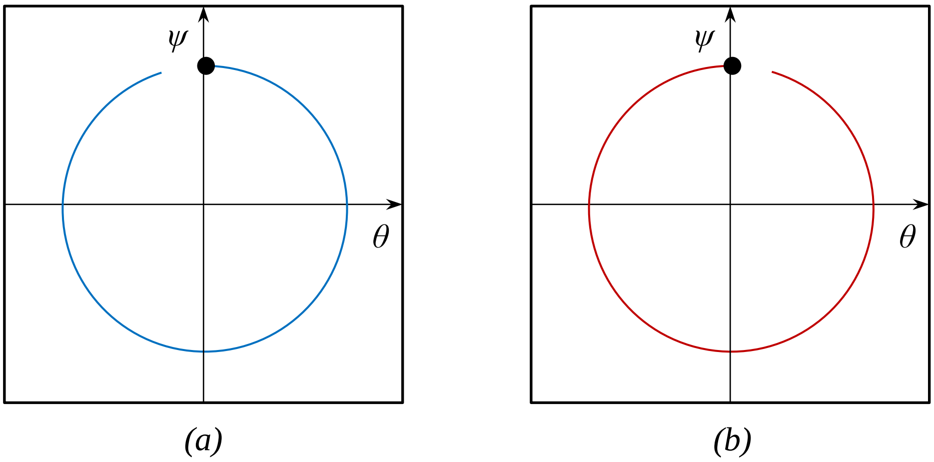


Figure 13 Backward and forward whirls

Example:

A uniform rotor of length 0.6 m and diameter 0.4 m is made of steel (density 7810 kg/m^3) is supported by identical short bearings of stiffness 1 MN/m in the horizontal and vertical directions. If the distance between the bearings is 0.7 m, determine the translational natural frequencies, plot whirl speed map and determine forward and backward whirl speeds.

Solution:

$$m = \rho\pi\frac{D^2}{4}L = 7810\pi\times\frac{(0.4)^2}{4}\times 0.6 = 588.8\,kg$$

$$I_p = m\frac{D^2}{8} = 588.8\times\frac{(0.4)^2}{8} = 11.77\,kg.m^2$$

$$I_d = m\frac{D^2}{16} + m\frac{L^2}{12} = 588.8\times\frac{(0.4)^2}{16} + 588.8\times\frac{(0.6)^2}{12} = 23.55\,kg.m^2$$

$$k_T = 1\times10^6 + 1\times10^6 = 2\times10^6\ N/m,$$

$$k_C = 0,$$

$$k_R = (0.35)^2\times1\times10^6 + (0.35)^2\times1\times10^6 = 245000\,N/m$$

Hence;

$$s_1 = s_2 = j\sqrt{\frac{k_T}{m}} = j\sqrt{\frac{2\times10^6}{588.8}} = j58.28$$

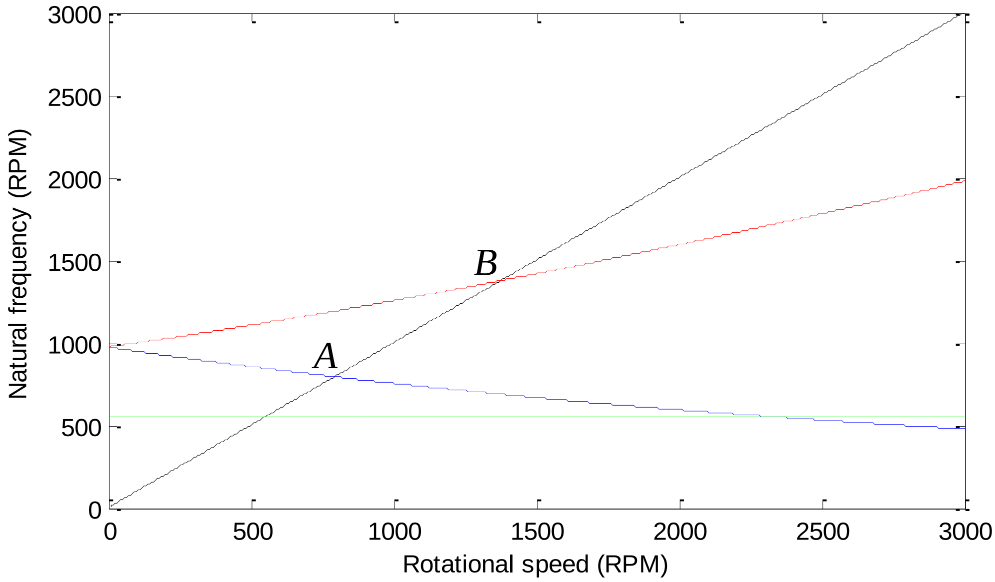
$$\therefore\ \omega_1 = \omega_2 = 58.28\ rad/s \Rightarrow N_{1,2} = 556.5\ RPM$$

$$\omega_3 = -\frac{I_p\Omega}{2I_d} + \sqrt{\left(\frac{I_p\Omega}{2I_d}\right)^2 + \frac{k_R}{I_d}} \text{ and } \omega_4 = \frac{I_p\Omega}{2I_d} + \sqrt{\left(\frac{I_p\Omega}{2I_d}\right)^2 + \frac{k_R}{I_d}}$$

$$\omega_3 = -\frac{11.77\Omega}{2\times 23.55} + \sqrt{\left(\frac{11.77\Omega}{2\times 23.55}\right)^2 + \frac{245000}{23.55}} = -0.2499\Omega + \sqrt{(0.2499\Omega)^2 + 10403.4}$$

$$\omega_4 = 0.2499\Omega + \sqrt{(0.2499\Omega)^2 + 10403.4}$$

With $\omega_3 = \omega_4 = 102\, rad/s$ when $\Omega = 0$.



Backward whirling speed at point A can be found by equating synchronous line with backward whirling line:

$$\Omega = -0.2499\Omega + \sqrt{(0.2499\Omega)^2 + 10403.4}$$

With few manipulation, $\Omega_A = 83.28\ rad/s\ \ or\ \ N_A = 795.3\ RPM$

Forward whirling speed at point B can be found by equating synchronous line with forward whirling line:

$$\Omega = 0.2499\Omega + \sqrt{(0.2499\Omega)^2 + 10403.4}$$

$$\Omega_B = 144.2\ rad/s\ \ or\ \ N_B = 1377\ RPM$$

H.W:

A uniform rotor of mass 600 kg and diametral moment of inertia of 24 kg.m^2, is supported by identical short bearings of stiffness 1 MN/m in the horizontal and vertical directions. If the distance between the bearings is 0.8 m, determine:

1. Determine the translational natural frequencies,
2. What is the corresponding polar moment of inertia if the forward whirling speed is 120% of the static structure tilting natural frequency?
3. Determine the backward whirling speed.
4. Plot the whirling map diagram.

7. Gyroscopic Effect in MDOF

The presence of gyroscopic effect will introduce skew-symmetric damping matrix. The linearized equation of motion can be written as:

$$\mathbf{M}\ddot{\mathbf{q}} + \Omega\mathbf{G}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0 \qquad (24)$$

Where **q** is the generalized coordinate matrix, **M** is the mass matrix, **G** is the gyroscopic skew-symmetric matrix, **K** is the symmetric stiffness matrix.

Equation (24) can be solved using state-space method discussed previously.

$$\begin{bmatrix} \Omega\mathbf{G} & \mathbf{M} \\ \mathbf{M} & 0 \end{bmatrix}\dot{\mathbf{y}} + \begin{bmatrix} \mathbf{K} & 0 \\ 0 & -\mathbf{M} \end{bmatrix}\mathbf{y} = \mathbf{0} \qquad (25)$$

Or equivalently:

$$\mathbf{A}\dot{\mathbf{y}} + \mathbf{B}\mathbf{y} = \mathbf{0} \qquad (26)$$

Where both A and B are square matrix with dimension $2n \times 2n$. Equation (26) can be cast into eigenvalue problem which is solved to find $2n$ eigenvalues and corresponding eigenvectors.

When damping is introduced in the system due to for example fluid film bearings, eq. (25) becomes:

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C} + \Omega\mathbf{G})\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0 \qquad (27)$$

Where **C** is the symmetric damping matrix. The steady-state oscillatory solution of eq. (27) is an eigenvalue problem and produce complex values for the natural frequencies and mode shapes. As the spin speed can appear explicitly in the equation of motion, the natural frequencies of a machine containing a rotor depend on the spin speed. When this occurs, the free behavior of the system is usually summarized by a plot of the natural frequencies ω_i as functions of Ω. Because in many cases the frequencies of the exciting forces also depend on the speed, they can be reported on the same plot, known as the Campbell Diagram as shown in Fig. 14.

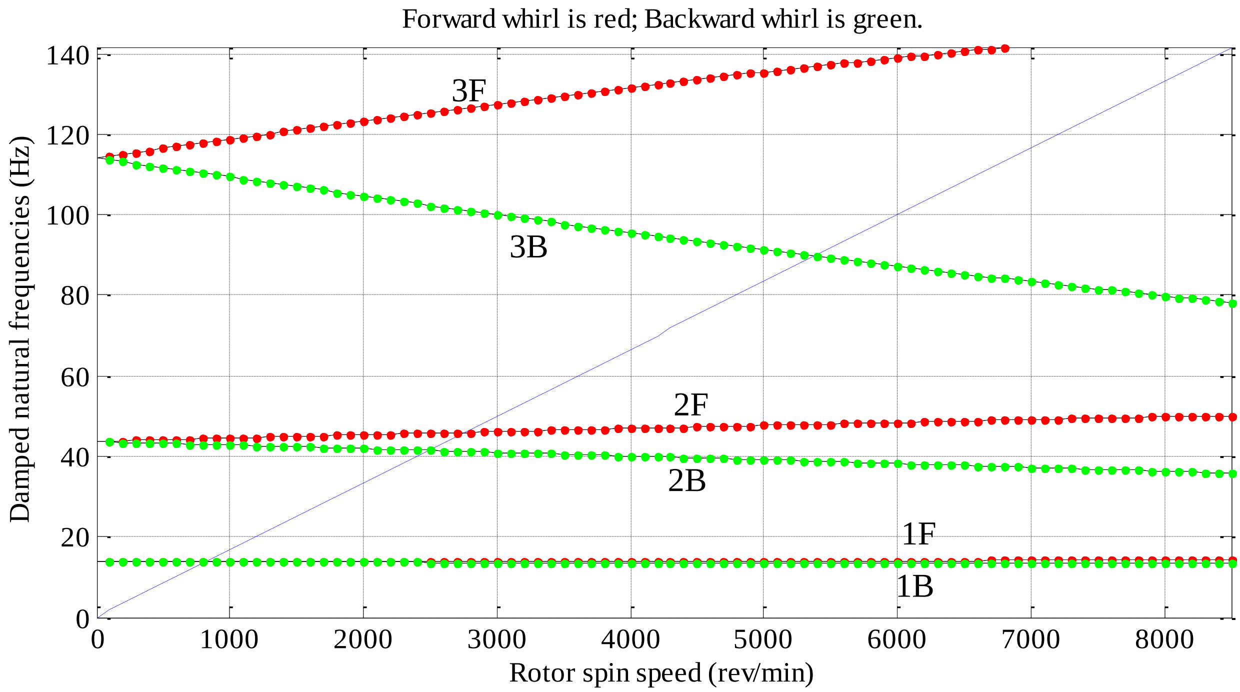


Figure 14 Campbell's Diagram

Remarks about Campbell's diagram:

(1) The intersections of the various branches of the Campbell diagram with the ω axis are the natural frequencies at standstill of the system. If the system is axially symmetrical (i.e., the characteristics are independent on the direction), the natural frequencies at standstill are pairs of coincident values. With increasing speed, the values are no more double: Two diverging branches start from each point on the ω axis of the Campbell diagram.

(2) The backward whirling frequency (as absolute value) decreases with increasing spin speed Ω

(3) The forward whirling frequency increases as the spin speed Ω increases

Critical speeds can easily be found from Campbell diagram. Often rotors are subjected to forces that vary in time, and sometimes their time history is harmonic. This is the case, for example, of forces caused by the unbalance of the rotor, which can be described as a vector rotating with the same angular speed as the rotor and whose components in the fixed reference frame vary harmonically in time with circular frequency equal to the rotational speed Ω. In other cases the time history is less regular, but if it is periodical, it can always be represented as the sum of harmonic components. In these cases, the frequency of the forcing function or of its harmonic components is often linked with the spin speed of the rotor and can be plotted on the Campbell diagram. In the case of the excitation caused by unbalance, for example, the forcing frequency can be represented on the Ωω-plane of the Campbell diagram by the straight line $\Omega = \omega$. In this case, the excitation is said to be synchronous. The relationship linking the frequency of the forcing function to the spin speed is often of simple proportionality and can be represented on the Campbell diagram by a straight line through the origin.

There are, however, cases in which a very strong resonance takes place and the rotor cannot operate at or near a critical speed without strong vibrations or even a catastrophic failure. In particular, the resonances caused by the coincidence of one of the flexural natural frequencies with the spin speed are particularly dangerous; they can be detected on the Campbell diagram by the intersection of the curves related to the natural frequencies with the straight line $\Omega = \omega$. They are usually referred to as flexural critical speeds, without further indications, and other critical speeds related to bending behavior, which are usually less dangerous, are often said to be secondary critical speeds. The speed range spanning from zero to the first critical speed is usually referred to as the subcritical range; above the first critical speed, the supercritical range starts. A growing number of machines work in the supercritical range, and then at least one of the critical speeds must be crossed during startup and shut-down procedures.

8. Fields of Instability

Rotors may develop an unstable behavior in well-defined velocity ranges. The velocities at which this unstable behavior occurs must not, however, be confused with the critical speeds of the rotor because the two phenomena are completely different. The term unstable can have several meanings, and different definitions of stability exist. The amplitude of free vibration of damped linear systems decays exponentially in time, because of the energy dissipation caused by damping. In the case of rotors, however, there is a source of energy, the centrifugal field that may in some cases cause an unbounded growth in time of the amplitude of free vibrations.

The ranges of the spin speed in which this growth occurs, i.e., in which self-excited vibrations can develop, are usually called instability fields or instability ranges, and the speed at which the first of such field starts is the threshold of instability.

Instability ranges must not be confused with critical speeds: Critical speeds are a sort of resonance between a natural frequency and a forcing function acting on the rotor, and in instability ranges, true self-excited vibrations occur. They need the presence of some source of energy to sustain the vibration with increasing amplitude, and in this case, the energy can be supplied by the kinetic energy linked with rotation at the spin speed Ω. It is easy to verify that the kinetic energy stored in the rotor is greater by some orders of magnitude than the elastic potential energy the rotor can store without failure. A structural system is inherently stable, as it can only dissipate energy, unless some mechanism that may supply energy to it is present. This is the case of the inverted pendulum energy supplied by the gravitational field), aeroelastic vibrations (energy supplied by the aerodynamic field and ultimately by the kinetic energy of the aircraft), rotors (energy supplied by rotation), and active controlled systems (energy supplied by the controller). In all of those cases, instability may occur and the designer must study very carefully the conditions that assure a stable working of the system.

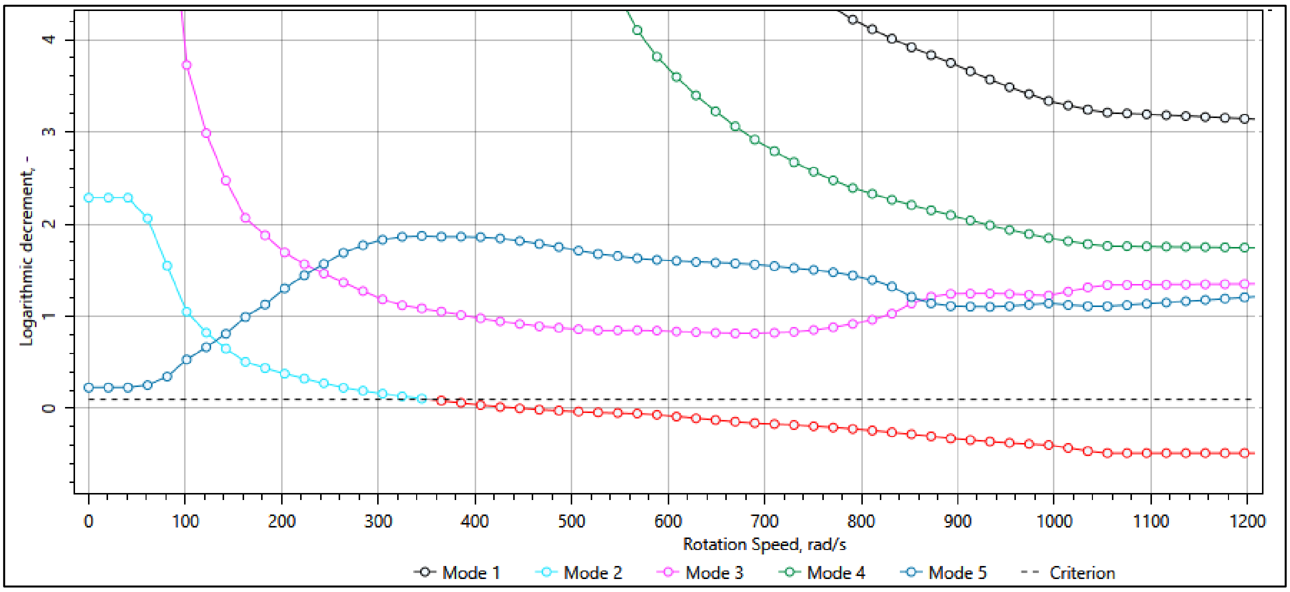


Figure 15 Instability of Mode 2

To make it easier to distinguish between critical speeds and fields of instability, the following features may be listed:

Table 1 Comparison between critical speeds and instability fields

Critical Speeds	Instability Fields
• They occur at well-defined values of the spin speed. • The amplitude grows linearly in time if no damping is present. It can be maintained within reasonable limits, and as a consequence, a critical speed can be passed. • The value of the speed is fixed, but that of the maximum amplitude depends on the amplitude of the perturbation causing it. In particular, the main flexural critical speeds do not depend on the amount of unbalance, but the amplitude increases with increasing	• Their span is usually large. Often, all speeds in excess of the threshold of instability give way to unstable behavior. • The threshold of instability, if it exists, is usually located in the supercritical range. • The amplitude grows exponentially in time. It grows in an uncontrollable way, and then working above the threshold of instability is impossible. When it falls within the working range, the system must be modified to raise it well above the maximum operating speed. Only

unbalance.	possible nonlinearities of the system can prevent the amplitude from growing without limits, giving way to a limit cycle.

Table 2 Characterization of forced and self-excited vibration in rotors

Criterion	Forced or resonant vibration	Self-excited vibration
Relationship between frequency and speed	Frequency is equal to (i.e., synchronous with) the spin speed or a whole number or rational fraction of spin speed.	Frequency is nearly constant and essentially independent of spin speed or any external excitation
Relationship between amplitude and speed	Amplitude will peak in a narrow band of spin speed wherein the rotor's natural frequency is equal to the spin speed or to a whole number multiple or a rational fraction of the spin speed and is independent on external excitation.	Amplitude will suddenly increase at a threshold speed and continues at high or increasing levels as spin speed is increased.
Whirl direction	Almost always forward whirling.	Generally forward whirling, but backward whirling has been reported.
Rotor Stressing	Static stressing in case of synchronous whirling.	Oscillatory stressing at frequency equal to $\omega - \Omega$
Correcting action	1. Introduce damping to limit peak amplitudes at critical speeds. 2. Tune the system's critical speeds to be outside the working range. 3. Eliminate all deviations from axial symmetry in the system as built or as induced during operation (e.g., balancing).	1. Increase damping to increase the threshold of instability above the operating speed range. 2. Raise the rotor natural frequencies as high as possible. 3. Identify and eliminate the instability mechanism.

Influence of damping	Addition of damping may reduce peak amplitude but does not affect the spin speed at which it occurs.	Addition of damping may raise the speed at which instability occurs but usually does not affect the amplitude after onset.

Table 3 Diagnostic table of self-excited vibration in rotating machinery

Mechanism	**Ratio ω/Ω**	**Direction**
Internal rotor damping	$0,2 < \omega/\Omega < 1$ ($\omega/\Omega = 0,5$)	Forward
Hydrodynamic bearings, labyrinth, or liquid seals	$\omega/\Omega < 0,5$ ($0,45 < \omega/\Omega < 0,48$)	Forward
Blade-tip clearance excitation	Dependent on fluid force levels	Forward
Centrifugal pump and compressor whirl	Dependent on fluid force levels	Forward
Propeller and turbomachinery whirl	Dependent on fluid force levels	Backward, if the vertex of the cone described in the whirl motion is after the rotor (referring to the direction of the fluid flow). Forward in the opposite case
Excitation due to fluid trapped in rotors	$0,5 < \omega/\Omega < 1,0$ ($0,7 < \omega/\Omega < 0,9$)	Forward