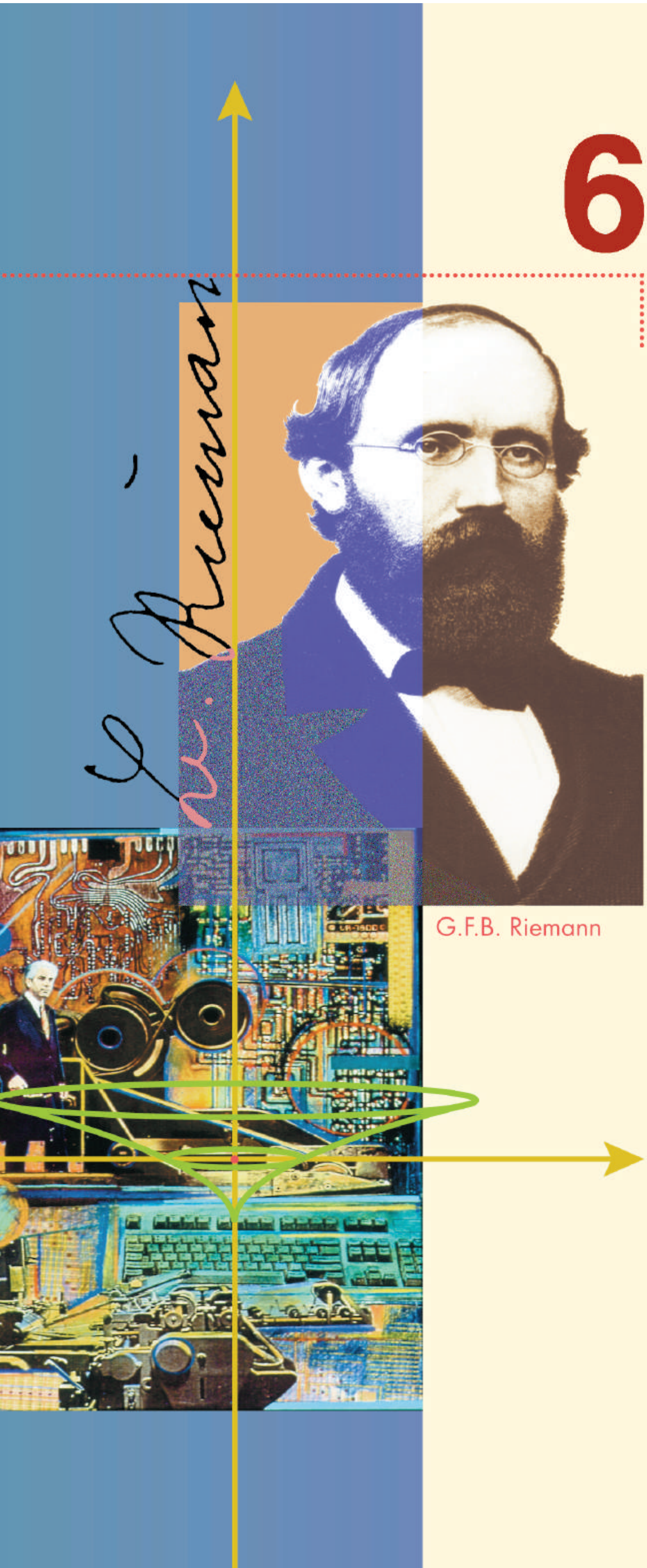


G.F.B. Riemann

6

APPLICATIONS OF THE DEFINITE INTEGRAL IN GEOMETRY, SCIENCE, AND ENGINEERING

In the last chapter we introduced the definite integral as the limit of Riemann sums in the context of finding areas. However, Riemann sums and definite integrals have applications that extend far beyond the area problem. In this chapter we will show how Riemann sums and definite integrals arise in such problems as finding the volume and surface area of a solid, finding the length of a plane curve, calculating the work done by a force, and finding the pressure and force exerted by a fluid on a submerged object.

Although these problems are diverse, the required calculations can all be approached by the same procedure that we used to find areas—breaking the required calculation into "small parts," making an approximation that is good because the part is small, adding the approximations from the parts to produce a Riemann sum that approximates the entire quantity to be calculated, and then taking the limit of the Riemann sums to produce an exact result.

6.1 AREA BETWEEN TWO CURVES

In the last chapter we showed how to find the area between a curve $y = f(x)$ and an interval on the x-axis. Here we will show how to find the area between two curves.

A REVIEW OF RIEMANN SUMS

Before we consider the problem of finding the area between two curves it will be helpful to review the basic principle that underlies the calculation of area as a definite integral. Recall that if f is continuous and nonnegative on $[a, b]$, then the definite integral for the area A under $y = f(x)$ over the interval $[a, b]$ is obtained in four steps (Figure 6.1.1):

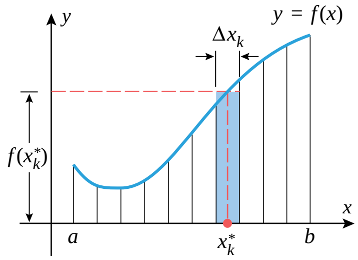


Figure 6.1.1

- Divide the interval $[a, b]$ into n subintervals, and use those subintervals to divide the area under the curve $y = f(x)$ into n strips.
- Assuming that the width of the kth strip is Δx_k, approximate the area of that strip by the area of a rectangle of width Δx_k and height $f(x_k^*)$, where x_k^* is a number in the kth subinterval.
- Add the approximate areas of the strips to approximate the entire area A by the Riemann sum:
$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$
- Take the limit of the Riemann sums as the number of subintervals increases and their widths approach zero. This causes the error in the approximations to approach zero and produces the following definite integral for the exact area A:
$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$$

Observe the effect that the limit process has on the various parts of the Riemann sum:

- The quantity x_k^* in the Riemann sum becomes the variable x in the definite integral.
- The interval width Δx_k in the Riemann sum becomes the dx in the definite integral.
- The interval $[a, b]$ is implicit in the Riemann sum as the aggregate of the subintervals with widths $\Delta x_1, \ldots, \Delta x_n$, but $[a, b]$ is explicitly represented by the upper and lower limits of integration in the definite integral.

AREA BETWEEN $y = f(x)$ AND $y = g(x)$

We will now consider the following extension of the area problem.

6.1.1 FIRST AREA PROBLEM. Suppose that f and g are continuous functions on an interval $[a, b]$ and

$$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$

[This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.1.2a).

To solve this problem we divide the interval $[a, b]$ into n subintervals, which has the effect of subdividing the region into n strips (Figure 6.1.2b). If we assume that the width of the kth strip is Δx_k, then the area of the strip can be approximated by the area of a rectangle of width Δx_k and height $f(x_k^*) - g(x_k^*)$, where x_k^* is a number in the kth subinterval. Adding these approximations yields the following Riemann sum that approximates the area A:

$$A \approx \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x_k$$

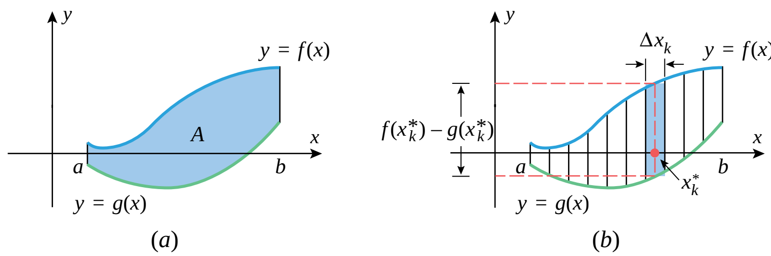


Figure 6.1.2

Taking the limit as n increases and the widths of the subintervals approach zero yields the following definite integral for the area A between the curves:

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x_k = \int_a^b [f(x) - g(x)]\, dx$$

In summary, we have the following result:

6.1.2 AREA FORMULA. If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx \qquad (1)$$

In the case where f and g are *nonnegative* on the interval $[a, b]$, the formula

$$A = \int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

states that the area A between the curves can be obtained by subtracting the area under $y = g(x)$ from the area under $y = f(x)$ (Figure 6.1.3).

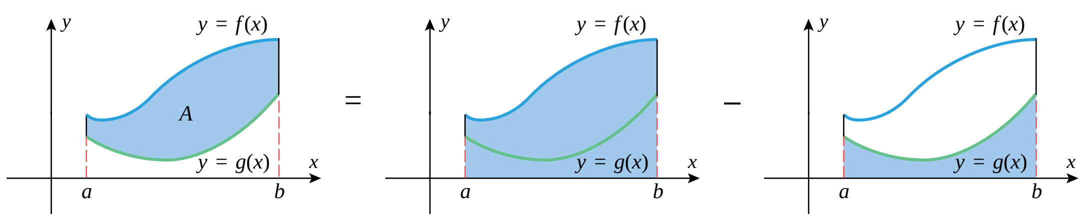


Figure 6.1.3

When the region is complicated, it may require some careful thought to determine the integrand and limits of integration in (1). Here is a systematic procedure that you can follow to set up this formula.

Step 1. Sketch the region and then draw a vertical line segment through the region at an arbitrary point x on the x-axis, connecting the top and bottom boundaries (Figure 6.1.4*a*).

Step 2. The y-coordinate of the top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand in (1).

Step 3. To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$ (Figures 6.1.4*b* and 6.1.4*c*).

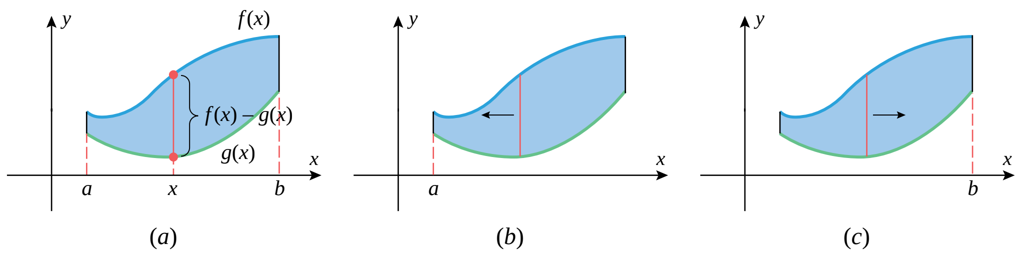


Figure 6.1.4

REMARK. It is not necessary to make an extremely accurate sketch in Step 1; the only purpose of the sketch is to determine which curve is the upper boundary and which is the lower boundary.

REMARK. There is a useful way of thinking about this procedure: If you view the vertical line segment as the "cross section" of the region at the point x, then Formula (1) states that the area between the curves is obtained by integrating the length of the cross section over the interval from a to b.

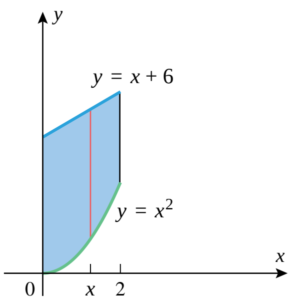


Figure 6.1.5

Example 1 Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

Solution. The region and a cross section are shown in Figure 6.1.5. The cross section extends from $g(x) = x^2$ on the bottom to $f(x) = x + 6$ on the top. If the cross section is moved through the region, then its leftmost position will be $x = 0$ and its rightmost position will be $x = 2$. Thus, from (1)

$$A = \int_0^2 [(x+6) - x^2]\,dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3} \qquad \blacktriangleleft$$

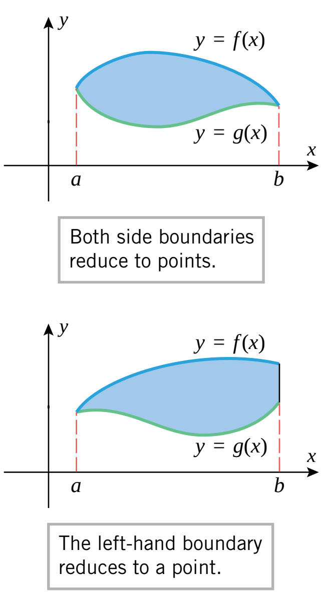


Figure 6.1.6

It is possible that the upper and lower boundaries of a region may intersect at one or both endpoints, in which case the sides of the region will be points, rather than vertical line segments (Figure 6.1.6). When that occurs you will have to determine the points of intersection to obtain the limits of integration.

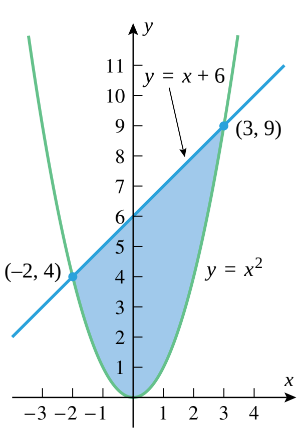


Figure 6.1.7

Example 2 Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution. A sketch of the region (Figure 6.1.7) shows that the lower boundary is $y = x^2$ and the upper boundary is $y = x + 6$. At the endpoints of the region, the upper and lower boundaries have the same y-coordinates; thus, to find the endpoints we equate

$$y = x^2 \quad \text{and} \quad y = x + 6 \tag{2}$$

This yields

$$x^2 = x + 6 \quad \text{or} \quad x^2 - x - 6 = 0 \quad \text{or} \quad (x+2)(x-3) = 0$$

from which we obtain

$$x = -2 \quad \text{and} \quad x = 3$$

Although the y-coordinates of the endpoints are not essential to our solution, they may be obtained from (2) by substituting $x = -2$ and $x = 3$ in either equation. This yields $y = 4$ and $y = 9$, so the upper and lower boundaries intersect at $(-2, 4)$ and $(3, 9)$.

From (1) with $f(x) = x + 6$, $g(x) = x^2$, $a = -2$, and $b = 3$, we obtain the area

$$A = \int_{-2}^3 [(x+6) - x^2]\,dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^3 = \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6} \qquad \blacktriangleleft$$

It is possible for the upper or lower boundary of a region to consist of two or more different curves, in which case it will be necessary to subdivide the region into smaller pieces in order to apply Formula (1). This is illustrated in the next example.

Example 3 Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

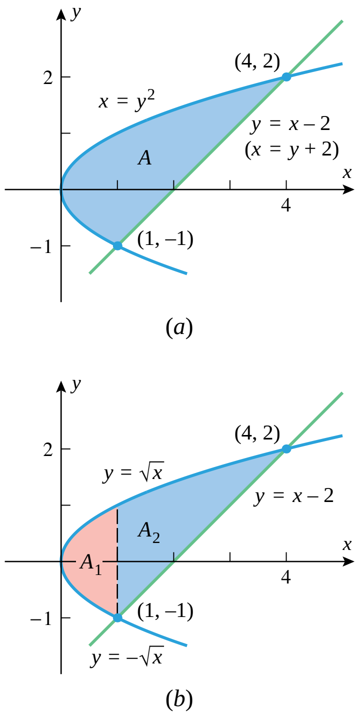


Figure 6.1.8

Solution. To make an accurate sketch of the region, we need to know where the curves $x = y^2$ and $y = x - 2$ intersect. In Example 2 we found intersections by equating the expressions for y. Here it is easier to rewrite the latter equation as $x = y + 2$ and equate the expressions for x, namely

$$x = y^2 \quad \text{and} \quad x = y + 2 \tag{3}$$

This yields

$$y^2 = y + 2 \quad \text{or} \quad y^2 - y - 2 = 0 \quad \text{or} \quad (y+1)(y-2) = 0$$

from which we obtain $y = -1$, $y = 2$. Substituting these values in either equation in (3) we see that the corresponding x-values are $x = 1$ and $x = 4$, respectively, so the points of intersection are $(1, -1)$ and $(4, 2)$ (Figure 6.1.8*a*).

To apply Formula (1), the equations of the boundaries must be written so that y is expressed explicitly as a function of x. The upper boundary can be written as $y = \sqrt{x}$ (rewrite $x = y^2$ as $y = \pm\sqrt{x}$ and choose the + for the upper portion of the curve). The lower portion of the boundary consists of two parts: $y = -\sqrt{x}$ for $0 \le x \le 1$ and $y = x - 2$ for $1 \le x \le 4$ (Figure 6.1.8*b*). Because of this change in the formula for the lower boundary, it is necessary to divide the region into two parts and find the area of each part separately.

From (1) with $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x}$, $a = 0$, and $b = 1$, we obtain

$$A_1 = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx = 2\int_0^1 \sqrt{x}\,dx = 2\left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{4}{3} - 0 = \frac{4}{3}$$

From (1) with $f(x) = \sqrt{x}$, $g(x) = x - 2$, $a = 1$, and $b = 4$, we obtain

$$\begin{aligned} A_2 &= \int_1^4 [\sqrt{x} - (x-2)]\,dx = \int_1^4 (\sqrt{x} - x + 2)\,dx \\ &= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4 = \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{19}{6} \end{aligned}$$

Thus, the area of the entire region is

$$A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}$$ ◀

FOR THE READER. It is assumed in Formula (1) that $f(x) \ge g(x)$ for all x in the interval $[a, b]$. What do you think that the integral represents if this condition is not satisfied, that is, the graphs of f and g cross one another over the interval? Explain your reasoning, and give an example to support your conclusion. Using definite integrals, write an expression for the area between the graphs of f and g in your example.

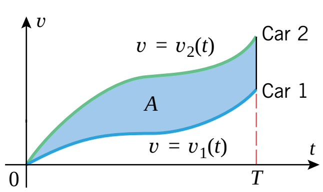


Figure 6.1.9

Example 4 Figure 6.1.9 shows velocity versus time curves for two race cars that move along a straight track, starting from rest at the same line. What does the area A between the curves over the interval $0 \le t \le T$ represent?

Solution. From (1)

$$A = \int_0^T [v_2(t) - v_1(t)]\,dt = \int_0^T v_2(t)\,dt - \int_0^T v_1(t)\,dt$$

But from 5.7.4, the first integral is the distance traveled by car 2 during the time interval, and the second integral is the distance traveled by car 1. Thus, A is the distance by which car 2 is ahead of car 1 at time T. ◀

REVERSING THE ROLES OF x AND y

Sometimes it is possible to avoid splitting a region into parts by integrating with respect to y rather than x. We will now show how this can be done.

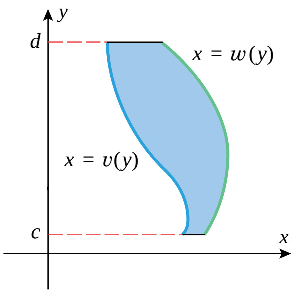


Figure 6.1.10

6.1.3 SECOND AREA PROBLEM. Suppose that w and v are continuous functions of y on an interval $[c, d]$ and that

$$w(y) \geq v(y) \quad \text{for} \quad c \leq y \leq d$$

[This means that the curve $x = w(y)$ lies to the right of the curve $x = v(y)$ and that the two can touch but not cross.] Find the area A of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, and above and below by the lines $y = d$ and $y = c$ (Figure 6.1.10).

Proceeding as in the derivation of (1), but with the roles of x and y reversed, leads to the following analog of 6.1.2.

6.1.4 AREA FORMULA. If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)]\, dy \tag{4}$$

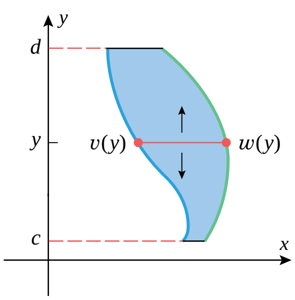


Figure 6.1.11

The guiding principle in applying this formula is the same as with (1): The integrand in (4) can be viewed as the length of the horizontal cross section at an arbitrary point y on the y-axis, in which case Formula (4) states that the area can be obtained by integrating the length of the horizontal cross section over the interval $[c, d]$ on the y-axis (Figure 6.1.11).

In Example 3, where we integrated with respect to x to find the area of the region enclosed by $x = y^2$ and $y = x - 2$, we had to split the region into parts and evaluate two integrals. In the next example we will see that by integrating with respect to y no splitting of the region is necessary.

Example 5 Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y.

Solution. From Figure 6.1.8 the left boundary is $x = y^2$, the right boundary is $y = x - 2$, and the region extends over the interval $-1 \leq y \leq 2$. However, to apply (4) the equations for the boundaries must be written so that x is expressed explicitly as a function of y. Thus, we rewrite $y = x - 2$ as $x = y + 2$. It now follows from (4) that

$$A = \int_{-1}^{2} [(y + 2) - y^2]\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

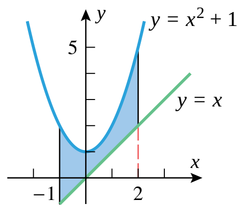
which agrees with the result obtained in Example 3. ◀

REMARK. The choice between Formulas (1) and (4) is generally dictated by the shape of the region, and one would usually choose the formula that requires the least amount of splitting. However, if the integral(s) resulting by one method are difficult to evaluate, then the other method might be preferable, even if it requires more splitting.

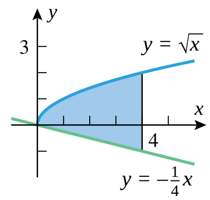
EXERCISE SET 6.1 Graphing Calculator C CAS

In Exercises 1–4, find the area of the shaded region.

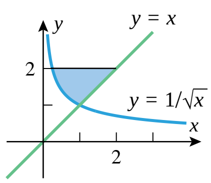
1.



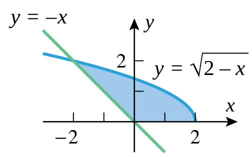
2.



3.



4.



5. Find the area of the region enclosed by the curves $y = x^2$ and $y = 4x$ by integrating
(a) with respect to x (b) with respect to y.

6. Find the area of the region enclosed by the curves $y^2 = 4x$ and $y = 2x - 4$ by integrating
(a) with respect to x (b) with respect to y.

In Exercises 7–14, sketch the region enclosed by the curves and find its area.

7. $y = x^2,\ \ y = \sqrt{x},\ \ x = 1/4,\ \ x = 1$

8. $y = x^3 - 4x,\ \ y = 0,\ \ x = 0,\ \ x = 2$

9. $y = \cos 2x,\ \ y = 0,\ \ x = \pi/4,\ \ x = \pi/2$

10. $y = \sec^2 x,\ \ y = 2,\ \ x = -\pi/4,\ \ x = \pi/4$

11. $x = \sin y,\ \ x = 0,\ \ y = \pi/4,\ \ y = 3\pi/4$

12. $x^2 = y,\ \ x = y - 2$

13. $y = 2 + |x - 1|,\ \ y = -\frac{1}{5}x + 7$

14. $y = x,\ \ y = 4x,\ \ y = -x + 2$

In Exercises 15–20, use a graphing utility, where helpful, to find the area of the region enclosed by the curves.

15. $y = x^3 - 4x^2 + 3x,\ \ y = 0$

16. $y = x^3 - 2x^2,\ \ y = 2x^2 - 3x$

17. $y = \sin x,\ \ y = \cos x,\ \ x = 0,\ \ x = 2\pi$

18. $y = x^3 - 4x,\ \ y = 0$

19. $x = y^3 - y,\ \ x = 0$

20. $x = y^3 - 4y^2 + 3y,\ \ x = y^2 - y$

C **21.** Use a CAS to find the area enclosed by $y = 3 - 2x$ and $y = x^6 + 2x^5 - 3x^4 + x^2$.

C **22.** Use a CAS to find the exact area enclosed by the curves $y = x^5 - 2x^3 - 3x$ and $y = x^3$.

23. Find a horizontal line $y = k$ that divides the area between $y = x^2$ and $y = 9$ into two equal parts.

24. Find a vertical line $x = k$ that divides the area enclosed by $x = \sqrt{y}$, $x = 2$, and $y = 0$ into two equal parts.

25. (a) Find the area of the region enclosed by the parabola $y = 2x - x^2$ and the x-axis.
(b) Find the value of m so that the line $y = mx$ divides the region in part (a) into two regions of equal area.

26. Find the area between the curve $y = \sin x$ and the line segment joining the points $(0, 0)$ and $(5\pi/6, 1/2)$ on the curve.

27. Suppose that f and g are integrable on $[a, b]$, but neither $f(x) \geq g(x)$ nor $g(x) \geq f(x)$ holds for all x in $[a, b]$ [i.e., the curves $y = f(x)$ and $y = g(x)$ are intertwined].
(a) What is the geometric significance of the integral

$$\int_a^b [f(x) - g(x)]\, dx?$$

(b) What is the geometric significance of the integral

$$\int_a^b |f(x) - g(x)|\, dx?$$

28. Let $A(n)$ be the area in the first quadrant enclosed by the curves $y = \sqrt[n]{x}$ and $y = x$.
(a) By considering how the graph of $y = \sqrt[n]{x}$ changes as n increases, make a conjecture about the limit of $A(n)$ as $n \to +\infty$.
(b) Confirm your conjecture by calculating the limit.

In Exercises 29 and 30, use Newton's Method (Section 4.7), where needed, to approximate the x-coordinates of the intersections of the curves to at least four decimal places, and then use those approximations to approximate the area of the region.

29. The region that lies below the curve $y = \sin x$ and above the line $y = 0.2x$, where $x \geq 0$.

30. The region enclosed by the graphs of $y = x^2$ and $y = \cos x$.

31. The accompanying figure shows velocity versus time curves for two cars that move along a straight track, accelerating from rest at a common starting line.
(a) How far apart are the cars after 60 seconds?
(b) How far apart are the cars after T seconds, where $0 \leq T \leq 60$?

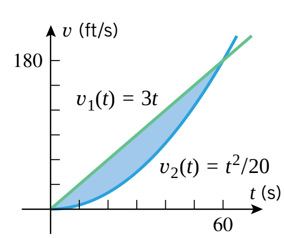


Figure Ex-31

32. The accompanying figure shows acceleration versus time curves for two cars that move along a straight track, accelerating from rest at the starting line. What does the area A between the curves over the interval $0 \le t \le T$ represent? Justify your answer.

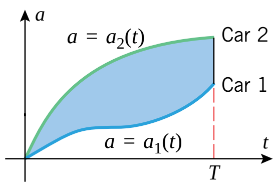


Figure Ex-32

33. Find the area of the region enclosed between the curve $x^{1/2} + y^{1/2} = a^{1/2}$ and the coordinate axes.

34. Show that the area of the ellipse in the accompanying figure is πab. [*Hint:* Use a formula from geometry.]

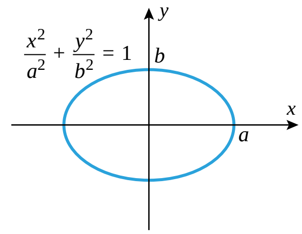


Figure Ex-34

35. A rectangle with edges parallel to the coordinate axes has one vertex at the origin and the diagonally opposite vertex on the curve $y = kx^m$ at the point where $x = b$ ($b > 0$, $k > 0$, and $m \ge 0$). Show that the fraction of the area of the rectangle that lies between the curve and the x-axis depends on m but not on k or b.

6.2 VOLUMES BY SLICING; DISKS AND WASHERS

In the last section we showed that the area of a plane region bounded by two curves can be obtained by integrating the length of a general cross section over an appropriate interval. In this section we will see that the same basic principle can be used to find volumes of certain three-dimensional solids.

VOLUMES BY SLICING

Recall that the underlying principle for finding the area of a plane region is to divide the region into thin strips, approximate the area of each strip by the area of a rectangle, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the area. Under appropriate conditions, the same strategy can be used to find the volume of a solid. The idea is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the volume (Figure 6.2.1).

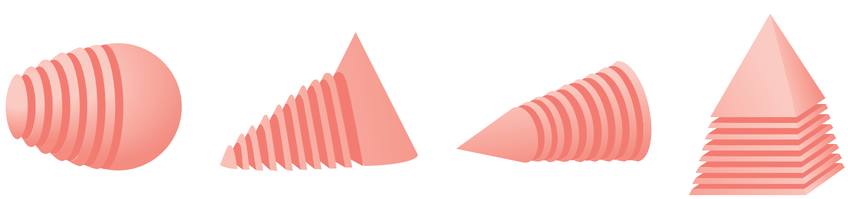
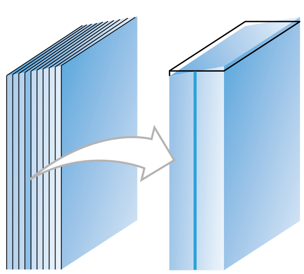


Figure 6.2.1



In a thin slab, the cross sections do not vary much in size and shape.

Figure 6.2.2

What makes this method work is the fact that a *thin* slab has cross sections that do not vary much in size or shape, which, as we will see, makes its volume easy to approximate (Figure 6.2.2). Moreover, the thinner the slab, the less variation in its cross sections and the better the approximation. Thus, once we approximate the volumes of the slabs, we can set

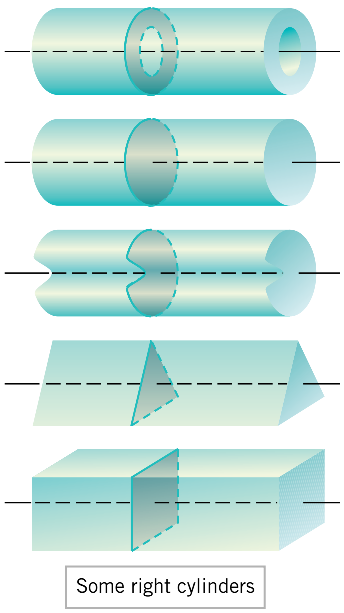


Figure 6.2.3

up a Riemann sum whose limit is the volume of the entire solid. We will give the details shortly, but first we need to discuss how to find the volume of a solid whose cross sections do not vary in size and shape (i.e., are congruent).

One of the simplest examples of a solid with congruent cross sections is a right circular cylinder of radius r, since all cross sections taken perpendicular to the central axis are circular regions of radius r. The volume V of a right circular cylinder of radius r and height h can be expressed in terms of the height and the area of a cross section as

$$V = \pi r^2 h = [\text{area of a cross section}] \times [\text{height}] \tag{1}$$

This is a special case of a more general volume formula that applies to solids called *right cylinders*. A ***right cylinder*** is a solid that is generated when a plane region is translated along a line or ***axis*** that is perpendicular to the region (Figure 6.2.3). The distance h that the region is translated is called the ***height*** or sometimes the ***width*** of the cylinder, and each cross section is a duplicate of the translated region. We will assume that the volume V of a right cylinder with cross-sectional area A and height h is given by

$$V = A \cdot h = [\text{area of a cross section}] \times [\text{height}] \tag{2}$$

(Figure 6.2.4). Note that this is consistent with Formula (1) for the volume of a right circular cylinder. We now have all of the tools required to solve the following problem.

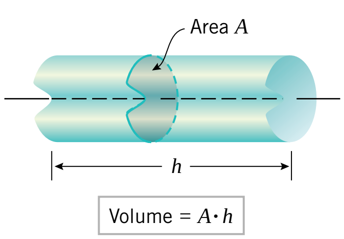


Figure 6.2.4

6.2.1 PROBLEM. Let S be a solid that extends along the x-axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the x-axis at $x = a$ and $x = b$ (Figure 6.2.5a). Find the volume V of the solid, assuming that its cross-sectional area $A(x)$ is known at each x in the interval $[a, b]$.

To solve this problem we divide the interval $[a, b]$ into n subintervals, which has the effect of dividing the solid into n slabs (Figure 6.2.5b).

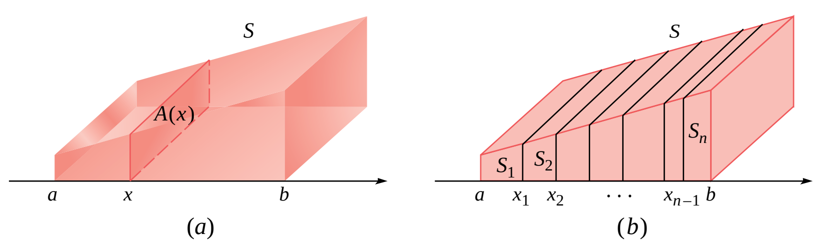


Figure 6.2.5

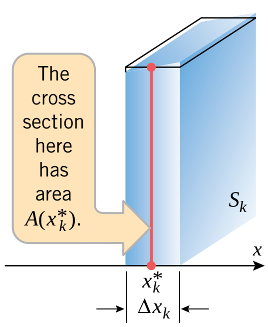


Figure 6.2.6

If we assume that the width of the kth slab is Δx_k, then the volume of the slab can be approximated by the volume of a right cylinder of width (height) Δx_k and cross-sectional area $A(x_k^*)$, where x_k^* is a number in the kth subinterval (Figure 6.2.6). Adding these approximations yields the following Riemann sum that approximates the volume V:

$$V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k = \int_a^b A(x)\,dx$$

In summary, we have the following result:

6.2.2 VOLUME FORMULA. Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\,dx \tag{3}$$

provided $A(x)$ is integrable.

There is a similar result for cross sections perpendicular to the y-axis.

6.2.3 VOLUME FORMULA. Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)\,dy \tag{4}$$

provided $A(y)$ is integrable.

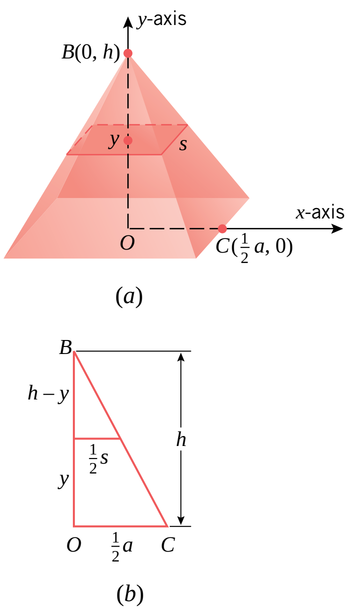


Figure 6.2.7

In words, these formulas state:

The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other.

Example 1 Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length a.

Solution. As illustrated in Figure 6.2.7*a*, we introduce a rectangular coordinate system in which the y-axis passes through the apex and is perpendicular to the base, and the x-axis passes through the base and is parallel to a side of the base.

At any y in the interval $[0, h]$ on the y-axis, the cross section perpendicular to the y-axis is a square. If s denotes the length of a side of this square, then by similar triangles (Figure 6.2.7*b*)

$$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h} \quad \text{or} \quad s = \frac{a}{h}(h-y)$$

Thus, the area $A(y)$ of the cross section at y is

$$A(y) = s^2 = \frac{a^2}{h^2}(h-y)^2$$

and by (4) the volume is

$$V = \int_0^h A(y)\,dy = \int_0^h \frac{a^2}{h^2}(h-y)^2\,dy = \frac{a^2}{h^2}\int_0^h (h-y)^2\,dy$$

$$= \frac{a^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_{y=0}^h = \frac{a^2}{h^2}\left[0 + \frac{1}{3}h^3\right] = \frac{1}{3}a^2h$$

That is, the volume is $\frac{1}{3}$ of the area of the base times the altitude. ◀

SOLIDS OF REVOLUTION

A ***solid of revolution*** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the ***axis of revolution***. Many familiar solids are of this type (Figure 6.2.8).

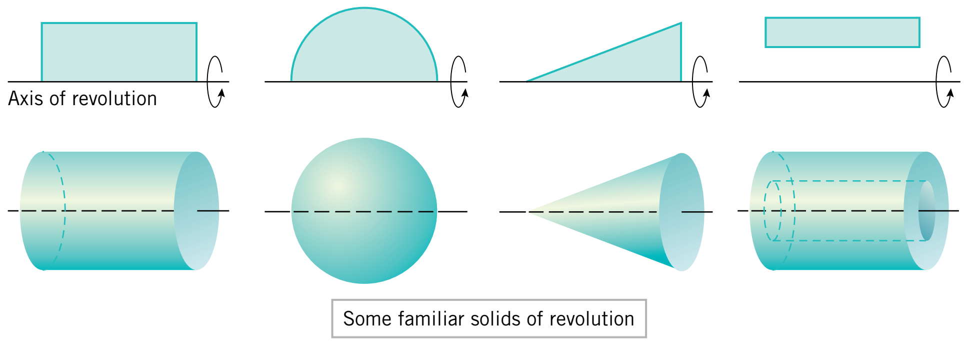


Figure 6.2.8

We will be interested in the following general problem:

VOLUMES BY DISKS PERPENDICULAR TO THE *x*-AXIS

6.2.4 PROBLEM. Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.9*a*). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is a circular disk of radius $f(x)$ (Figure 6.2.9*b*). The area of this region is

$$A(x) = \pi[f(x)]^2$$

Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi[f(x)]^2\,dx \tag{5}$$

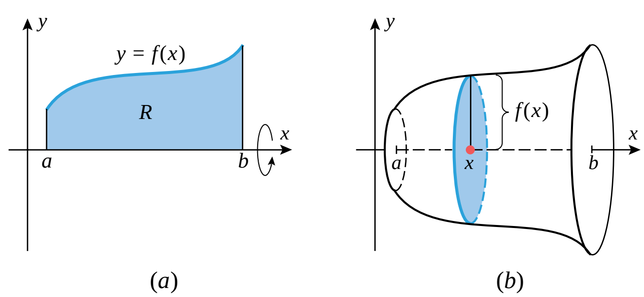


Figure 6.2.9

Because the cross sections are disk shaped, the application of this formula is called the ***method of disks***.

Example 2 Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis (Figure 6.2.10).

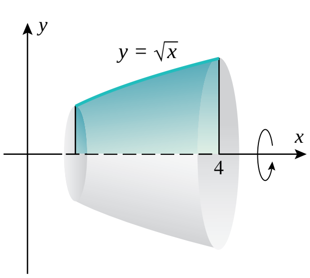


Figure 6.2.10

Solution. From (5), the volume is

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_1^4 \pi x\,dx = \frac{\pi x^2}{2}\bigg]_1^4 = 8\pi - \frac{\pi}{2} = \frac{15\pi}{2}$$ ◄

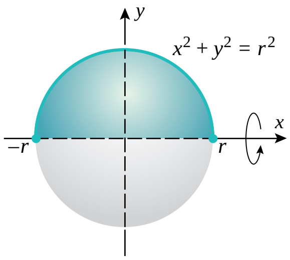


Figure 6.2.11

Example 3 Derive the formula for the volume of a sphere of radius r.

Solution. As indicated in Figure 6.2.11, a sphere of radius r can be generated by revolving the upper semicircular disk enclosed between the x-axis and

$$x^2 + y^2 = r^2$$

about the x-axis. Since the upper half of this circle is the graph of $y = f(x) = \sqrt{r^2 - x^2}$, it follows from (5) that the volume of the sphere is

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4}{3}\pi r^3 \quad \blacktriangleleft$$

VOLUMES BY WASHERS PERPENDICULAR TO THE x-AXIS

Not all solids of revolution have solid interiors; some have holes or channels that create interior surfaces, as in the last part of Figure 6.2.8. Thus, we will be interested in problems of the following type.

> **6.2.5 PROBLEM.** Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.12a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.

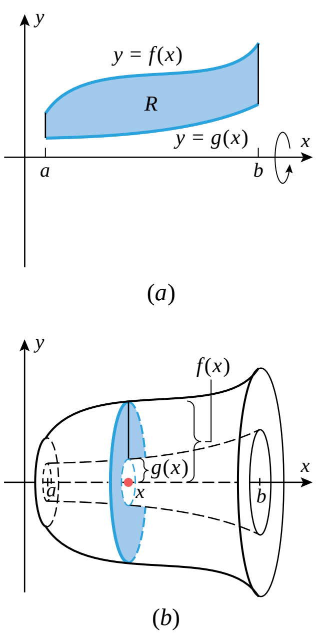


Figure 6.2.12

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is the annular or "washer-shaped" region with inner radius $g(x)$ and outer radius $f(x)$ (Figure 6.2.12b); hence its area is

$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi([f(x)]^2 - [g(x)]^2)$$

Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx \tag{6}$$

Because the cross sections are washer shaped, the application of this formula is called the ***method of washers***.

Example 4 Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis (Figure 6.2.13).

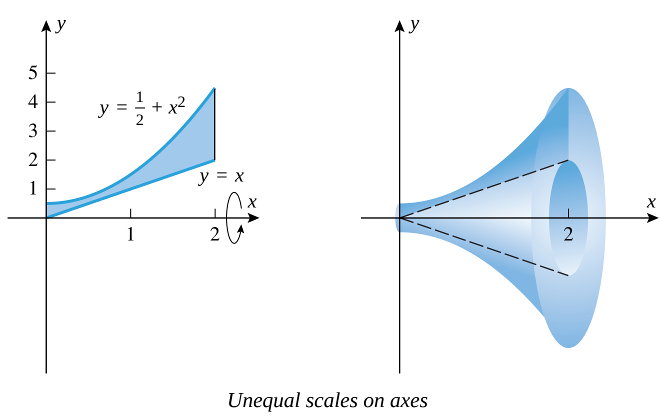


Figure 6.2.13

Solution. From (6) the volume is

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx = \int_0^2 \pi\left(\left[\tfrac{1}{2} + x^2\right]^2 - x^2\right)\,dx$$

$$= \int_0^2 \pi\left(\frac{1}{4} + x^4\right)\,dx = \pi\left[\frac{x}{4} + \frac{x^5}{5}\right]_0^2 = \frac{69\pi}{10} \qquad \blacktriangleleft$$

VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE *y*-AXIS

The methods of disks and washers have analogs for regions that are revolved about the y-axis (Figures 6.2.14 and 6.2.15). Using the method of slicing and Formula (4), you should have no trouble deducing the following formulas for the volumes of the solids in the figures.

$$V = \int_c^d \pi[u(y)]^2\,dy \quad \text{Disks} \qquad\qquad V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy \quad \text{Washers} \tag{7–8}$$

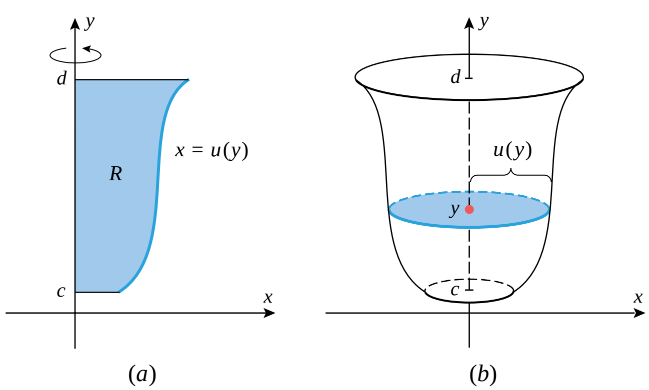


Figure 6.2.14

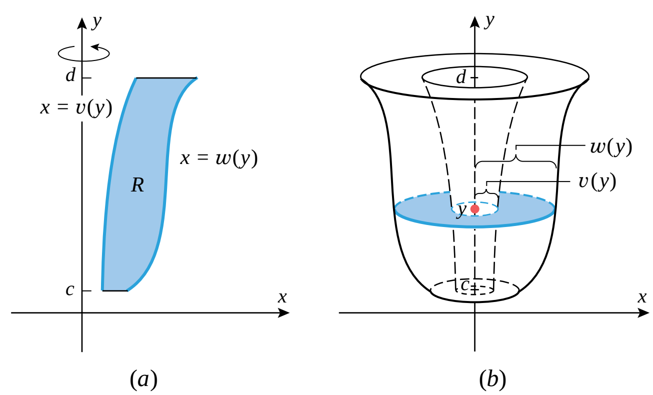


Figure 6.2.15

Example 5 Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis (Figure 6.2.16).

Solution. The cross sections taken perpendicular to the y-axis are disks, so we will apply (7). But first we must rewrite $y = \sqrt{x}$ as $x = y^2$. Thus, from (7) with $u(y) = y^2$, the volume is

$$V = \int_c^d \pi[u(y)]^2\,dy = \int_0^2 \pi y^4\,dy = \frac{\pi y^5}{5}\bigg]_0^2 = \frac{32\pi}{5} \qquad \blacktriangleleft$$

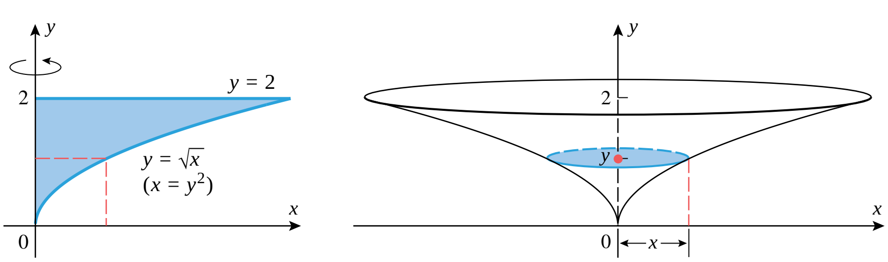
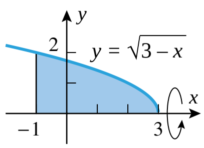


Figure 6.2.16

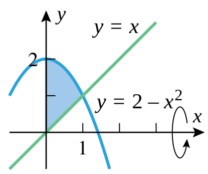
EXERCISE SET 6.2 [C] CAS

In Exercises 1–4, find the volume of the solid that results when the shaded region is revolved about the indicated axis.

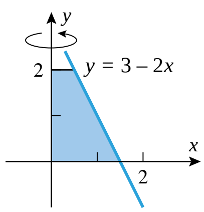
1.



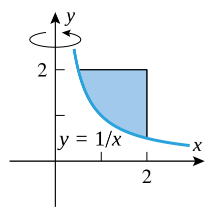
2.



3.



4.



In Exercises 5–12, find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

5. $y = x^2,\ x = 0,\ x = 2,\ y = 0$

6. $y = \sec x,\ x = \pi/4,\ x = \pi/3,\ y = 0$

7. $y = \sqrt{\cos x},\ x = \pi/4,\ x = \pi/2,\ y = 0$

8. $y = x^2,\ y = x^3$

9. $y = \sqrt{25 - x^2},\ y = 3$

10. $y = 9 - x^2,\ y = 0$

11. $x = \sqrt{y},\ x = y/4$

12. $y = \sin x,\ y = \cos x,\ x = 0,\ x = \pi/4$. [*Hint:* Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.]

In Exercises 13–20, find the volume of the solid that results when the region enclosed by the given curves is revolved about the y-axis.

13. $y = x^3,\ x = 0,\ y = 1$

14. $x = 1 - y^2,\ x = 0$

15. $x = \sqrt{1 + y},\ x = 0,\ y = 3$

16. $y = x^2 - 1,\ x = 2,\ y = 0$

17. $x = \csc y,\ y = \pi/4,\ y = 3\pi/4,\ x = 0$

18. $y = x^2,\ x = y^2$

19. $x = y^2,\ x = y + 2$

20. $x = 1 - y^2,\ x = 2 + y^2,\ y = -1,\ y = 1$

21. Find the volume of the solid that results when the region above the x-axis and below the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$$

is revolved about the x-axis.

22. Let V be the volume of the solid that results when the region enclosed by $y = 1/x$, $y = 0$, $x = 2$, and $x = b$ $(0 < b < 2)$ is revolved about the x-axis. Find the value of b for which $V = 3$.

23. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x + 1}$, $y = \sqrt{2x}$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

24. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

25. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 9$ is revolved about the line $x = 9$.

26. Find the volume of the solid that results when the region in Exercise 25 is revolved about the line $y = 3$.

27. Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved about the line $y = -1$.

28. Find the volume of the solid that results when the region in Exercise 27 is revolved about the line $x = -1$.

29. A nose cone for a space reentry vehicle is designed so that a cross section, taken x ft from the tip and perpendicular to the axis of symmetry, is a circle of radius $\frac{1}{4}x^2$ ft. Find the volume of the nose cone given that its length is 20 ft.

30. A certain solid is 1 ft high, and a horizontal cross section taken x ft above the bottom of the solid is an annulus of inner radius x^2 and outer radius $\sqrt{x}$. Find the volume of the solid.

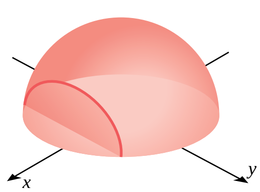
31. Find the volume of the solid whose base is the region bounded between the curves $y = x$ and $y = x^2$, and whose cross sections perpendicular to the x-axis are squares.

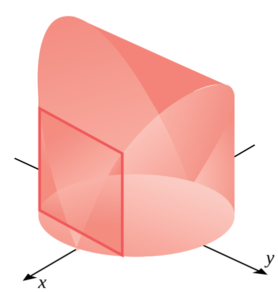
32. The base of a certain solid is the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 4$. Every cross section perpendicular to the x-axis is a semicircle with its diameter across the base. Find the volume of the solid.

33. Find the volume of the solid whose base is enclosed by the circle $x^2 + y^2 = 1$ and whose cross sections taken perpendicular to the base are

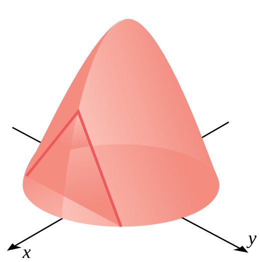
(a) semicircles

(b) squares





(c) equilateral triangles.



34. Derive the formula for the volume of a right circular cone with radius r and height h.

In Exercises 35 and 36, use a CAS to estimate the volume of the solid that results when the region enclosed by the curves is revolved about the stated axis.

C **35.** $y = \sin^8 x$, $y = 2x/\pi$, $x = 0$, $x = \pi/2$; x-axis

C **36.** $y = \pi^2 \sin x \cos^3 x$, $y = 4x^2$, $x = 0$, $x = \pi/4$; x-axis

37. The accompanying figure shows a ***spherical cap*** of radius ρ and height h cut from a sphere of radius r. Show that the volume V of the spherical cap can be expressed as
(a) $V = \frac{1}{3}\pi h^2(3r - h)$ (b) $V = \frac{1}{6}\pi h(3\rho^2 + h^2)$.

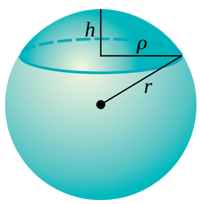


Figure Ex-37

38. If fluid enters a hemispherical bowl with a radius of 10 ft at a rate of $\frac{1}{2}$ ft^3/min, how fast will the fluid be rising when the depth is 5 ft? [*Hint:* See Exercise 37.]

39. The accompanying figure shows the dimensions of a small lightbulb at 10 equally spaced points.
(a) Use formulas from geometry to make a rough estimate of the volume enclosed by the glass portion of the bulb.
(b) Use the average of left and right endpoint approximations to approximate the volume.

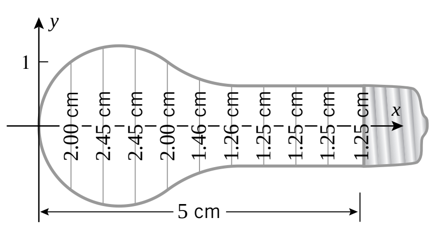


Figure Ex-39

40. Use the result in Exercise 37 to find the volume of the solid that remains when a hole of radius $r/2$ is drilled through the center of a sphere of radius r, and then check your answer by integrating.

41. As shown in the accompanying figure, a cocktail glass with a bowl shaped like a hemisphere of diameter 8 cm contains a cherry with a diameter of 2 cm. If the glass is filled to a depth of h cm, what is the volume of liquid it contains? [*Hint:* First consider the case where the cherry is partially submerged, then the case where it is totally submerged.]

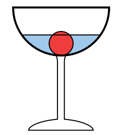


Figure Ex-41

42. Find the volume of the torus that results when the region enclosed by the circle of radius r with center at $(h, 0)$, $h > r$, is revolved about the y-axis. [*Hint:* Use an appropriate formula from plane geometry to help evaluate the definite integral.]

43. A wedge is cut from a right circular cylinder of radius r by two planes, one perpendicular to the axis of the cylinder and the other making an angle θ with the first. Find the volume of the wedge by slicing perpendicular to the y-axis as shown in the accompanying figure.

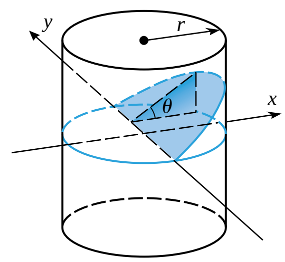


Figure Ex-43

44. Find the volume of the wedge described in Exercise 43 by slicing perpendicular to the x-axis.

45. Two right circular cylinders of radius r have axes that intersect at right angles. Find the volume of the solid common to the two cylinders. [*Hint:* One-eighth of the solid is sketched in the accompanying figure.]

46. In 1635 Bonaventura Cavalieri, a student of Galileo, stated the following result, called ***Cavalieri's principle***: *If two solids have the same height, and if the areas of their cross sections taken parallel to and at equal distances from their bases are always equal, then the solids have the same volume.* Use this result to find the volume of the oblique cylinder in the accompanying figure.

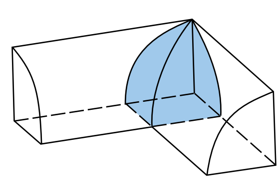


Figure Ex-45

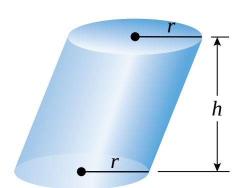


Figure Ex-46

6.3 VOLUMES BY CYLINDRICAL SHELLS

The methods for computing volumes that have been discussed so far depend on our ability to compute the cross-sectional area of the solid and to integrate that area across the solid. In this section we will develop another method for finding volumes that may be applicable when the cross-sectional area cannot be found or the integration is too difficult.

CYLINDRICAL SHELLS

In this section we will be interested in the following problem:

> **6.3.1** PROBLEM. Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Find the volume V of the solid of revolution S that is generated by revolving the region R about the y-axis (Figure 6.3.1).

Sometimes problems of this type can be solved by the method of disks or washers perpendicular to the y-axis, but when that method is not applicable or the resulting integral is difficult, the *method of cylindrical shells*, which we will discuss here, will often work.

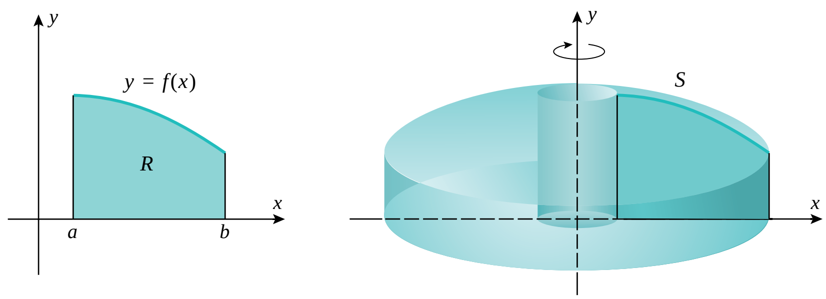


Figure 6.3.1

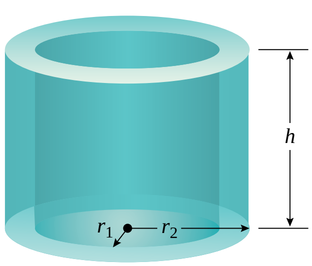


Figure 6.3.2

A ***cylindrical shell*** is a solid enclosed by two concentric right circular cylinders (Figure 6.3.2). The volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h can be written as

$$\begin{aligned} V &= [\text{area of cross section}] \cdot [\text{height}] = (\pi r_2^2 - \pi r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h = 2\pi \cdot \left[\tfrac{1}{2}(r_1 + r_2)\right] \cdot h \cdot (r_2 - r_1) \end{aligned}$$

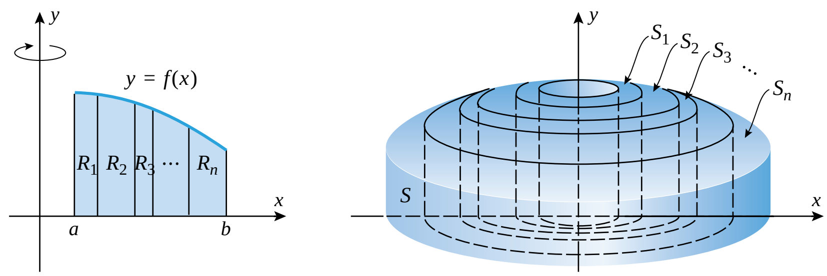
But $\frac{1}{2}(r_1 + r_2)$ is the average radius of the shell and $r_2 - r_1$ is its thickness, so

$$V = 2\pi \cdot [\text{average radius}] \cdot [\text{height}] \cdot [\text{thickness}] \tag{1}$$

We will now show how this formula can be used to solve the problem posed above. The underlying idea is to divide the interval $[a, b]$ into n subintervals, thereby subdividing the region R into n strips, $R_1, R_2, \ldots, R_n$ (Figure 6.3.3*a*). When the region R is revolved about the y-axis, these strips generate "tube-like" solids $S_1, S_2, \ldots, S_n$ that are nested one inside the other and together comprise the entire solid S (Figure 6.3.3*b*). Thus, the volume V of the solid can be obtained by adding together the volumes of the tubes; that is,

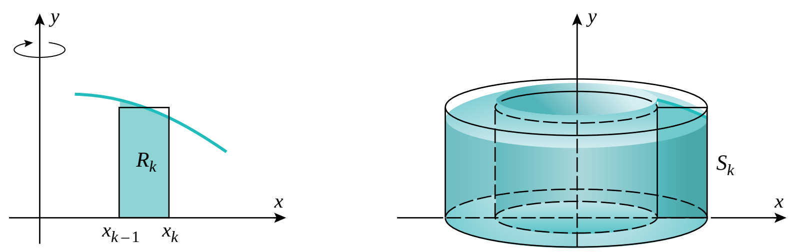
$$V = V(S_1) + V(S_2) + \cdots + V(S_n)$$

As a rule, the tubes will have curved upper surfaces, so there will be no simple formulas for their volumes. However, if the strips are thin, then we can approximate each strip by a rectangle (Figure 6.3.4*a*). These rectangles, when revolved about the y-axis, will produce cylindrical shells whose volumes closely approximate the volumes generated by the original strips (Figure 6.3.4*b*). We will show that by adding the volumes of the cylindrical shells we can obtain a Riemann sum that approximates the volume V, and by taking the limit of the Riemann sums we can obtain an integral for the exact volume V.



(*a*) (*b*)

Figure 6.3.3



(*a*) (*b*)

Figure 6.3.4

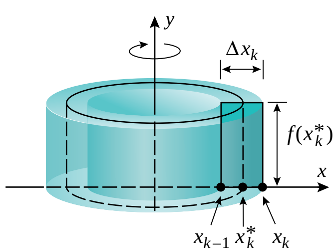


Figure 6.3.5

To implement this idea, suppose that the kth strip extends from x_{k-1} to x_k and that the width of this strip is

$$\Delta x_k = x_k - x_{k-1}$$

If we let x_k^* be the *midpoint* of the interval $[x_{k-1}, x_k]$, and if we construct a rectangle of height $f(x_k^*)$ over the interval, then revolving this rectangle about the y-axis produces a cylindrical shell of height $f(x_k^*)$, average radius x_k^*, and thickness Δx_k (Figure 6.3.5). From (1), the volume V_k of this cylindrical shell is

$$V_k = 2\pi x_k^* f(x_k^*)\Delta x_k$$

Adding the volumes of the n cylindrical shells yields the following Riemann sum that approximates the volume V:

$$V \approx \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*)\Delta x_k = \int_a^b 2\pi x f(x)\,dx$$

In summary, we have the following result.

6.3.2 VOLUME BY CYLINDRICAL SHELLS ABOUT THE ***y***-AXIS. Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution that is generated by revolving the region R about the y-axis is given by

$$V = \int_a^b 2\pi x f(x)\,dx \qquad (2)$$

Example 1 Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x-axis is revolved about the y-axis (Figure 6.3.6).

Solution. Since $f(x) = \sqrt{x}$, $a = 1$, and $b = 4$, Formula (2) yields

$$V = \int_1^4 2\pi x\sqrt{x}\,dx = 2\pi\int_1^4 x^{3/2}\,dx = \left[2\pi\cdot\frac{2}{5}x^{5/2}\right]_1^4 = \frac{4\pi}{5}[32-1] = \frac{124\pi}{5}$$ ◀

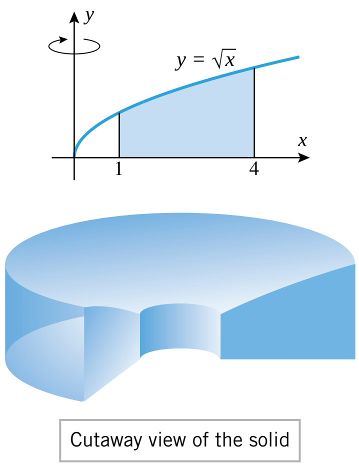


Figure 6.3.6

VARIATIONS OF THE METHOD OF CYLINDRICAL SHELLS

The method of cylindrical shells is applicable in a variety of situations that do not fit the conditions required by Formula (2). For example, the region may be enclosed between two curves, or the axis of revolution may be some line other than the y-axis. However, rather than develop a separate formula for every possible situation, we will give a general way of thinking about the method of cylindrical shells that can be adapted to each new situation as it arises.

For this purpose, we will need to reexamine the integrand in Formula (2): At each x in the interval $[a, b]$, the vertical line segment from the x-axis to the curve $y = f(x)$ can be viewed as the cross section of the region R at x (Figure 6.3.7a). When the region R is revolved about the y-axis, the cross section at x sweeps out the *surface* of a right circular cylinder of height $f(x)$ and radius x (Figure 6.3.7b). The area of this surface is

$$2\pi x f(x)$$

(Figure 6.3.7c), which is the integrand in (2). Thus, Formula (2) can be viewed informally in the following way.

> **6.3.3 AN INFORMAL VIEWPOINT ABOUT CYLINDRICAL SHELLS.** The volume V of a solid of revolution that is generated by revolving a region R about an axis can be obtained by integrating the area of the surface generated by an arbitrary cross section of R taken parallel to the axis of revolution.

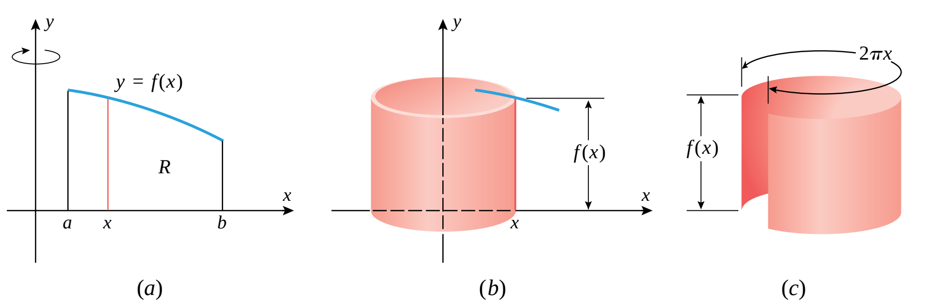


Figure 6.3.7

The following examples illustrate how to apply this result in situations where Formula (2) is not applicable.

Example 2 Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed between $y = x$ and $y = x^2$ is revolved about the y-axis (Figure 6.3.8).

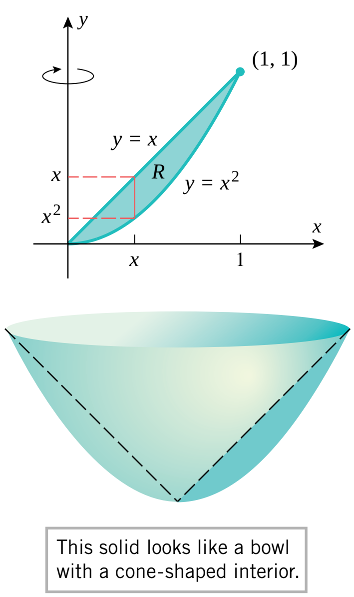


Figure 6.3.8

Solution. At each x in $[0, 1]$ the cross section of R parallel to the y-axis generates a cylindrical surface of height $x - x^2$ and radius x. Since the area of this surface is

$$2\pi x(x - x^2)$$

the volume of the solid is

$$V = \int_0^1 2\pi x(x-x^2)\,dx = 2\pi\int_0^1 (x^2-x^3)\,dx = 2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = 2\pi\left[\frac{1}{3}-\frac{1}{4}\right] = \frac{\pi}{6}$$ ◀

FOR THE READER. The volume in this example can also be obtained by the method of washers. Confirm that the volume produced by that method agrees with the volume obtained by cylindrical shells.

Example 3 Use cylindrical shells to find the volume of the solid generated when the region R under $y = x^2$ over the interval $[0, 2]$ is revolved about the x-axis (Figure 6.3.9).

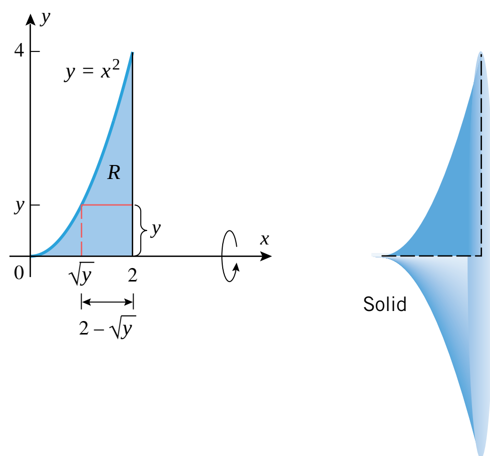


Figure 6.3.9

Solution. At each y in the interval $0 \le y \le 4$, the cross section of R parallel to the x-axis generates a cylindrical surface of height $2 - \sqrt{y}$ and radius y. Since the area of this surface is $2\pi y(2 - \sqrt{y}\,)$, the volume of the solid is

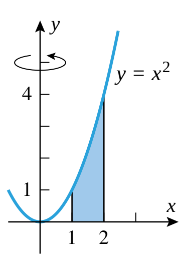
$$V = \int_0^4 2\pi y(2 - \sqrt{y}\,)\,dy = 2\pi \int_0^4 (2y - y^{3/2})\,dy = 2\pi\left[y^2 - \frac{2}{5}y^{5/2}\right]_0^4 = \frac{32\pi}{5} \quad ◀$$

FOR THE READER. The volume in this example can also be obtained by the method of disks. Confirm that the volume produced by that method agrees with the volume obtained by cylindrical shells.

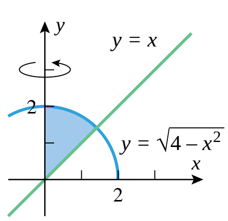
EXERCISE SET 6.3 C CAS

In Exercises 1–4, use cylindrical shells to find the volume of the solid generated when the shaded region is revolved about the indicated axis.

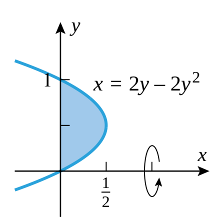
1.



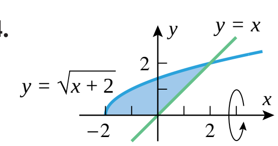
2.



3.



4.



In Exercises 5–10, use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the y-axis.

5. $y = x^3,\ x = 1,\ y = 0$

6. $y = \sqrt{x},\ x = 4,\ x = 9,\ y = 0$

7. $y = 1/x,\ y = 0,\ x = 1,\ x = 3$

8. $y = \cos(x^2),\ x = 0,\ x = \frac{1}{2}\sqrt{\pi},\ y = 0$

9. $y = 2x - 1,\ y = -2x + 3,\ x = 2$

10. $y = 2x - x^2,\ y = 0$

In Exercises 11–14, use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the x-axis.

11. $y^2 = x,\ y = 1,\ x = 0$

12. $x = 2y,\ y = 2,\ y = 3,\ x = 0$

13. $y = x^2,\ x = 1,\ y = 0$

14. $xy = 4,\ x + y = 5$

C **15.** Use a CAS to find the volume of the solid generated when the region enclosed by $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$ is revolved about the y-axis.

C **16.** Use a CAS to find the volume of the solid generated when the region enclosed by $y = \cos x$, $y = 0$, and $x = 0$ for $0 \le x \le \pi/2$ is revolved about the y-axis.

17. (a) Use cylindrical shells to find the volume of the solid that is generated when the region under the curve

$$y = x^3 - 3x^2 + 2x$$

over $[0, 1]$ is revolved about the y-axis.

(b) For this problem, is the method of cylindrical shells easier or harder than the method of slicing discussed in the last section? Explain.

18. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = 1/x^3$, $x = 1$, $x = 2$, $y = 0$ is revolved about the line $x = -1$.

19. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = x^3$, $y = 1$, $x = 0$ is revolved about the line $y = 1$.

20. Let R_1 and R_2 be regions of the form shown in the accompanying figure. Use cylindrical shells to find a formula for the volume of the solid that results when

(a) region R_1 is revolved about the y-axis

(b) region R_2 is revolved about the x-axis.

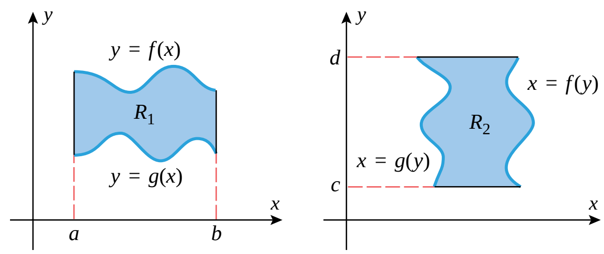


Figure Ex-20

21. Use cylindrical shells to find the volume of the cone generated when the triangle with vertices $(0, 0)$, $(0, r)$, $(h, 0)$, where $r > 0$ and $h > 0$, is revolved about the x-axis.

22. The region enclosed between the curve $y^2 = kx$ and the line $x = \frac{1}{4}k$ is revolved about the line $x = \frac{1}{2}k$. Use cylindrical shells to find the volume of the resulting solid. (Assume $k > 0$.)

23. A round hole of radius a is drilled through the center of a solid sphere of radius r. Use cylindrical shells to find the volume of the portion removed. (Assume $r > a$.)

24. Use cylindrical shells to find the volume of the torus obtained by revolving the circle $x^2 + y^2 = a^2$ about the line $x = b$, where $b > a > 0$. [*Hint:* It may help in the integration to think of an integral as an area.]

25. Let V_x and V_y be the volumes of the solids that result when the region enclosed by $y = 1/x$, $y = 0$, $x = \frac{1}{2}$, and $x = b$ $\left(b > \frac{1}{2}\right)$ is revolved about the x-axis and y-axis, respectively. Is there a value of b for which $V_x = V_y$?

6.4 LENGTH OF A PLANE CURVE

In this section we will consider the problem of finding the length of a plane curve.

ARC LENGTH

Although formulas for lengths of circular arcs appear in early historical records, very little was known about the lengths of more general curves until the mid-seventeenth century. About that time formulas were discovered for a few specific curves such as the length of an arch of a cycloid. However, such basic problems as finding the length of an ellipse defied the mathematicians of that period, and almost no progress was made on the general problem of finding lengths of curves until the advent of calculus in the next century.

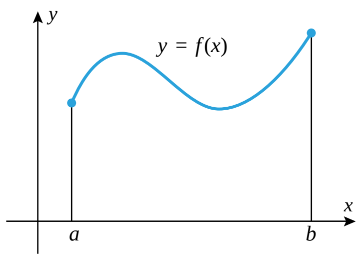


Figure 6.4.1

Our first objective in this section is to *define* what we mean by the length (also called the ***arc length***) of a plane curve $y = f(x)$ over an interval $[a, b]$ (Figure 6.4.1). Once that is done we will be able to focus on computational matters. To avoid some complications that would otherwise occur, we will impose the requirement that f' be continuous on $[a, b]$, in which case we will say that $y = f(x)$ is a ***smooth curve*** on $[a, b]$ (or that f is a ***smooth function*** on $[a, b]$).

We will be concerned with the following problem:

> **6.4.1** ARC LENGTH PROBLEM. Suppose that $y = f(x)$ is a smooth curve on the interval $[a, b]$. Define and find a formula for the arc length L of the curve $y = f(x)$ over the interval $[a, b]$.

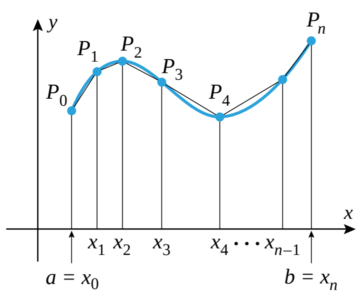


Figure 6.4.2

The basic idea for defining arc length is to break up the curve into small segments, approximate the curve segments by line segments, add the lengths of the line segments to form a Riemann sum that approximates the arc length L, and take the limit of the Riemann sums to obtain an integral for L.

To implement this idea, divide the interval $[a, b]$ into n subintervals by inserting numbers $x_1, x_2, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. As shown in Figure 6.4.2, let $P_0, P_1, \ldots, P_n$ be the points on the curve with x-coordinates $a = x_0, x_1, x_2, \ldots, x_{n-1}, b = x_n$ and join these points with straight line segments. These line segments form a ***polygonal path*** that we can regard as an approximation to the curve $y = f(x)$. As suggested by Figure 6.4.3, the length L_k of the kth line segment in the polygonal path is

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \tag{1}$$

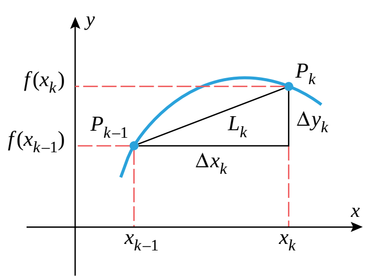


Figure 6.4.3

If we now add the lengths of these line segments, we obtain the following approximation to the length L of the curve

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \tag{2}$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a number x_k^* between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*) \quad \text{or} \quad f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$$

and hence we can rewrite (2) as

$$L \approx \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k$$

Thus, taking the limit as n increases and the widths of the subintervals approach zero yields the following integral that defines the arc length L:

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

In summary, we have the following definition:

> **6.4.2** DEFINITION. If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as
>
> $$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \tag{3}$$

This result provides both a definition and a formula for computing arc lengths. Where convenient, (3) can also be expressed as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{4}$$

Moreover, for a curve expressed in the form $x = g(y)$, where g' is continuous on $[c, d]$, the arc length L from $y = c$ to $y = d$ can be expressed as

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \tag{5}$$

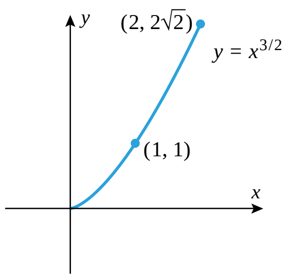


Figure 6.4.4

Example 1 Find the arc length of the curve $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$ (Figure 6.4.4) in two ways: (a) using Formula (4) and (b) using Formula (5).

Solution (a). Since

$$\frac{dy}{dx} = \tfrac{3}{2}x^{1/2}$$

and since the curve extends from $x = 1$ to $x = 2$, it follows from (4) that

$$L = \int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx$$

To evaluate this integral we make the u-substitution

$$u = 1 + \tfrac{9}{4}x, \quad du = \tfrac{9}{4}\, dx$$

and then change the x-limits of integration $(x = 1, x = 2)$ to the corresponding u-limits $\left(u = \frac{13}{4}, u = \frac{22}{4}\right)$:

$$L = \frac{4}{9}\int_{13/4}^{22/4} u^{1/2}\, du = \frac{8}{27}u^{3/2}\bigg]_{13/4}^{22/4} = \frac{8}{27}\left[\left(\frac{22}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

$$= \frac{22\sqrt{22} - 13\sqrt{13}}{27} \approx 2.09$$

Solution (b). To apply Formula (5) we must first rewrite the equation $y = x^{3/2}$ so that x is expressed as a function of y. This yields $x = y^{2/3}$ and

$$\frac{dx}{dy} = \tfrac{2}{3}y^{-1/3}$$

Since the curve extends from $y = 1$ to $y = 2\sqrt{2}$, it follows from (5) that

$$L = \int_1^{2\sqrt{2}} \sqrt{1 + \frac{4}{9}y^{-2/3}}\, dy = \frac{1}{3}\int_1^{2\sqrt{2}} y^{-1/3}\sqrt{9y^{2/3} + 4}\, dy$$

To evaluate this integral we make the u-substitution

$$u = 9y^{2/3} + 4, \quad du = 6y^{-1/3}\, dy$$

and change the y-limits of integration $(y = 1, y = 2\sqrt{2})$ to the corresponding u-limits $(u = 13, u = 22)$. This gives

$$L = \frac{1}{18}\int_{13}^{22} u^{1/2}\, du = \frac{1}{27}u^{3/2}\bigg]_{13}^{22} = \frac{1}{27}[(22)^{3/2} - (13)^{3/2}] = \frac{22\sqrt{22} - 13\sqrt{13}}{27}$$

This result agrees with that in part (a); however, the integration here is more tedious. In problems where there is a choice between using (4) or (5), it is often the case that one of the formulas leads to a simpler integral than the other. ◀

ARC LENGTH OF PARAMETRIC CURVES

The following result provides a formula for finding the arc length of a curve from parametric equations for the curve. Its derivation is similar to that of Formula (3) and will be omitted.

6.4.3 ARC LENGTH FORMULA FOR PARAMETRIC CURVES. If no segment of the curve represented by the parametric equations

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b)$$

is traced more than once as t increases from a to b, and if dx/dt and dy/dt are continuous functions for $a \le t \le b$, then the arc length L of the curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{6}$$

REMARK. Note that Formulas (4) and (5) are special cases of (6). For example, Formula (4) can be obtained from (6) by writing $y = f(x)$ parametrically as $x = t$, $y = f(t)$; similarly, Formula (5) can be obtained from (6) by writing $x = g(y)$ parametrically as $x = g(t)$, $y = t$. We leave the details as exercises.

Example 2 Use (6) to find the circumference of a circle of radius a from the parametric equations

$$x = a\cos t, \quad y = a\sin t \qquad (0 \le t \le 2\pi)$$

Solution.

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt$$
$$= \int_0^{2\pi} a\, dt = at\Big]_0^{2\pi} = 2\pi a$$ ◀

FINDING ARC LENGTH BY NUMERICAL METHODS

As a rule, the integrals that arise in calculating arc length tend to be impossible to evaluate in terms of elementary functions, so it will often be necessary to approximate the integral using a numerical method such as the midpoint approximation (discussed in Section 5.4) or some other comparable method. Examples 1 and 2 are rare exceptions.

Example 3 From (4), the arc length of $y = \sin x$ from $x = 0$ to $x = \pi$ is given by the integral

$$L = \int_0^{\pi} \sqrt{1 + (\cos x)^2}\, dx$$

This integral cannot be evaluated in terms of elementary functions; however, using a calculating utility with a numerical integration capability yields the approximation $L \approx 3.8202$. ◀

FOR THE READER. In Figure 6.4.5, the scale on both axes is 2 centimeters per unit. Confirm that the result in Example 3 is reasonable by laying a piece of string as closely as possible along the curve in the figure and measuring its length in centimeters.

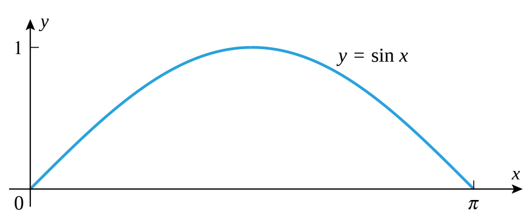


Figure 6.4.5

FOR THE READER. Computer algebra systems and some scientific calculators have commands for evaluating integrals numerically, and some scientific calculators have built-in commands for approximating arc lengths. If you have a scientific calculator with one of these capabilities or a CAS, read the documentation, and then use your calculator or CAS to check the result in Example 3.

EXERCISE SET 6.4 Graphing Calculator C CAS

1. Use the Theorem of Pythagoras to find the length of the line segment $y = 2x$ from (1, 2) to (2, 4), and confirm that the value is consistent with the length computed using
   (a) Formula (4) (b) Formula (5).
2. Use the Theorem of Pythagoras to find the length of the line segment $x = t, y = 5t$ $(0 \le t \le 1)$, and confirm that the value is consistent with the length computed using Formula (6).

In Exercises 3–8, find the exact arc length of the curve over the stated interval.

3. $y = 3x^{3/2} - 1$ from $x = 0$ to $x = 1$
4. $x = \frac{1}{3}(y^2 + 2)^{3/2}$ from $y = 0$ to $y = 1$
5. $y = x^{2/3}$ from $x = 1$ to $x = 8$
6. $y = (x^6 + 8)/(16x^2)$ from $x = 2$ to $x = 3$
7. $24xy = y^4 + 48$ from $y = 2$ to $y = 4$
8. $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ from $y = 1$ to $y = 4$

In Exercises 9–12, find the exact arc length of the parametric curve without eliminating the parameter.

9. $x = \frac{1}{3}t^3,\ \ y = \frac{1}{2}t^2 \quad (0 \le t \le 1)$
10. $x = (1+t)^2,\ \ y = (1+t)^3 \quad (0 \le t \le 1)$
11. $x = \cos 2t,\ \ y = \sin 2t \quad (0 \le t \le \pi/2)$
12. $x = \cos t + t\sin t,\ \ y = \sin t - t\cos t \quad (0 \le t \le \pi)$

C 13. (a) Recall from Section 1.8 that a cycloid is the path traced by a point on the rim of a wheel that rolls along a line (Figure 1.8.13). Use the parametric equations in Formula (9) of that section to show that the length L of one arch of a cycloid is given by the integral

$$L = a\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\,d\theta$$

(b) Use a CAS to show that L is eight times the radius of the wheel (see the accompanying figure).

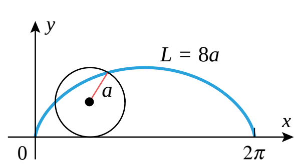


Figure Ex-13

14. It was stated in Exercise 41 of Section 1.8 that the curve given parametrically by the equations

$$x = a\cos^3\phi, \quad y = a\sin^3\phi$$

is called a *four-cusped hypocycloid* (also called an ***astroid***).
   (a) Use a graphing utility to generate the graph in the case where $a = 1$, so that it is traced exactly once.
   (b) Find the exact arc length of the curve in part (a).
15. Consider the curve $y = x^{2/3}$.
   (a) Sketch the portion of the curve between $x = -1$ and $x = 8$.
   (b) Explain why Formula (4) cannot be used to find the arc length of the curve sketched in part (a).
   (c) Find the arc length of the curve sketched in part (a).
16. Derive Formulas (4) and (5) from Formula (6) by choosing appropriate parametrizations of the curves.

In Exercises 17 and 18, use the midpoint approximation with $n = 20$ subintervals to approximate the arc length of the curve over the given interval.

17. $y = x^2$ from $x = 0$ to $x = 2$
18. $x = \sin y$ from $y = 0$ to $y = \pi$

C 19. Use a CAS or a scientific calculator with numerical integration capabilities to approximate the arc lengths in Exercises 17 and 18.
20. Let $y = f(x)$ be a smooth curve on the closed interval $[a, b]$. Prove that if there are nonnegative numbers m and M such that $m \le f'(x) \le M$ for all x in $[a, b]$, then the arc length L of $y = f(x)$ over the interval $[a, b]$ satisfies the inequalities

$$(b-a)\sqrt{1+m^2} \le L \le (b-a)\sqrt{1+M^2}$$

21. Use the result of Exercise 20 to show that the arc length L of $y = \sin x$ over the interval $0 \le x \le \pi/4$ satisfies

$$\frac{\pi}{4}\sqrt{\frac{3}{2}} \le L \le \frac{\pi}{4}\sqrt{2}$$

22. Show that the total arc length of the ellipse $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$ for $a > b > 0$ is given by

$$4a\int_0^{\pi/2} \sqrt{1 - k^2\cos^2 t}\,dt$$

where $k = \sqrt{a^2 - b^2}/a$.

C **23.** (a) Show that the total arc length of the ellipse

$$x = 2\cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

is given by

$$4\int_0^{\pi/2} \sqrt{1 + 3\sin^2 t}\, dt$$

(b) Use a CAS or a scientific calculator with numerical integration capabilities to approximate the arc length in part (a). Round your answer to two decimal places.

(c) Suppose that the parametric equations in part (a) describe the path of a particle moving in the xy-plane, where t is time in seconds and x and y are in centimeters. Use a CAS or a scientific calculator with numerical integration capabilities to approximate the distance traveled by the particle from $t = 1.5$ s to $t = 4.8$ s. Round your answer to two decimal places.

C **24.** A basketball player makes a successful shot from the free throw line. Suppose that the path of the ball from the moment of release to the moment it enters the hoop is described by

$$y = 2.15 + 2.09x - 0.41x^2, \quad 0 \le x \le 4.6$$

where x is the horizontal distance (in meters) from the point of release, and y is the vertical distance (in meters) above the floor. Use a CAS or a scientific calculator with numerical integration capabilities to approximate the distance the ball travels from the moment it is released to the moment it enters the hoop. Round your answer to two decimal places.

C **25.** Find a positive value of k (to two decimal places) such that the curve $y = k\sin x$ has an arc length of $L = 5$ units over the interval from $x = 0$ to $x = \pi$. [*Hint:* Find an integral for the arc length L in terms of k, and then use a CAS or a scientific calculator with a numeric integration capability to find integer values of k at which the values of $L - 5$ have opposite signs. Complete the solution by using the Intermediate-Value Theorem (2.5.8) to approximate the value of k to two decimal places.]

6.5 AREA OF A SURFACE OF REVOLUTION

In this section we will consider the problem of finding the area of a surface that is generated by revolving a plane curve about a line.

SURFACE AREA

A ***surface of revolution*** is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve. For example, the surface of a sphere can be generated by revolving a semicircle about its diameter, and the lateral surface of a right circular cylinder can be generated by revolving a line segment about an axis that is parallel to it (Figure 6.5.1).

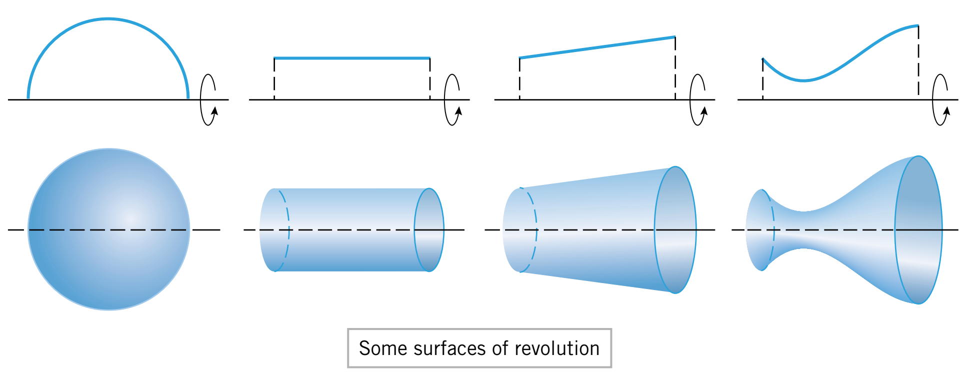


Figure 6.5.1

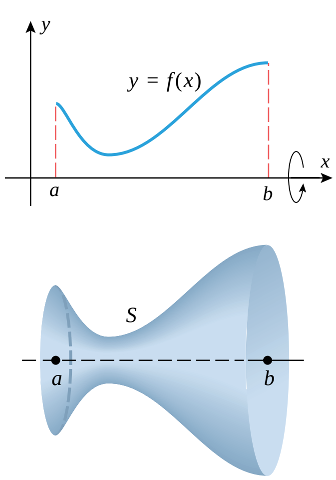


Figure 6.5.2

In this section we will be concerned with the following problem:

6.5.1 SURFACE AREA PROBLEM. Suppose that f is a smooth, nonnegative function on $[a, b]$ and that a surface of revolution is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis (Figure 6.5.2). Define what is meant by the *area* S of the surface, and find a formula for computing it.

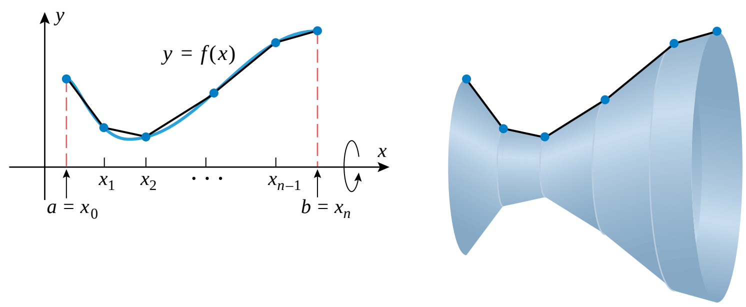


Figure 6.5.3

To motivate an appropriate definition for the area S of a surface of revolution, we will decompose the surface into small sections whose areas can be approximated by elementary formulas, add the approximations of the areas of the sections to form a Riemann sum that approximates S, and then take the limit of the Riemann sums to obtain an integral for the exact value of S.

To implement this idea, divide the interval $[a, b]$ into n subintervals by inserting numbers $x_1, x_2, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. As illustrated in Figure 6.5.3*a*, these points define a polygonal path that approximates the curve $y = f(x)$ over the interval $[a, b]$. When this polygonal path is revolved about the x-axis, it generates a surface consisting of n parts, each of which is a frustum of a right circular cone (Figure 6.5.3*b*). Thus, the area of each part of the approximating surface can be obtained from the formula

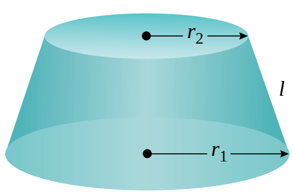


Figure 6.5.4

$$S = \pi(r_1 + r_2)l \tag{1}$$

for the lateral area S of a frustum of slant height l and base radii r_1 and r_2 (Figure 6.5.4). As suggested by Figure 6.5.5, the kth frustum has radii $f(x_{k-1})$ and $f(x_k)$ and height Δx_k. Its slant height is the length L_k of the kth line segment in the polygonal path, which from Formula (1) of Section 6.4 is

$$L_k = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

Thus, the lateral area S_k of the kth frustum is

$$S_k = \pi[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

If we add these areas, we obtain the following approximation to the area S of the entire surface:

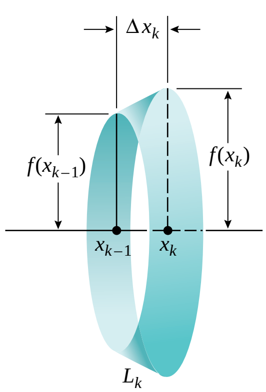


Figure 6.5.5

$$S \approx \sum_{k=1}^{n} \pi[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \tag{2}$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a number x_k^* between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*) \quad \text{or} \quad f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$$

and hence we can rewrite (2) as

$$S \approx \sum_{k=1}^{n} \pi[f(x_{k-1}) + f(x_k)]\sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k \tag{3}$$

However, this is not yet a Riemann sum because it involves the variables x_{k-1} and x_k. To eliminate these variables from the expression, observe that the average value of the numbers $f(x_{k-1})$ and $f(x_k)$ lies between these numbers, so the continuity of f and the Intermediate-Value Theorem (2.5.8) imply that there is a number x_k^{**} between x_{k-1} and x_k such that

$$\tfrac{1}{2}[f(x_{k-1}) + f(x_k)] = f(x_k^{**})$$

Thus, (2) can be expressed as

$$S \approx \sum_{k=1}^{n} 2\pi f(x_k^{**})\sqrt{1+[f'(x_k^*)]^2}\Delta x_k$$

Although this expression is close to a Riemann sum in form, it is not a true Riemann sum because it involves two variables x_k^* and x_k^{**}, rather than x_k^* alone. However, it is proved in advanced calculus courses that this has no effect on the limit because of the continuity of f. Thus, we can assume that $x_k^{**} = x_k^*$ when taking the limit, and this suggests that S can be defined as

$$S = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi f(x_k^{**})\sqrt{1+[f'(x_k^*)]^2}\Delta x_k = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

In summary, we have the following definition:

6.5.2 DEFINITION. If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is defined as

$$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

This result provides both a definition and a formula for computing surface areas. Where convenient, this formula can also be expressed as

$$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{4}$$

Moreover, if g is nonnegative and $x = g(y)$ is a smooth curve on the interval $[c, d]$, then the area of the surface that is generated by revolving the portion of a curve $x = g(y)$ between $y = c$ and $y = d$ about the y-axis can be expressed as

$$S = \int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \tag{5}$$

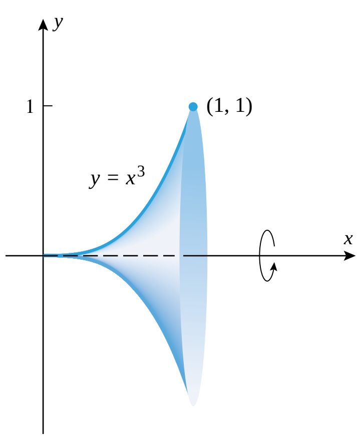


Figure 6.5.6

Example 1 Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis (Figure 6.5.6).

Solution. Since $y = x^3$, we have $dy/dx = 3x^2$, and hence from (4) the surface area S is

$$\begin{aligned} S &= \int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_0^1 2\pi x^3\sqrt{1+(3x^2)^2}\,dx \\ &= 2\pi\int_0^1 x^3(1+9x^4)^{1/2}\,dx \\ &= \frac{2\pi}{36}\int_1^{10} u^{1/2}\,du \qquad \begin{bmatrix} u = 1+9x^4 \\ du = 36x^3\,dx \end{bmatrix} \\ &= \frac{2\pi}{36}\cdot\frac{2}{3}u^{3/2}\Big]_{u=1}^{10} = \frac{\pi}{27}(10^{3/2}-1) \approx 3.56 \end{aligned}$$

◀

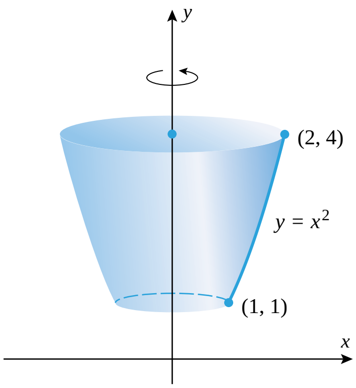


Figure 6.5.7

Example 2 Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis (Figure 6.5.7).

Solution. Because the curve is revolved about the y-axis we will apply Formula (5). Toward this end, we rewrite $y = x^2$ as $x = \sqrt{y}$ and observe that the y-values corresponding to $x = 1$ and $x = 2$ are $y = 1$ and $y = 4$. Since $x = \sqrt{y}$, we have $dx/dy = 1/(2\sqrt{y})$, and hence from (5) the surface area S is

$$\begin{aligned} S &= \int_1^4 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \\ &= \int_1^4 2\pi\sqrt{y}\sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\,dy \\ &= \pi\int_1^4 \sqrt{4y+1}\,dy \\ &= \frac{\pi}{4}\int_5^{17} u^{1/2}\,du \qquad \begin{bmatrix} u = 4y+1 \\ du = 4\,dy \end{bmatrix} \\ &= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big]_{u=5}^{17} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.85 \end{aligned}$$

◀

EXERCISE SET 6.5 C CAS

In Exercises 1–4, find the area of the surface generated by revolving the given curve about the x-axis.

1. $y = 7x,\ 0 \le x \le 1$

2. $y = \sqrt{x},\ 1 \le x \le 4$

3. $y = \sqrt{4 - x^2},\ -1 \le x \le 1$

4. $x = \sqrt[3]{y},\ 1 \le y \le 8$

In Exercises 5–8, find the area of the surface generated by revolving the given curve about the y-axis.

5. $x = 9y + 1,\ 0 \le y \le 2$

6. $x = y^3,\ 0 \le y \le 1$

7. $x = \sqrt{9 - y^2},\ -2 \le y \le 2$

8. $x = 2\sqrt{1 - y},\ -1 \le y \le 0$

In Exercises 9–12, use a CAS to find the exact area of the surface generated by revolving the curve about the stated axis.

C **9.** $y = \sqrt{x} - \frac{1}{3}x^{3/2},\ 1 \le x \le 3$; x-axis

C **10.** $y = \frac{1}{3}x^3 + \frac{1}{4}x^{-1},\ 1 \le x \le 2$; x-axis

C **11.** $8xy^2 = 2y^6 + 1,\ 1 \le y \le 2$; y-axis

C **12.** $x = \sqrt{16 - y},\ 0 \le y \le 15$; y-axis

In Exercises 13 and 14, use a CAS or a calculator with numerical integration capabilities to approximate the area of the surface generated by revolving the curve about the stated axis. Round your answer to two decimal places.

C **13.** $y = \sin x,\ 0 \le x \le \pi$; x-axis

C **14.** $x = \tan y,\ 0 \le y \le \pi/4$; y-axis

15. Use Formula (4) to show that the lateral area S of a right circular cone with height h and base radius r is

$$S = \pi r\sqrt{r^2 + h^2}$$

16. Show that the area of the surface of a sphere of radius r is $4\pi r^2$. [*Hint:* Revolve the semicircle $y = \sqrt{r^2 - x^2}$ about the x-axis.]

17. (a) The figure in Exercise 37 of Section 6.2 shows a spherical cap of height h cut from a sphere of radius r. Show that the surface area S of the cap is $S = 2\pi rh$. [*Hint:* Revolve an appropriate portion of the circle $x^2 + y^2 = r^2$ about the y-axis.]

(b) The portion of a sphere that is cut by two parallel planes is called a ***zone***. Use the result in part (a) to show that the surface area of a zone depends on the radius of the sphere and the distance between the planes, but not on the location of the zone.

Exercises 18–24 require the formulas developed in the following discussion: If $x'(t)$ and $y'(t)$ are continuous functions and if no segment of the curve

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b)$$

is traced more than once, then it can be shown that the area of the surface generated by revolving this curve about the x-axis is

$$S = \int_a^b 2\pi y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt \qquad \text{(A)}$$

and the area of the surface generated by revolving the curve about the y-axis is

$$S = \int_a^b 2\pi x(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt \qquad \text{(B)}$$

18. Derive Formulas (4) and (5) from Formulas (A) and (B) above by choosing appropriate parametrizations for the curves $y = f(x)$ and $x = g(y)$.

19. Find the area of the surface generated by revolving the parametric curve $x = t^2$, $y = 2t$, $0 \le t \le 4$ about the x-axis.

C **20.** Use a CAS to find the area of the surface generated by revolving the parametric curve

$$x = \cos^2 t, \quad y = 5\sin t \qquad 0 \le t \le \pi/2$$

about the x-axis.

21. Find the area of the surface generated by revolving the parametric curve $x = t$, $y = 2t^2$, $0 \le t \le 1$ about the y-axis.

22. Find the area of the surface generated by revolving the parametric curve $x = \cos^2 t$, $y = \sin^2 t$, $0 \le t \le \pi/2$ about the y-axis.

23. By revolving the semicircle

$$x = r\cos t, \quad y = r\sin t \qquad (0 \le t \le \pi)$$

about the x-axis, show that the surface area of a sphere of radius r is $4\pi r^2$.

24. The equations

$$x = a\phi - a\sin\phi, \quad y = a - a\cos\phi \qquad (0 \le \phi \le 2\pi)$$

represent one arch of a cycloid. Show that the surface area generated by revolving this curve about the x-axis is $S = 64\pi a^2/3$. [*Hint:* Use the identities $\sin^2\dfrac{\phi}{2} = \dfrac{1-\cos\phi}{2}$ and $\sin^3\phi = (1-\cos^2\phi)\sin\phi$ to help with the integration.]

25. (a) If a cone of slant height l and base radius r is cut along a lateral edge and laid flat, then as shown in the accompanying figure it becomes a sector of a circle of radius l. Use the formula $A = \frac{1}{2}l^2\theta$ for the area of a sector with radius l and central angle θ (in radians) to show that the lateral surface area of the cone is $\pi r l$.

(b) Use the result in part (a) to obtain Formula (1) for the lateral surface area of a frustum.

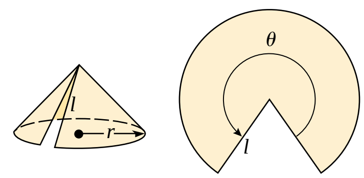


Figure Ex-25

26. Assume that $y = f(x)$ is a smooth curve on the interval $[a, b]$ and assume that $f(x) \ge 0$ for $a \le x \le b$. Derive a formula for the surface area generated when the curve $y = f(x)$, $a \le x \le b$, is revolved about the line $y = -k$ $(k > 0)$.

27. Let $y = f(x)$ be a smooth curve on the interval $[a, b]$ and assume that $f(x) \ge 0$ for $a \le x \le b$. By the Extreme-Value Theorem (4.5.3), the function f has a maximum value K and a minimum value k on $[a, b]$. Prove: If L is the arc length of the curve $y = f(x)$ between $x = a$ and $x = b$ and if S is the area of the surface that is generated by revolving this curve about the x-axis, then

$$2\pi kL \le S \le 2\pi KL$$

28. Let $y = f(x)$ be a smooth curve on $[a, b]$ and assume that $f(x) \ge 0$ for $a \le x \le b$. Let A be the area under the curve $y = f(x)$ between $x = a$ and $x = b$ and let S be the area of the surface obtained when this section of curve is revolved about the x-axis.

(a) Prove that $2\pi A \le S$.

(b) For what functions f is $2\pi A = S$?

6.6 WORK

In this section we will use the integration tools developed in the preceding chapter to study some of the basic principles of "work," which is one of the fundamental concepts in physics and engineering.

THE ROLE OF WORK IN PHYSICS AND ENGINEERING

In this section we will be concerned with two related concepts, *work* and *energy*. To put these ideas in a familiar setting, when you push a stalled car for a certain distance you are performing work, and the effect of your work is to make the car move. The energy of motion caused by the work is called the *kinetic energy* of the car. The exact connection between work and kinetic energy is governed by a principle of physics, called the *work–energy* *relationship*. Although we will touch on this idea in this section, a detailed study of the relationship between work and energy will be left for courses in physics and engineering. Our primary goal here will be to explain the role of integration in the study of work.

WORK DONE BY A CONSTANT FORCE APPLIED IN THE DIRECTION OF MOTION

When a stalled car is pushed, the speed that the car attains depends on the force F with which it is pushed and the distance d over which that force is applied (Figure 6.6.1). Thus, force and distance are the ingredients of work in the following definition.

6.6.1 DEFINITION. If a constant force of magnitude F is applied in the direction of motion of an object, and if that object moves a distance d, then we define the ***work*** W performed by the force on the object to be

$$W = F \cdot d \tag{1}$$

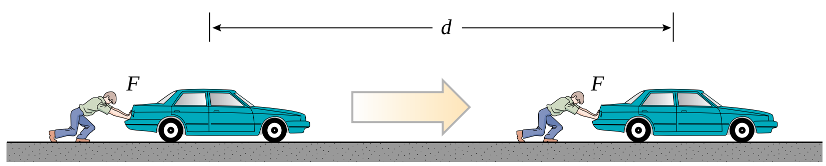


Figure 6.6.1

FOR THE READER. If you push against an immovable object, such as a brick wall, you may tire yourself out, but you will perform no work. Why?

Common units for measuring force are newtons (N) in the International System of Units (SI), dynes (dyn) in the centimeter-gram-second (CGS) system, and pounds (lb) in the British Engineering (BE) system. One newton is the force required to give a mass of 1 kg an acceleration of 1 m/s^2, one dyne is the force required to give a mass of 1 g an acceleration of 1 cm/s^2, and one pound of force is the force required to give a mass of 1 slug an acceleration of 1 ft/s^2.

It follows from Definition 6.6.1 that work has units of force times distance. The most common units of work are newton-meters (N·m), dyne-centimeters (dyn·cm), and foot-pounds (ft·lb). As indicated in Table 6.6.1, one newton-meter is also called a ***joule*** (J), and one dyne-centimeter is also called an ***erg***. One foot-pound is approximately 1.36 J.

Table 6.6.1

SYSTEM	FORCE	×	DISTANCE	=	WORK
SI	newton (N)		meter (m)		joule (J)
CGS	dyne (dyn)		centimeter (cm)		erg
BE	pound (lb)		foot (ft)		foot-pound (ft·lb)

CONVERSION FACTORS:
1 N = 10^5 dyn ≈ 0.225 lb — 1 lb ≈ 4.45 N
1 J = 10^7 erg ≈ 0.738 ft·lb — 1 ft·lb ≈ 1.36 J = 1.36 × 10^7 erg

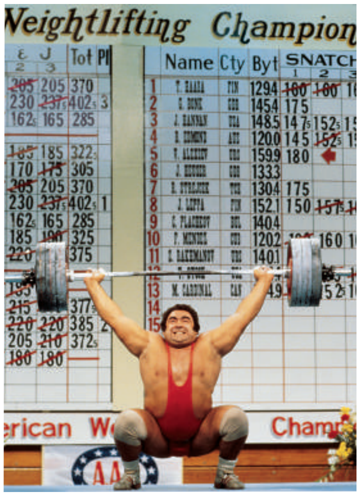
Example 1 An object moves 5 ft along a line while subjected to a constant force of 100 lb in its direction of motion. The work done is

$$W = F \cdot d = 100 \cdot 5 = 500 \text{ ft·lb}$$

An object moves 25 m along a line while subjected to a constant force of 4 N in its direction of motion. The work done is

$$W = F \cdot d = 4 \cdot 25 = 100 \text{ N·m} = 100 \text{ J}$$

◀



Vasili Alexeev lifting a record-breaking 562 lb in the 1976 Olympics

Example 2 In the 1976 Olympics, Vasili Alexeev astounded the world by lifting a record-breaking 562 lb from the floor to above his head (about 2 m). Equally astounding was the feat of strongman Paul Anderson, who in 1957 braced himself on the floor and used his back to lift 6270 lb of lead and automobile parts a distance of 1 cm. Who did more work?

Solution. To lift an object one must apply sufficient force to overcome the gravitational force that the Earth exerts on that object. The force that the Earth exerts on an object is that object's weight; thus, in performing their feats, Alexeev applied a force of 562 lb over a distance of 2 m and Anderson applied a force of 6270 lb over a distance of 1 cm. Pounds are units in the BE system, meters are units in SI, and centimeters are units in the CGS system, we will need to decide on the measurement system we want to use and be consistent. Let us agree to use SI and express the work of the two men in joules. Using the conversion factor in Table 6.6.1 we obtain

$$562\text{ lb} \approx 562\text{ lb} \times 4.45\text{ N/lb} = 2500.9\text{ N}$$

$$6270\text{ lb} \approx 6270\text{ lb} \times 4.45\text{ N/lb} = 27{,}901.5\text{ N}$$

Using these values and the fact that 1 cm = 0.01 m we obtain

$$\text{Alexeev's work} = (2500.9\text{ N}) \times (2\text{ m}) \approx 5002\text{ J}$$

$$\text{Anderson's work} = (27{,}901.5\text{ N}) \times (0.01\text{ m}) \approx 279\text{ J}$$

Therefore, even though Anderson's lift required a tremendous upward force, it was applied over such a short distance that Alexeev did more work. ◀

WORK DONE BY A VARIABLE FORCE APPLIED IN THE DIRECTION OF MOTION

Many important problems are concerned with finding the work done by a *variable* force that is applied in the direction of motion. For example, Figure 6.6.2*a* shows a spring in its natural state (neither compressed nor stretched). If we want to pull the block horizontally (Figure 6.6.2*b*), then we would have to apply more and more force to the block to overcome the increasing force of the stretching spring. Thus, our next objective is to define what is meant by the work performed by a variable force and to find a formula for computing it. This will require calculus.

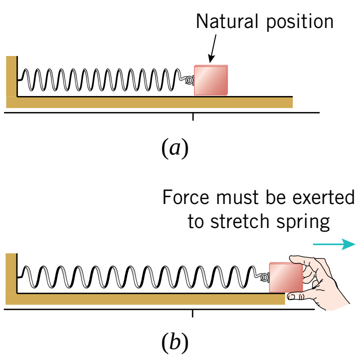


Figure 6.6.2

> **6.6.2** PROBLEM. Suppose that an object moves in the positive direction along a coordinate line while subjected to a variable force $F(x)$ that is applied in the direction of motion. Define what is meant by the *work* W performed by the force on the object as the object moves from $x = a$ to $x = b$, and find a formula for computing the work.

The basic idea for solving this problem is to break up the interval $[a, b]$ into subintervals that are sufficiently small that the force does not vary much on each subinterval. This will allow us to treat the force as constant on each subinterval and to approximate the work on each subinterval using Formula (1). By adding the approximations to the work on the subintervals, we will obtain a Riemann sum that approximates the work W over the entire interval, and by taking the limit of the Riemann sums we will obtain an integral for W.

To implement this idea, divide the interval $[a, b]$ into n subintervals by inserting numbers $x_1, x_2, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. We can use Formula (1) to approximate the work W_k done in the kth subinterval by choosing any number x_k^* in this interval and regarding the force to have a constant value $F(x_k^*)$ throughout the interval. Since the width of the kth subinterval is $x_k - x_{k-1} = \Delta x_k$, this yields the approximation

$$W_k \approx F(x_k^*)\Delta x_k$$

Adding these approximations yields the following Riemann sum that approximates the work W done over the entire interval:

$$W \approx \sum_{k=1}^{n} F(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of the subintervals approach zero yields the definite integral

$$W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} F(x_k^*)\Delta x_k = \int_a^b F(x)\,dx$$

In summary, we have the following result:

6.6.3 DEFINITION. Suppose that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a variable force $F(x)$ that is applied in the direction of motion. Then we define the ***work*** W performed by the force on the object to be

$$W = \int_a^b F(x)\,dx \tag{2}$$

Hooke's law [Robert Hooke (1635–1703), English physicist] states that under appropriate conditions a spring that is stretched x units beyond its natural length pulls back with a force

$$F(x) = kx$$

where k is a constant (called the ***spring constant*** or ***spring stiffness***). The value of k depends on such factors as the thickness of the spring and the material used in its composition. Since $k = F(x)/x$, the constant k has units of force per unit length.

Example 3 A spring exerts a force of 5 N when stretched 1 m beyond its natural length.

(a) Find the spring constant k.

(b) How much work is required to stretch the spring 1.8 m beyond its natural length?

Solution (a). From Hooke's law,

$$F(x) = kx$$

From the data, $F(x) = 5$ N when $x = 1$ m, so $5 = k \cdot 1$. Thus, the spring constant is $k = 5$ newtons per meter (N/m). This means that the force $F(x)$ required to stretch the spring x meters is

$$F(x) = 5x \tag{3}$$

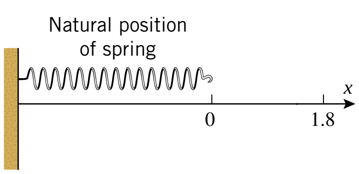


Figure 6.6.3

Solution (b). Place the spring along a coordinate line as shown in Figure 6.6.3. We want to find the work W required to stretch the spring over the interval from $x = 0$ to $x = 1.8$. From (2) and (3) the work W required is

$$W = \int_a^b F(x)\,dx = \int_0^{1.8} 5x\,dx = \frac{5x^2}{2}\bigg]_0^{1.8} = 8.1 \text{ J}$$ ◀

Example 4 An astronaut's *weight* (or more precisely, *Earth weight*) is the force exerted on the astronaut by the Earth's gravity. As the astronaut moves upward into space, the gravitational pull of the Earth decreases, and hence so does his or her weight. We will show later in the text that if the Earth is assumed to be a sphere of radius 4000 mi, then an astronaut who weighs 150 lb on Earth will have a weight of

$$w(x) = \frac{2{,}400{,}000{,}000}{x^2} \text{ lb}, \quad x \geq 4000$$

at a distance of x mi from the Earth's center. Use this formula to determine the work in foot-pounds required to lift the astronaut to a point that is 800 mi above the surface of the Earth (Figure 6.6.4).

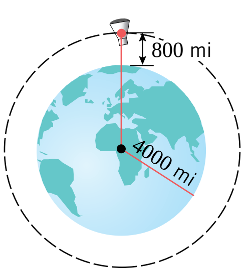


Figure 6.6.4

Solution. Since the Earth has a radius of 4000 mi, the astronaut is lifted from a point that is 4000 mi from the Earth's center to a point that is 4800 mi from the Earth's center. Thus, from (2), the work W required to lift the astronaut is

$$
\begin{aligned}
W &= \int_{4000}^{4800} \frac{2{,}400{,}000{,}000}{x^2}\,dx \\
&= -\frac{2{,}400{,}000{,}000}{x}\Bigg]_{4000}^{4800} \\
&= -500{,}000 + 600{,}000 \\
&= 100{,}000 \text{ mile-pounds} \\
&= (100{,}000 \text{ mi}\cdot\text{lb}) \times (5280 \text{ ft/mi}) \\
&= 5.28 \times 10^8 \text{ ft}\cdot\text{lb}
\end{aligned}
$$

◀

CALCULATING WORK FROM BASIC PRINCIPLES

Some problems cannot be solved by mechanically substituting into formulas, and one must return to basic principles to obtain solutions. This is illustrated in the next example.

Example 5 A conical water tank of radius 10 ft and height 30 ft is filled with water to a depth of 15 ft (Figure 6.6.5*a*). How much work is required to pump all of the water out through a hole in the top of the tank?

Solution. Our strategy will be to divide the water into thin layers, approximate the work required to move each layer to the top of the tank, add the approximations for the layers to obtain a Riemann sum that approximates the total work, and then take the limit of the Riemann sums to produce an integral for the total work.

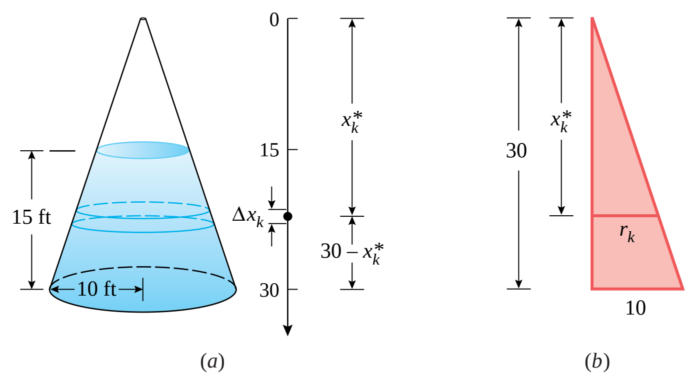


Figure 6.6.5

To implement this idea, introduce an x-axis as shown in Figure 6.6.5*a*, and divide the water into n layers with Δx_k denoting the thickness of the kth layer. This division induces a partition of the interval [15, 30] into n subintervals. Although the upper and lower surfaces of the kth layer are at different distances from the top, the difference will be small if the layer is thin, and we can reasonably assume that the entire layer is concentrated at a single point x_k^* (Figure 6.6.5*a*). Thus, the work W_k required to move the kth layer to the top of the tank is approximately

$$W_k \approx F_k x_k^* \tag{4}$$

where F_k is the force required to lift the kth layer. But the force required to lift the kth layer is the force needed to overcome gravity, and this is the same as the weight of the layer. If the layer is very thin, we can approximate the volume of the kth layer with the volume of a cylinder of height Δx_k and radius r_k, where (by similar triangles)

$$\frac{r_k}{x_k^*} = \frac{10}{30} = \frac{1}{3}$$

or equivalently, $r_k = x_k^*/3$ (Figure 6.6.5*b*). Therefore, the volume of the kth layer of water is approximately

$$\pi r_k^2 \Delta x_k = \pi (x_k^*/3)^2 \Delta x_k = \frac{\pi}{9}(x_k^*)^2 \Delta x_k$$

Since the weight density of water is 62.4 lb/ft^3, it follows that

$$F_k \approx \frac{62.4\pi}{9}(x_k^*)^2 \Delta x_k$$

Thus, from (4)

$$W_k \approx \left(\frac{62.4\pi}{9}(x_k^*)^2 \Delta x_k\right) x_k^* = \frac{62.4\pi}{9}(x_k^*)^3 \Delta x_k$$

and hence the work W required to move all n layers has the approximation

$$W = \sum_{k=1}^{n} W_k \approx \sum_{k=1}^{n} \frac{62.4\pi}{9}(x_k^*)^3 \Delta x_k$$

To find the *exact* value of the work we take the limit as $\max \Delta x_k \to 0$. This yields

$$W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \frac{62.4\pi}{9}(x_k^*)^3 \Delta x_k = \int_{15}^{30} \frac{62.4\pi}{9} x^3\, dx$$

$$= \frac{62.4\pi}{9}\left(\frac{x^4}{4}\right)\Bigg]_{15}^{30} = 1{,}316{,}250\pi \approx 4{,}135{,}000 \text{ ft·lb} \quad ◀$$

THE WORK–ENERGY RELATIONSHIP

When you see an object in motion, you can be certain that somehow work has been expended to create that motion. For example, when you drop a stone from a building, the stone gathers speed because the force of the Earth's gravity is performing work on it, and when a hockey player strikes a puck with a hockey stick, the work performed on the puck during the brief period of contact with the stick creates the enormous speed of the puck across the ice. However, experience shows that the speed obtained by an object depends not only on the amount of work done, but also on the mass of the object. For example, the work required to throw a 5-oz baseball 50 mi/h would accelerate a 10-lb bowling ball to less than 9 mi/h.

Using the method of substitution for definite integrals, we will derive a simple equation that relates the work done on an object to the object's mass and velocity. Furthermore, this equation will allow us to motivate an appropriate definition for the "energy of motion" of an object. As in Definition 6.6.3, we will assume that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a force $F(x)$ that is applied in the direction of motion. We let $x = x(t)$, $v = v(t) = x'(t)$, and $v'(t)$ denote the respective position, velocity, and acceleration of the object at time t. It follows from Newton's Second Law of Motion that

$$F(x(t)) = mv'(t)$$

where m is the mass of the object. Assume that

$$x(t_0) = a \quad \text{and} \quad x(t_1) = b$$

with

$$v(t_0) = v_i \quad \text{and} \quad v(t_1) = v_f$$

the initial and final velocities of the object, respectively. Then

$$W = \int_a^b F(x)\, dx = \int_{x(t_0)}^{x(t_1)} F(x)\, dx$$

$$= \int_{t_0}^{t_1} F(x(t))x'(t)\, dt \qquad \text{By Theorem 5.8.1 with } x = x(t),\ dx = x'(t)\, dt$$

$$= \int_{t_0}^{t_1} mv'(t)v(t)\, dt = \int_{t_0}^{t_1} mv(t)v'(t)\, dt$$

$$= \int_{v(t_0)}^{v(t_1)} mv\, dv \qquad \text{By Theorem 5.8.1 with } v = v(t),\ dv = v'(t)\, dt$$

$$= \int_{v_i}^{v_f} mv\, dv = \tfrac{1}{2}mv^2\Big|_{v_i}^{v_f} = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

We see from the equation

$$W = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 \tag{5}$$

that the work done on the object is equal to the change in the quantity $\frac{1}{2}mv^2$ from its initial value to its final value. We will refer to Equation (5) as the ***work-energy relationship***. If we define the "energy of motion" or ***kinetic energy*** of our object to be given by

$$K = \tfrac{1}{2}mv^2 \tag{6}$$

then Equation (5) tells us that the work done on an object is equal to the *change* in the object's kinetic energy. Loosely speaking, we may think of work done on an object as being "transformed" into kinetic energy of the object. The units of kinetic energy are the same as the units of work. For example, in SI kinetic energy is measured in joules (J).

Example 6 A space probe of mass $m = 5.00 \times 10^4$ kg travels in deep space subjected only to the force of its own engine. Starting at a time when the speed of the probe is $v = 1.10 \times 10^4$ m/s, the engine is fired continuously over a distance of 2.50×10^6 m with a constant force of 4.00×10^5 N in the direction of motion. What is the final speed of the probe?

Solution. Since the force applied by the engine is constant and in the direction of motion, the work W expended by the engine on the probe is

$$W = \text{force} \times \text{distance} = (4.00 \times 10^5 \text{ N}) \times (2.50 \times 10^6 \text{ m}) = 1.00 \times 10^{12} \text{ J}$$

From (5), the final kinetic energy $K_f = \frac{1}{2}mv_f^2$ of the probe can be expressed in terms of the work W and the initial kinetic energy $K_i = \frac{1}{2}mv_i^2$ as

$$K_f = W + K_i$$

Thus, from the known mass and initial speed we have

$$K_f = (1.00 \times 10^{12} \text{ J}) + \tfrac{1}{2}(5.00 \times 10^4 \text{ kg})(1.10 \times 10^4 \text{ m/s})^2 = 4.025 \times 10^{12} \text{ J}$$

The final kinetic energy is $K_f = \frac{1}{2}mv_f^2$, so the final speed of the probe is

$$v_f = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2(4.025 \times 10^{12})}{5.00 \times 10^4}} \approx 1.27 \times 10^4 \text{ m/s}$$ ◀

EXERCISE SET 6.6

1. Find the work done when
(a) a constant force of 30 lb in the positive x-direction moves an object from $x = -2$ to $x = 5$ ft
(b) a variable force of $F(x) = 1/x^2$ lb in the positive x-direction moves an object from $x = 1$ to $x = 6$ ft.

2. A variable force $F(x)$ in the positive x-direction is graphed in the accompanying figure. Find the work done by the force on a particle that moves from $x = 0$ to $x = 5$.

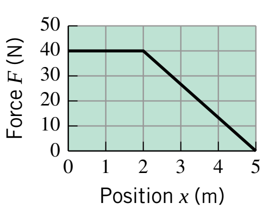


Figure Ex-2

3. A constant force of 10 lb in the positive x-direction is applied to a particle whose velocity versus time curve is shown in the accompanying figure. Find the work done by the force on the particle from time $t = 0$ to $t = 5$.

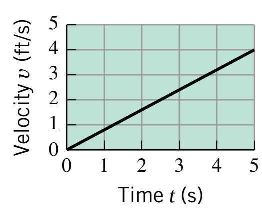


Figure Ex-3

4. A spring whose natural length is 15 cm exerts a force of 45 N when stretched to a length of 20 cm.
(a) Find the spring constant (in newtons/meter).
(b) Find the work that is done in stretching the spring 3 cm beyond its natural length.
(c) Find the work done in stretching the spring from a length of 20 cm to a length of 25 cm.

5. A spring exerts a force of 100 N when it is stretched 0.2 m beyond its natural length. How much work is required to stretch the spring 0.8 m beyond its natural length?

6. Assume that a force of 6 N is required to compress a spring from a natural length of 4 m to a length of $3\frac{1}{2}$ m. Find the work required to compress the spring from its natural length to a length of 2 m. (Hooke's law applies to compression as well as extension.)

7. Assume that 10 ft·lb of work is required to stretch a spring 1 ft beyond its natural length. What is the spring constant?

8. A cylindrical tank of radius 5 ft and height 9 ft is two-thirds filled with water. Find the work required to pump all the water over the upper rim.

9. Solve Exercise 8 assuming that the tank is two-thirds filled with a liquid that weighs ρ lb/ft^3.

10. A cone-shaped water reservoir is 20 ft in diameter across the top and 15 ft deep. If the reservoir is filled to a depth of 10 ft, how much work is required to pump all the water to the top of the reservoir?

11. The vat shown in the accompanying figure contains water to a depth of 2 m. Find the work required to pump all the water to the top of the vat. [Use 9810 N/m^3 as the weight density of water.]

12. The cylindrical tank shown in the accompanying figure is filled with a liquid weighing 50 lb/ft^3. Find the work required to pump all the liquid to a level 1 ft above the top of the tank.

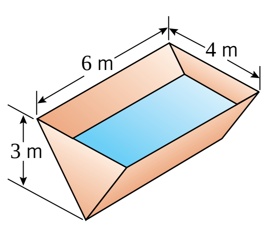


Figure Ex-11

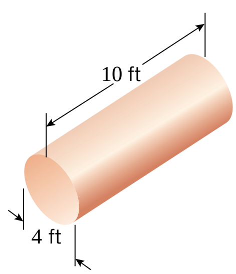


Figure Ex-12

13. A swimming pool is built in the shape of a rectangular parallelepiped 10 ft deep, 15 ft wide, and 20 ft long.
(a) If the pool is filled to 1 ft below the top, how much work is required to pump all the water into a drain at the top edge of the pool?
(b) A one-horsepower motor can do 550 ft·lb of work per second. What size motor is required to empty the pool in 1 hour?

14. How much work is required to fill the swimming pool in Exercise 13 to 1 ft below the top if the water is pumped in through an opening located at the bottom of the pool?

15. A 100-ft length of steel chain weighing 15 lb/ft is dangling from a pulley. How much work is required to wind the chain onto the pulley?

16. A 3-lb bucket containing 20 lb of water is hanging at the end of a 20-ft rope that weighs 4 oz/ft. The other end of the rope is attached to a pulley. How much work is required to wind the length of rope onto the pulley, assuming that the rope is wound onto the pulley at a rate of 2 ft/s and that as the bucket is being lifted, water leaks from the bucket at a rate of 0.5 lb/s?

17. A rocket weighing 3 tons is filled with 40 tons of liquid fuel. In the initial part of the flight, fuel is burned off at a constant rate of 2 tons per 1000 ft of vertical height. How much work is done in lifting the rocket to 3000 ft?

18. It follows from Coulomb's law in physics that two like electrostatic charges repel each other with a force inversely proportional to the square of the distance between them. Suppose that two charges A and B repel with a force of k newtons when they are positioned at points $A(-a, 0)$ and $B(a, 0)$, where a is measured in meters. Find the work W required to move charge A along the x-axis to the origin if charge B remains stationary.

19. It is a law of physics that the gravitational force exerted by the Earth on an object varies inversely as the square of its distance from the Earth's center. Thus, an object's weight $w(x)$ is related to its distance x from the Earth's center by a formula of the form

$$w(x) = \frac{k}{x^2}$$

where k is a constant of proportionality that depends on the mass of the object.
(a) Use this fact and the assumption that the Earth is a sphere of radius 4000 mi to obtain the formula for $w(x)$ in Example 4.
(b) Find a formula for the weight $w(x)$ of a satellite that is x mi from the Earth's surface if its weight on Earth is 6000 lb.
(c) How much work is required to lift the satellite from the surface of the Earth to an orbital position that is 1000 mi high?

20. (a) The formula $w(x) = k/x^2$ in Exercise 19 is applicable to all celestial bodies. Assuming that the Moon is a sphere of radius 1080 mi, find the force that the Moon exerts on an astronaut who is x mi from the surface of the Moon if her weight on the Moon's surface is 20 lb.
(b) How much work is required to lift the astronaut to a point that is 10.8 mi above the Moon's surface?

21. The Yamanashi Maglev Test Line in Japan that runs between Sakaigawa and Akiyama is currently testing magnetic levitation (MAGLEV) trains that are designed to levitate inches above powerful magnetic fields. Suppose that a MAGLEV train has a mass of $m = 4.00 \times 10^5$ kg and that starting at a time when the train has a speed of 20 m/s the engine applies a force of 6.40×10^5 N in the direction of motion over a distance of 3.00×10^3 m. Use the work–energy relationship (5) to find the final speed of the train.

22. Assume that a Mars probe of mass $m = 2.00 \times 10^3$ kg is subjected only to the force of its own engine. Starting at a time when the speed of the probe is $v = 1.00 \times 10^4$ m/s, the engine is fired continuously over a distance of 2.00×10^5 m with a constant force of 2.00×10^5 N in the direction of motion. Use the work–energy relationship (5) to find the final speed of the probe.

23. On August 10, 1972 a meteorite with an estimated mass of 4×10^6 kg and an estimated speed of 15 km/s skipped across the atmosphere above the western United States and Canada but fortunately did not hit the Earth.

(a) Assuming that the meteorite had hit the Earth with a speed of 15 km/s, what would have been its change in kinetic energy in joules (J)?

(b) Express the energy as a multiple of the explosive energy of 1 megaton of TNT, which is 4.2×10^{15} J.

(c) The energy associated with the Hiroshima atomic bomb was 13 kilotons of TNT. To how many such bombs would the meteorite impact have been equivalent?

6.7 FLUID PRESSURE AND FORCE

In this section we will use the integration tools developed in the preceding chapter to study the pressures and forces exerted by fluids on submerged objects.

WHAT IS A FLUID?

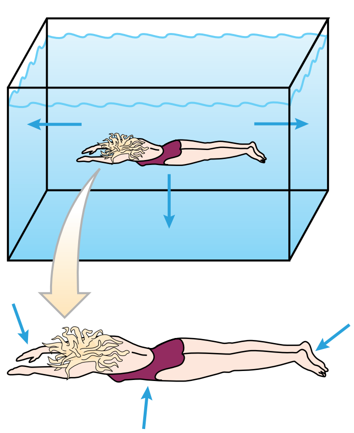
A ***fluid*** is a substance that flows to conform to the boundaries of any container in which it is placed. Fluids include *liquids*, such as water, oil, and mercury, as well as *gases*, such as helium, oxygen, and air. The study of fluids falls into two categories: *fluid statics* (the study of fluids at rest) and *fluid dynamics* (the study of fluids in motion). In this section we will be concerned only with fluid statics; toward the end of this text we will investigate problems in fluid dynamics.

THE CONCEPT OF PRESSURE

The effect that a force has on an object depends on how that force is spread over the surface of the object. For example, when you walk on soft snow with boots, the weight of your body crushes the snow and you sink into it. However, if you put on a pair of skis to spread the weight of your body over a greater surface area, then the weight of your body has less of a crushing effect on the snow, and you are able to glide across the surface. The concept that accounts for both the magnitude of a force and the area over which it is applied is called *pressure*.

6.7.1 DEFINITION. If a force of magnitude F is applied to a surface of area A, then we define the ***pressure*** P exerted by the force on the surface to be

$$P = \frac{F}{A} \tag{1}$$



Fluid forces always act perpendicular to the surface of a submerged object.

Figure 6.7.1

It follows from this definition that pressure has units of force per unit area. The most common units of pressure are newtons per square meter (N/m^2) in SI and pounds per square inch (lb/in^2) or pounds per square foot (lb/ft^2) in the BE system. As indicated in Table 6.7.1, one newton per square meter is called a *pascal** (see page 436) (Pa). A pressure of 1 Pa is quite small ($1 \text{ Pa} = 1.45 \times 10^{-4} \text{ lb/in}^2$), so in countries using SI, tire pressure gauges are usually calibrated in kilopascals (kPa), which is 1000 pascals.

In this section we will be interested in pressures and forces on objects submerged in fluids. Pressures themselves have no directional characteristics, but the forces that they create always act perpendicular to the face of the submerged object. Thus, in Figure 6.7.1 the water pressure creates horizontal forces on the sides of the tank, vertical forces on the bottom of the tank, and forces that vary in direction, so as to be perpendicular to the different parts of the swimmer's body.

Table 6.7.1

SYSTEM	FORCE ÷	AREA =	PRESSURE
SI	newton (N)	square meter (m^2)	pascal (Pa)
BE	pound (lb)	square foot (ft^2)	lb/ft^2
BE	pound (lb)	square inch (in^2)	lb/in^2 (psi)

CONVERSION FACTORS:
1 Pa $\approx 1.45 \times 10^{-4}$ lb/in^2 $\approx 2.09 \times 10^{-2}$ lb/ft^2
1 $lb/in^2 \approx 6.89 \times 10^3$ Pa   1 $lb/ft^2 \approx 47.9$ Pa

Example 1 Referring to Figure 6.7.1, suppose that the back of the swimmer's hand has a surface area of 8.4×10^{-3} m^2 and that the pressure acting on it is 5.1×10^4 Pa (a realistic value near the bottom of a deep diving pool). Find the force that acts on the swimmer's hand.

Solution. From (1), the force F is

$$F = PA = (5.1 \times 10^4 \text{ N/m}^2)(8.4 \times 10^{-3} \text{ m}^2) \approx 4.3 \times 10^2 \text{ N}$$

This is quite a large force (nearly 100 lb in the BE system). ◀

FLUID DENSITY

Scuba divers know that the deeper they dive, the greater the pressure and the forces that they feel on their bodies. This sense of pressure and force is caused by the weight of the water and air above—the deeper the diver goes, the greater the weight above and hence the greater the pressure and force that he or she feels.

To calculate pressures and forces on submerged objects, we need to know something about the characteristics of the fluids in which they are submerged. For simplicity, we will assume that the fluids under consideration are *homogeneous*, by which we mean that any two samples of the fluid with the same volume have the same mass. It follows from this assumption that the mass per unit volume is a constant δ that depends on the physical characteristics of the fluid but not on the size or location of the sample; we call

$$\delta = \frac{m}{V} \tag{2}$$

the ***mass density*** of the fluid. Sometimes it is more convenient to work with weight per unit volume than with mass per unit volume. Thus, we define the ***weight density*** ρ of a fluid to be

$$\rho = \frac{w}{V} \tag{3}$$

where w is the weight of a fluid sample of volume V. Thus, if the weight density of a fluid is known, then the weight w of a fluid sample of volume V can be computed from the formula $w = \rho V$. Table 6.7.2 shows some typical weight densities.

Table 6.7.2

WEIGHT DENSITIES	
SI	N/m^3
Machine oil	4,708
Gasoline	6,602
Fresh water	9,810
Seawater	10,045
Mercury	133,416
BE SYSTEM	lb/ft^3
Machine oil	30.0
Gasoline	42.0
Fresh water	62.4
Seawater	64.0
Mercury	849.0

All densities are affected by variations in temperature and pressure. Weight densities are also affected by variations in g.

* BLAISE PASCAL (1623–1662). French mathematician and scientist. Pascal's mother died when he was three years old and his father, a highly educated magistrate, personally provided the boy's early education. Although Pascal showed an inclination for science and mathematics, his father refused to tutor him in those subjects until he mastered Latin and Greek. Pascal's sister and primary biographer claimed that he independently discovered the first thirty-two propositions of Euclid without ever reading a book on geometry. (However, it is generally agreed that the story is apocryphal.) Nevertheless, the precocious Pascal published a highly respected essay on conic sections by the time he was sixteen years old. Descartes, who read the essay, thought it so brilliant that he could not believe that it was written by such a young man. By age 18 his health began to fail and until his death he was in frequent pain. However, his creativity was unimpaired.

Pascal's contributions to physics include the discovery that air pressure decreases with altitude and the principle of fluid pressure that bears his name. However, the originality of his work is questioned by some historians. Pascal made major contributions to a branch of mathematics called "projective geometry," and he helped to develop probability theory through a series of letters with Fermat.

In 1646, Pascal's health problems resulted in a deep emotional crisis that led him to become increasingly concerned with religious matters. Although born a Catholic, he converted to a religious doctrine called Jansenism and spent most of his final years writing on religion and philosophy.

FLUID PRESSURE

To calculate fluid pressures and forces we will need to make use of an experimental observation. Suppose that a flat surface of area A is submerged in a homogeneous fluid of weight density ρ such that the entire surface lies between depths h_1 and h_2, where $h_1 \le h_2$ (Figure 6.7.2). Experiments show that on both sides of the surface, the fluid exerts a force that is perpendicular to the surface and whose magnitude F satisfies the inequalities

$$\rho h_1 A \le F \le \rho h_2 A \tag{4}$$

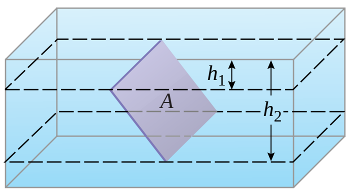


Figure 6.7.2

Thus, it follows from (1) that the pressure $P = F/A$ on a given side of the surface satisfies the inequalities

$$\rho h_1 \le P \le \rho h_2 \tag{5}$$

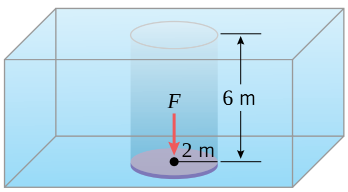
Note that it is now a straightforward matter to calculate fluid force and pressure on a flat surface that is submerged *horizontally* at depth h, for then $h = h_1 = h_2$ and inequalities (4) and (5) become the *equalities*

$$F = \rho h A \tag{6}$$

and

$$P = \rho h \tag{7}$$

Example 2 Find the fluid pressure and force on the top of a flat circular plate of radius 2 m that is submerged horizontally in water at a depth of 6 m (Figure 6.7.3).



The fluid force is the fluid pressure times the area.

Figure 6.7.3

Solution. Since the weight density of water is $\rho = 9810\ \text{N/m}^3$, it follows from (7) that the fluid pressure is

$$P = \rho h = (9810)(6) = 58{,}860\ \text{Pa}$$

and it follows from (6) that the fluid force is

$$F = \rho h A = \rho h(\pi r^2) = (9810)(6)(4\pi) = 235{,}440\pi \approx 739{,}700\ \text{N}$$ ◀

FLUID FORCE ON A VERTICAL SURFACE

It was easy to calculate the fluid force on the horizontal plate in Example 2 because each point on the plate was at the same depth. The problem of finding the fluid force on a vertical surface is more complicated because the depth, and hence the pressure, is not constant over the surface. To find the fluid force on a vertical surface we will need calculus.

6.7.2 PROBLEM. Suppose that a flat surface is immersed vertically in a fluid of weight density ρ and that the submerged portion of the surface extends from $x = a$ to $x = b$ along an x-axis whose positive direction is down (Figure 6.7.4*a*). For $a \le x \le b$, suppose that $w(x)$ is the width of the surface and that $h(x)$ is the depth of the point x. Define what is meant by the *fluid force F* on the surface, and find a formula for computing it.

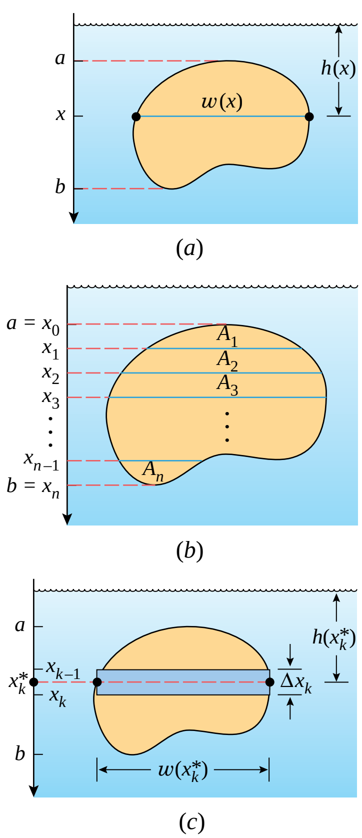


Figure 6.7.4

The basic idea for solving this problem is to divide the surface into horizontal strips whose areas may be approximated by areas of rectangles. These area approximations, along with inequalities (4), will allow us to create a Riemann sum that approximates the total force on the surface. By taking a limit of Riemann sums we will then obtain an integral for F.

To implement this idea, we divide the interval $[a, b]$ into n subintervals by inserting the numbers $x_1, x_2, \dots, x_{n-1}$ between $a = x_0$ and $b = x_n$. This has the effect of dividing the surface into n strips of area A_k, $k = 1, 2, \dots, n$ (Figure 6.7.4*b*). It follows from (4) that the force F_k on the kth strip satisfies the inequalities

$$\rho h(x_{k-1})A_k \le F_k \le \rho h(x_k)A_k$$

or equivalently,

$$h(x_{k-1}) \le \frac{F_k}{\rho A_k} \le h(x_k)$$

Since the depth function $h(x)$ increases linearly, there must exist a number x_k^* between x_{k-1} and x_k such that

$$h(x_k^*) = \frac{F_k}{\rho A_k}$$

or equivalently,

$$F_k = \rho h(x_k^*)A_k$$

We now approximate the area A_k of the kth strip of the surface by the area of a rectangle of width $w(x_k^*)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.7.4*c*). It follows that F_k may be approximated as

$$F_k = \rho h(x_k^*)A_k \approx \rho h(x_k^*) \cdot \underbrace{w(x_k^*)\Delta x_k}_{\text{Area of rectangle}}$$

Adding these approximations yields the following Riemann sum that approximates the total force F on the surface:

$$F = \sum_{k=1}^{n} F_k \approx \sum_{k=1}^{n} \rho h(x_k^*)w(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of the subintervals approach zero yields the definite integral

$$F = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \rho h(x_k^*)w(x_k^*)\Delta x_k = \int_a^b \rho h(x)w(x)\,dx$$

In summary, we have the following result:

6.7.3 DEFINITION. Suppose that a flat surface is immersed vertically in a fluid of weight density ρ and that the submerged portion of the surface extends from $x = a$ to $x = b$ along an x-axis whose positive direction is down (Figure 6.7.4*a*). For $a \le x \le b$, suppose that $w(x)$ is the width of the surface and that $h(x)$ is the depth of the point x. Then we define the ***fluid force*** F on the surface to be

$$F = \int_a^b \rho h(x)w(x)\,dx \tag{8}$$

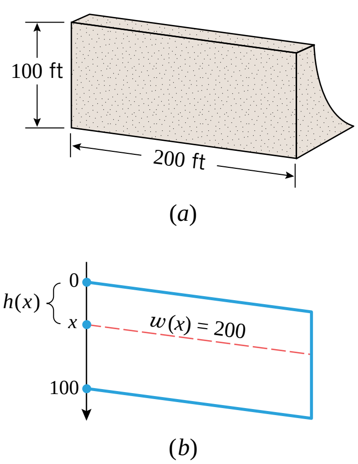


Figure 6.7.5

Example 3 The face of a dam is a vertical rectangle of height 100 ft and width 200 ft (Figure 6.7.5*a*). Find the total fluid force exerted on the face when the water surface is level with the top of the dam.

Solution. Introduce an x-axis with its origin at the water surface as shown in Figure 6.7.5*b*. At a point x on this axis, the width of the dam in feet is $w(x) = 200$ and the depth in feet is $h(x) = x$. Thus, from (8) with $\rho = 62.4$ lb/ft^3 (the weight density of water) we obtain as the total force on the face

$$F = \int_0^{100} (62.4)(x)(200)\,dx = 12{,}480 \int_0^{100} x\,dx = 12{,}480 \left.\frac{x^2}{2}\right]_0^{100} = 62{,}400{,}000 \text{ lb}$$

◀

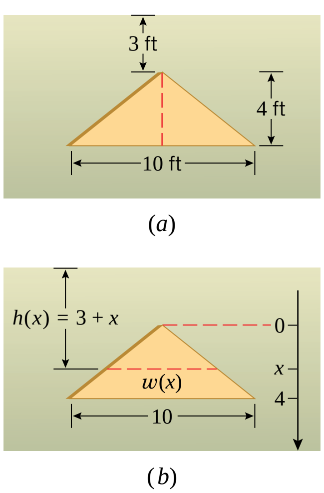


Figure 6.7.6

Example 4 A plate in the form of an isosceles triangle with base 10 ft and altitude 4 ft is submerged vertically in machine oil as shown in Figure 6.7.6*a*. Find the fluid force F against the plate surface if the oil has weight density $\rho = 30$ lb/ft^3.

Solution. Introduce an x-axis as shown in Figure 6.7.6*b*. By similar triangles, the width of the plate, in feet, at a depth of $h(x) = (3 + x)$ ft satisfies

$$\frac{w(x)}{10} = \frac{x}{4}, \quad \text{so} \quad w(x) = \frac{5}{2}x$$

Thus, it follows from (8) that the force on the plate is

$$F = \int_a^b \rho h(x)w(x)\,dx = \int_0^4 (30)(3 + x)\left(\frac{5}{2}x\right)dx$$

$$= 75\int_0^4 (3x + x^2)\,dx = 75\left[\frac{3x^2}{2} + \frac{x^3}{3}\right]_0^4 = 3400 \text{ lb}$$

◀

EXERCISE SET 6.7

In this exercise set, refer to Table 6.7.2 for weight densities of fluids, when needed.

1. A flat rectangular plate is submerged horizontally in water.
(a) Find the force (in lb) and the pressure (in lb/ft^2) on the top surface of the plate if its area is 100 ft^2 and the surface is at a depth of 5 ft.
(b) Find the force (in N) and the pressure (in Pa) on the top surface of the plate if its area is 25 m^2 and the surface is at a depth of 10 m.

2. (a) Find the force (in N) on the deck of a sunken ship if its area is 160 m^2 and the pressure acting on it is 6.0×10^5 Pa.
(b) Find the force (in lb) on a diver's face mask if its area is 60 in^2 and the pressure acting on it is 100 lb/in^2.

In Exercises 3–8, the flat surfaces shown are submerged vertically in water. Find the fluid force against the surface.

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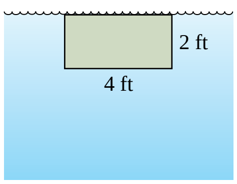

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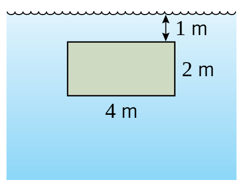

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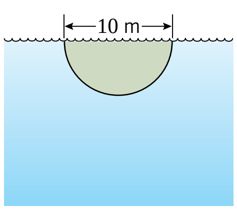

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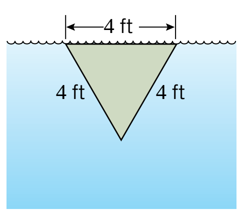

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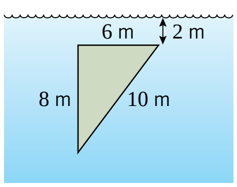

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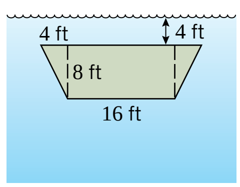

9. Suppose that a flat surface is immersed vertically in a fluid of weight density ρ. If ρ is doubled, is the force on the plate also doubled? Explain your reasoning.

10. An oil tank is shaped like a right circular cylinder of diameter 4 ft. Find the total fluid force against one end when the axis is horizontal and the tank is half filled with oil of weight density 50 lb/ft^3.

11. A square plate of side a feet is dipped in a liquid of weight density ρ lb/ft^3. Find the fluid force on the plate if a vertex is at the surface and a diagonal is perpendicular to the surface.

Formula (8) gives the fluid force on a flat surface immersed vertically in a fluid. More generally, if a flat surface is immersed so that it makes an angle of $0 \le \theta < \pi/2$ with the vertical, then the fluid force on the surface is given by

$$F = \int_a^b \rho h(x) w(x) \sec\theta \, dx$$

Use this formula in Exercises 12–15.

12. Derive the formula given above for the fluid force on a flat surface immersed at an angle in a fluid.

13. The accompanying figure shows a rectangular swimming pool whose bottom is an inclined plane. Find the fluid force on the bottom when the pool is filled to the top.

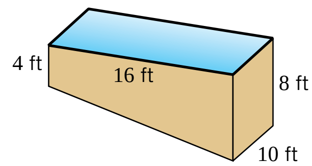

Figure Ex-13

14. By how many feet should the water in the pool of Exercise 13 be lowered in order for the force on the bottom to be reduced by a factor of 1/2?

15. The accompanying figure shows a dam whose face is an inclined rectangle. Find the fluid force on the face when the water is level with the top of this dam.

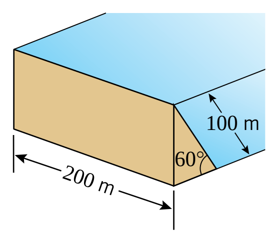

Figure Ex-15

16. An observation window on a submarine is a square with 2-ft sides. Using ρ_0 for the weight density of seawater, find the fluid force on the window when the submarine has descended so that the window is vertical and its top is at a depth of h feet.

17. (a) Show: If the submarine in Exercise 14 descends vertically at a constant rate, then the fluid force on the window increases at a constant rate.
(b) At what rate is the force on the window increasing if the submarine is descending vertically at 20 ft/min?

18. (a) Let $D = D_a$ denote a disk of radius a submerged in a fluid of weight density ρ such that the center of D is h units below the surface of the fluid. For each value of

r in the interval $(0, a]$, let D_r denote the disk of radius r that is concentric with D. Select a side of the disk D and define $P(r)$ to be the fluid pressure on the chosen side of D_r. Use (5) to prove that

$$\lim_{r \to 0^+} P(r) = \rho h$$

(b) Explain why the result in part (a) may be interpreted to mean that *fluid pressure at a given depth is the same in all directions*. (This statement is one version of a result known as ***Pascal's Principle***.)

SUPPLEMENTARY EXERCISES

C CAS

1. State an integral formula for finding the arc length of a smooth curve $y = f(x)$ over an interval $[a, b]$, and use Riemann sums to derive the formula.

2. Describe the method of slicing for finding volumes, and use that method to derive an integral formula for finding volumes by the method of disks.

3. State an integral formula for finding a volume by the method of cylindrical shells, and use Riemann sums to derive the formula.

4. State an integral formula for the work W done by a variable force $F(x)$ applied in the direction of motion to an object moving from $x = a$ to $x = b$, and use Riemann sums to derive the formula.

5. State an integral formula for the fluid force F exerted on a vertical flat surface immersed in a fluid of weight density ρ, and use Riemann sums to derive the formula.

6. Let R be the region in the first quadrant enclosed by $y = x^2$, $y = 2 + x$, and $x = 0$. In each part, set up, but *do not evaluate*, an integral or a sum of integrals that will solve the problem.
(a) Find the area of R by integrating with respect to x.
(b) Find the area of R by integrating with respect to y.
(c) Find the volume of the solid generated by revolving R about the x-axis by integrating with respect to x.
(d) Find the volume of the solid generated by revolving R about the x-axis by integrating with respect to y.
(e) Find the volume of the solid generated by revolving R about the y-axis by integrating with respect to x.
(f) Find the volume of the solid generated by revolving R about the y-axis by integrating with respect to y.

7. (a) Set up a sum of definite integrals that represents the total shaded area between the curves $y = f(x)$ and $y = g(x)$ in the accompanying figure.
(b) Find the total area enclosed between $y = x^3$ and $y = x$ over the interval $[-1, 2]$.

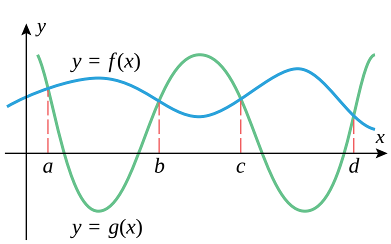

Figure Ex-7

8. Let C be the curve $27x - y^3 = 0$ between $y = 0$ and $y = 2$. In each part, set up, but *do not evaluate*, an integral or a sum of integrals that solves the problem.
(a) Find the area of the surface generated by revolving C about the y-axis by integrating with respect to x.
(b) Find the area of the surface generated by revolving C about the y-axis by integrating with respect to y.
(c) Find the area of the surface generated by revolving C about the line $y = -2$ by integrating with respect to y.

9. Find the arc length in the second quadrant of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ from $x = -a$ to $x = -\frac{1}{8}a$, where $a > 0$.

10. As shown in the accompanying figure, a cathedral dome is designed with three semicircular supports of radius r so that each horizontal cross section is a regular hexagon. Show that the volume of the dome is $r^3\sqrt{3}$.

11. As shown in the accompanying figure, a cylindrical hole is drilled all the way through the center of a sphere. Show that the volume of the remaining solid depends only on the length L of the hole, not on the size of the sphere.

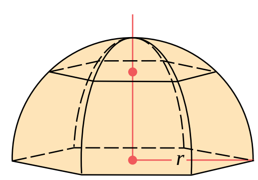

Figure Ex-10

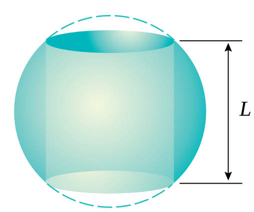

Figure Ex-11

12. A football has the shape of the solid generated by revolving the region bounded between the x-axis and the parabola $y = 4R(x^2 - \frac{1}{4}L^2)/L^2$ about the x-axis. Find its volume.

C **13.** As shown in the accompanying figure, a horizontal beam with dimensions 2 in × 6 in × 16 ft is fixed at both ends and is subjected to a uniformly distributed load of 120 lb/ft. As a result of the load, the centerline of the beam undergoes a deflection that is described by

$$y = -1.67 \times 10^{-8}(x^4 - 2Lx^3 + L^2x^2)$$

$(0 \le x \le 192)$, where $L = 192$ inches is the length of the unloaded beam, x is the horizontal distance along the beam measured in inches from the left end, and y is the deflection of the centerline in inches.

(a) Graph y versus x for $0 \leq x \leq 192$.
(b) Find the maximum deflection of the centerline.
(c) Use a CAS or a calculator with a numerical integration capability to find the length of the centerline of the loaded beam. Round your answer to two decimal places.

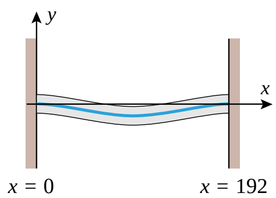

Figure Ex-13

C **14.** A golfer makes a successful chip shot to the green. Suppose that the path of the ball from the moment it is struck to the moment it hits the green is described by

$$y = 12.54x - 0.41x^2$$

where x is the horizontal distance (in yards) from the point where the ball is struck, and y is the vertical distance (in yards) above the fairway. Use a CAS or a calculator or program with a numerical integration capability to find the distance the ball travels from the moment it is struck to the moment it hits the green. Assume that the fairway and green are at the same level and round your answer to two decimal places.

15. (a) A spring exerts a force of 0.5 N when stretched 0.25 m beyond its natural length. Assuming that Hooke's law applies, how much work was performed in stretching the spring to this length?
(b) How far beyond its natural length can the spring be stretched with 25 J of work?

16. A boat is anchored so that the anchor is 150 ft below the surface of the water. In the water, the anchor weighs 2000 lb and the chain weighs 30 lb/ft. How much work is required to raise the anchor to the surface?

17. In each part, set up, but *do not evaluate*, an integral that solves the problem.
(a) Find the fluid force exerted on a side of a box that has a 3-m-square base and is filled to a depth of 1 m with a liquid of weight density ρ N/m^3.
(b) Find the fluid force exerted by a liquid of weight density ρ lb/ft^3 on a face of the vertical plate shown in part (*a*) of the accompanying figure.
(c) Find the fluid force exerted on the parabolic dam in part (*b*) of the accompanying figure by water that extends to the top of the dam.

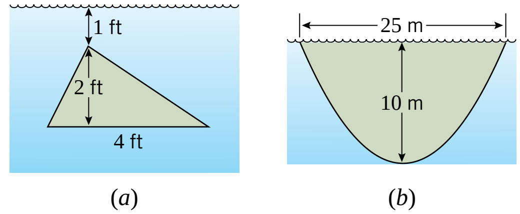

Figure Ex-17

Exercises 18–20 lead to equations that cannot be solved exactly. Use any method you choose to approximate the solutions of those equations, and round your answers to two decimal places.

18. Find the area of the region enclosed by the curves $y = x^2 - 1$ and $y = 2\sin x$.

19. Referring to the accompanying figure, find the value of k so that the areas of the shaded regions are equal. [*Note:* This exercise is based on Problem A1 of the Fifty-Fourth Annual William Lowell Putnam Mathematical Competition.]

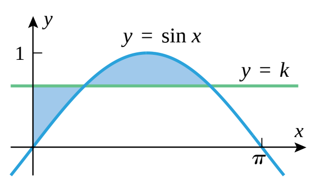

Figure Ex-19

C **20.** Consider the region to the left of the vertical line $x = k$ $(0 < k < \pi)$ and between the curve $y = \sin x$ and the x-axis. Use a CAS to find the value of k so that the solid generated by revolving the region about the y-axis has a volume of 8 cubic units.