5

INTEGRATION

Gottfried Leibniz

Traditionally, that portion of calculus concerned with finding tangent lines and rates of change is called ***differential calculus*** and that portion concerned with finding areas is called ***integral calculus***. However, we will see in this chapter that the two problems are so closely related that the distinction between differential and integral calculus is often hard to discern.

In this chapter we will begin with an overview of the problem of finding areas—we will discuss what the term "area" means, and we will outline two approaches to defining and calculating areas. Following this overview, we will discuss the "Fundamental Theorem of Calculus," which is the theorem that relates the problems of finding tangent lines and areas, and we will discuss techniques for calculating areas. Finally, we will use the ideas in this chapter to continue our study of rectilinear motion and to examine some consequences of the chain rule in integral calculus.

5.1 AN OVERVIEW OF THE AREA PROBLEM

In this introductory section we will consider the problem of calculating areas of plane regions with curvilinear boundaries. All of the results in this section will be reexamined in more detail later in this chapter, so our purpose here is simply to introduce the fundamental concepts.

FINDING AREA

The main goal of this chapter is to study the following major problem of calculus:

5.1.1 THE AREA PROBLEM. Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x-axis (Figure 5.1.1).

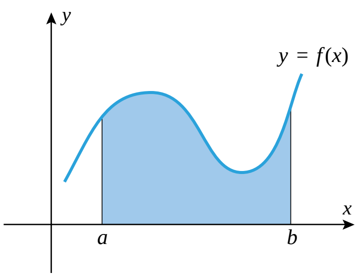


Figure 5.1.1

Of course, from a strictly logical point of view, we should first provide a precise definition of the term *area* before discussing methods for calculating areas. However, in this section we will treat the concept of area intuitively, postponing a more formal definition until Section 5.4.

Formulas for the areas of plane regions with straight-line boundaries (squares, rectangles, triangles, trapezoids, etc.) were well known in many early civilizations. On the other hand, obtaining formulas for regions with curvilinear boundaries (a circle being the simplest case) caused problems for early mathematicians. The first real progress on such problems was made by the Greek mathematician, Archimedes,* who obtained the areas of regions bounded by arcs of circles, parabolas, spirals, and various other curves by ingenious use of a procedure later known as the *method of exhaustion*. That method, when applied to a circle of radius r, consists of inscribing a succession of regular polygons in the circle and allowing the number of sides n to increase indefinitely (Figure 5.1.2). As n increases, the polygons tend to "exhaust" the region inside the circle, and the areas of those polygons become better and better approximations to the exact area of the circle.

*ARCHIMEDES (287 B.C.–212 B.C.). Greek mathematician and scientist. Born in Syracuse, Sicily, Archimedes was the son of the astronomer Pheidias and possibly related to Heiron II, king of Syracuse. Most of the facts about his life come from the Roman biographer, Plutarch, who inserted a few tantalizing pages about him in the massive biography of the Roman soldier, Marcellus. In the words of one writer, "the account of Archimedes is slipped like a tissue-thin shaving of ham in a bull-choking sandwich."

Archimedes ranks with Newton and Gauss as one of the three greatest mathematicians who ever lived, and he is certainly the greatest mathematician of antiquity. His mathematical work is so modern in spirit and technique that it is barely distinguishable from that of a seventeenth-century mathematician, yet it was all done without benefit of algebra or a convenient number system. Among his mathematical achievements, Archimedes developed a general method (exhaustion) for finding areas and volumes, and he used the method to find areas bounded by parabolas and spirals and to find volumes of cylinders, paraboloids, and segments of spheres. He gave a procedure for approximating π and bounded its value between $3\frac{10}{71}$ and $3\frac{1}{7}$. In spite of the limitations of the Greek numbering system, he devised methods for finding square roots and invented a method based on the Greek myriad (10,000) for representing numbers as large as 1 followed by 80 million billion zeros.

Of all his mathematical work, Archimedes was most proud of his discovery of the method for finding the volume of a sphere—he showed that the volume of a sphere is two-thirds the volume of the smallest cylinder that can contain it. At his request, the figure of a sphere and cylinder was engraved on his tombstone.

In addition to mathematics, Archimedes worked extensively in mechanics and hydrostatics. Nearly every schoolchild knows Archimedes as the absent-minded scientist who, on realizing that a floating object displaces its weight of liquid, leaped from his bath and ran naked through the streets of Syracuse shouting, "Eureka, Eureka!"—(meaning, "I have found it!"). Archimedes actually created the discipline of hydrostatics and used it to find equilibrium positions for various floating bodies. He laid down the fundamental postulates of mechanics, discovered the laws of levers, and calculated centers of gravity for various flat surfaces and solids. In the excitement of discovering the mathematical laws of the lever, he is said to have declared, "Give me a place to stand and I will move the earth."

Although Archimedes was apparently more interested in pure mathematics than its applications, he was an engineering genius. During the second Punic war, when Syracuse was attacked by the Roman fleet under the command of Marcellus, it was reported by Plutarch that Archimedes' military inventions held the fleet at bay for three years. He invented super catapults that showered the Romans with rocks weighing a quarter ton or more,

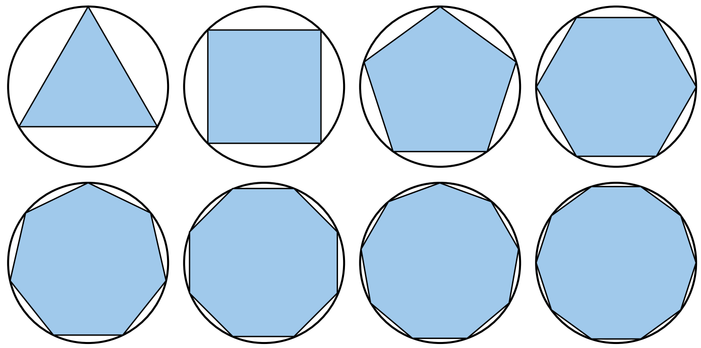


Figure 5.1.2

Table 5.1.1

n	$A(n)$
100	3.13952597647
200	3.14107590781
300	3.14136298250
400	3.14146346236
500	3.14150997084
600	3.14153523487
700	3.14155046835
800	3.14156035548
900	3.14156713408
1000	3.14157198278
2000	3.14158748588
3000	3.14159035683
4000	3.14159136166
5000	3.14159182676
6000	3.14159207940
7000	3.14159223174
8000	3.14159233061
9000	3.14159239839
10000	3.14159244688

To see how this works numerically, let $A(n)$ denote the area of a regular n-sided polygon inscribed in a circle of radius 1. Table 5.1.1 shows the values of $A(n)$ for various choices of n. Note that for large values of n the area $A(n)$ appears to be close to π (square units), as one would expect. This suggests that for a circle of radius 1, the method of exhaustion is equivalent to an equation of the form

$$\lim_{n\to\infty} A(n) = \pi$$

However, Greek mathematicians were very suspicious of the concept of "infinity" and intentionally avoided explanations that referred to the "limiting behavior" of some quantity. As a consequence, obtaining exact answers by the classical method of exhaustion was a cumbersome procedure. In our discussion of the area problem, we will consider a more modern version of the method of exhaustion that explicitly incorporates the notion of a limit. Because our approach uses a collection of rectangles to "exhaust" an area, we will refer to it as the *rectangle method*.

THE RECTANGLE METHOD FOR FINDING AREAS

There are two basic methods for finding the area of the region having the form shown in Figure 5.1.1—the *rectangle method* and the *antiderivative method*. The idea behind the rectangle method is as follows:

- Divide the interval $[a, b]$ into n equal subintervals, and over each subinterval construct a rectangle that extends from the x-axis to any point on the curve $y = f(x)$ that is above the subinterval; the particular point does not matter—it can be above the center, above an endpoint, or above any other point in the subinterval. In Figure 5.1.3 it is above the center.
- For each n, the total area of the rectangles can be viewed as an *approximation* to the exact area under the curve over the interval $[a, b]$. Moreover, it is evident intuitively that as n increases these approximations will get better and better and will approach the exact area as a limit (Figure 5.1.4).

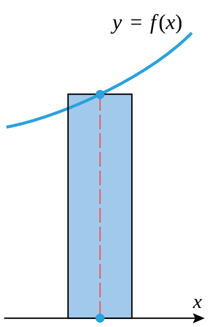


Figure 5.1.3

Later, this procedure will serve both as a mathematical definition and a method of computation—we will *define* the area under $y = f(x)$ over the interval $[a, b]$ as the limit of the areas of the approximating rectangles, and we will use the method itself to approximate this area.

and fearsome mechanical devices with iron "beaks and claws" that reached over the city walls, grasped the ships, and spun them against the rocks. After the first repulse, Marcellus called Archimedes a "geometrical Briareus (a hundred-armed mythological monster) who uses our ships like cups to ladle water from the sea."

Eventually the Roman army was victorious and contrary to Marcellus' specific orders the 75-year-old Archimedes was killed by a Roman soldier. According to one report of the incident, the soldier cast a shadow across the sand in which Archimedes was working on a mathematical problem. When the annoyed Archimedes yelled, "Don't disturb my circles," the soldier flew into a rage and cut the old man down.

With his death the Greek gift of mathematics passed into oblivion, not to be fully resurrected again until the sixteenth century. Unfortunately, there is no known accurate likeness or statue of this great man.

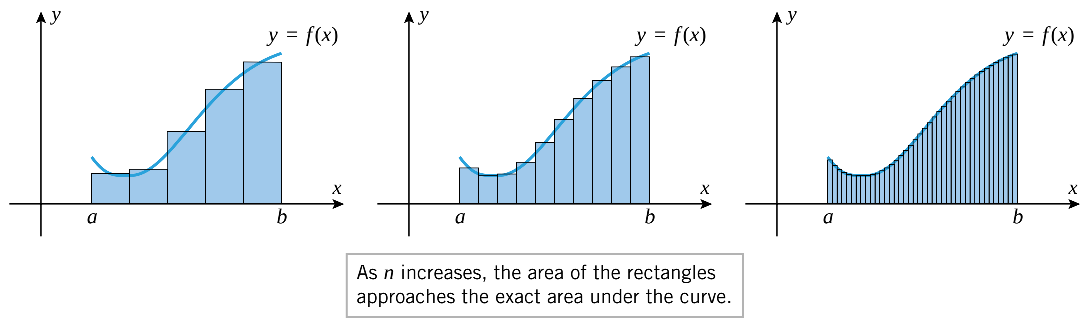


Figure 5.1.4

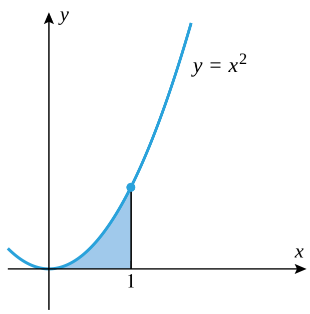


Figure 5.1.5

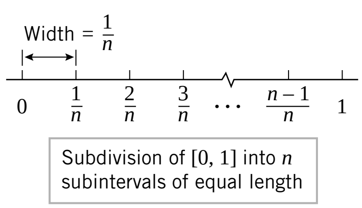


Figure 5.1.6

To illustrate this idea, we will use the rectangle method to approximate the area under the curve $y = x^2$ over the interval [0, 1] (Figure 5.1.5). We will begin by dividing the interval [0, 1] into n equal subintervals, from which it follows that each subinterval has length $1/n$; the endpoints of the subintervals occur at

$$0,\quad \frac{1}{n},\quad \frac{2}{n},\quad \frac{3}{n}, \ldots,\quad \frac{n-1}{n},\quad 1$$

(Figure 5.1.6). We want to construct a rectangle over each of these subintervals whose height is the value of the function $f(x) = x^2$ at some number in the subinterval. To be specific, let us use the right endpoints, in which case the heights of our rectangles will be

$$\left(\frac{1}{n}\right)^2, \quad \left(\frac{2}{n}\right)^2, \quad \left(\frac{3}{n}\right)^2, \ldots, \quad 1^2$$

and since each rectangle has a base of width $1/n$, the total area A_n of the n rectangles will be

$$A_n = \left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \cdots + 1^2\right]\left(\frac{1}{n}\right) \tag{1}$$

For example, if $n = 4$, then the total area of the four approximating rectangles would be

$$A_4 = \left[\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{2}{4}\right)^2 + \left(\tfrac{3}{4}\right)^2 + 1^2\right]\left(\tfrac{1}{4}\right) = \tfrac{15}{32} = 0.46875$$

Table 5.1.2 shows the result of evaluating (1) on a computer for some increasingly large values of n. These computations suggest that the exact area is close to $\frac{1}{3}$. In Section 5.4 we will prove that this area is exactly $\frac{1}{3}$ by showing that

$$\lim_{n\to\infty} A_n = \tfrac{1}{3}$$

Table 5.1.2

n	4	10	100	1000	10,000	100,000
A_n	0.468750	0.385000	0.338350	0.333834	0.333383	0.333338

Equation (1) may be written more concisely by using *sigma notation*, which is discussed in Section 5.4 in detail. [Sigma (Σ) is an uppercase letter in the Greek alphabet used to denote sums.] With sigma notation, the sum

$$\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \cdots + 1^2$$

may be expressed simply as

$$\sum_{k=1}^{n} \left(\frac{k}{n}\right)^2$$

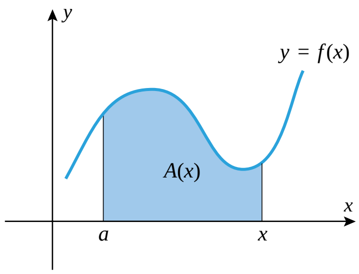


Figure 5.1.7

This notation tells us to form the sum of the terms that result when we substitute successive integers for k in the expression $(k/n)^2$, starting with $k = 1$ and ending with $k = n$. Each value of a positive integer n then determines a value of the sum. For example, if $n = 4$, then

$$\sum_{k=1}^{4}\left(\frac{k}{4}\right)^2 = \left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{4}\right)^2 = \frac{30}{16} = \frac{15}{8}$$

In general, using sigma notation we write

$$A_n = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2$$

FOR THE READER. Many calculating utilities perform automatic summations for expressions that involve some version of the sigma notation. If your calculating utility performs such summations, use it to verify the value of A_{100} given in Table 5.1.2. (Otherwise, use it to confirm A_{10}.)

THE ANTIDERIVATIVE METHOD FOR FINDING AREAS

Despite the intuitive appeal of the rectangle method, the limits involved can be evaluated directly only in certain special cases. For this reason, work on the area problem remained at a rudimentary level until the latter half of the seventeenth century. Two results that were to prove to be a major breakthrough in the area problem were discovered by mathematicians Isaac Barrow and Isaac Newton in Great Britain, and Gottfried Leibniz in Germany. These results appeared, without fanfare, as a proposition in Issac Barrow's *Lectiones geometricae*. Each of the two results can be used to solve the area problem.

The solution based on Proposition 11 was preferred by Isaac Newton and provides us with a paradoxically effective *indirect* approach to the area problem. According to this line of argument, to find the area under the curve in Figure 5.1.1, one should first consider the seemingly *harder* problem of finding the area $A(x)$ between the graph of f and the interval $[a, x]$, where x denotes an arbitrary number in $[a, b]$ (Figure 5.1.7). If one can discover a formula for the area function $A(x)$, then the area under the curve from a to b can be obtained simply by substituting $x = b$ into this formula.

This may seem to be a surprising approach to the area problem. After all, why should the problem of determining the area $A(x)$ for *every* x in the interval $[a, b]$ be more tractable than the problem of computing a *single* value $A(b)$? However, the basis for this approach is the observation that although the area function $A(x)$ may be difficult to compute, its *derivative* $A'(x)$ is easy to find. To illustrate, let us consider some examples of area functions $A(x)$ that *can* be computed from simple geometry.

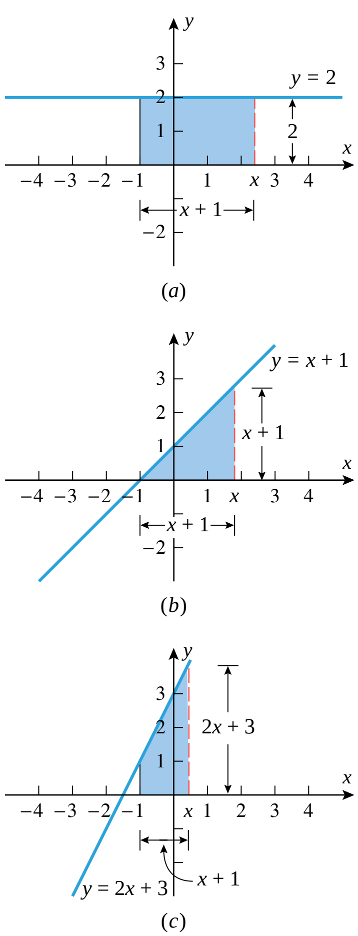


Figure 5.1.8

Example 1 For each of the functions f, find the area $A(x)$ between the graph of f and the interval $[a, x] = [-1, x]$, and find the derivative $A'(x)$ of this area function.

(a) $f(x) = 2$ (b) $f(x) = x + 1$ (c) $f(x) = 2x + 3$

Solution (a). From Figure 5.1.8*a* we see that

$$A(x) = 2(x - (-1)) = 2(x + 1) = 2x + 2$$

is the area of a rectangle of height 2 and base $x + 1$. For this area function,

$$A'(x) = 2 = f(x)$$

Solution (b). From Figure 5.1.8*b* we see that

$$A(x) = \frac{1}{2}(x + 1)(x + 1) = \frac{x^2}{2} + x + \frac{1}{2}$$

is the area of an isosceles right triangle with base and height equal to $x + 1$. For this area function,

$$A'(x) = x + 1 = f(x)$$

Solution (c). Recall that the formula for the area of a trapezoid is $A = \frac{1}{2}(b+b')h$, where b and b' denote the lengths of the parallel sides of the trapezoid, and the altitude h denotes the distance between the parallel sides. From Figure 5.1.8*c* we see that

$$A(x) = \tfrac{1}{2}((2x+3)+1)(x-(-1)) = x^2+3x+2$$

is the area of a trapezoid with parallel sides of lengths 1 and $2x+3$ and with altitude $x-(-1) = x+1$. For this area function,

$$A'(x) = 2x+3 = f(x)$$ ◀

Note that in every case in Example 1,

$$A'(x) = f(x) \tag{2}$$

That is, *the derivative of the area function $A(x)$ is the function whose graph forms the upper boundary of the region*. We will show in Section 5.6 that Equation (2) is valid not simply for linear functions such as those in Example 1, but for any continuous function. Thus, to find the area function $A'(x)$, we can look instead for a (particular) function whose *derivative* is $f(x)$. This is called an ***antidifferentiation*** problem because we are trying to find $A(x)$ by "undoing" a differentiation. Whereas earlier in the text we were concerned with the process of differentiation, we will now also be concerned with the process of antidifferentiation.

To see how this *antiderivative method* applies to a specific example, let us return to the problem of finding the area between the graph of $f(x) = x^2$ and the interval $[0, 1]$. If we let $A(x)$ denote the area between the graph of f and the interval $[0, x]$, then (2) tells us that $A'(x) = f(x) = x^2$. By simple guesswork, we see that one function whose derivative is $f(x) = x^2$ is $\frac{1}{3}x^3$. It then follows from Theorem 4.8.3 that $A(x) = \frac{1}{3}x^3 + C$ for some constant C. This is where the decision to solve the area problem for a general right-hand endpoint helps. If we consider the case $x = 0$, then the interval $[0, x]$ reduces to a single point. If we agree that the area above a single point should be taken as zero, then it follows that

$$0 = A(0) = \tfrac{1}{3}0^3 + C = 0 + C = C \quad \text{or} \quad C = 0$$

Therefore, $A(x) = \frac{1}{3}x^3$ and the area between the graph of f and the interval $[0, 1]$ is $A(1) = \frac{1}{3}$. Note that this conclusion agrees with our numerical estimates in Table 5.1.2.

Although the antiderivative method provides us with a convenient solution to the area problem, it appears to have little to do with the rectangle method. It would be nice to have a solution that more clearly elucidates the connection between the operation of summing areas of rectangles on the one hand and the operation of antidifferentiation on the other. Fortunately, the solution to the area problem based on Barrow's Proposition 19 reveals just this connection. In addition, it allows us to formulate in modern language the approach to the area problem preferred by Leibniz. We will provide this solution in Section 5.6 (Theorem 5.6.1), as well as develop a modern version of Barrow's Proposition 11 (Theorem 5.6.3). Together, these two approaches to the area problem comprise what is now known as the *Fundamental Theorem of Calculus*.

INTEGRAL CALCULUS

We see that the rectangle method and the use of antidifferentiation provide us with quite different approaches to the area problem. The rectangle method is a frontal assault on the problem, whereas antidifferentiation is more in the form of a sneak attack. In this chapter we will carefully study both approaches to the problem.

In Sections 5.2 and 5.3 we will begin to develop some techniques for the process of antidifferentiation, a process that is also known as *integration*. Later, in Section 5.5 we will discuss a more general version of the rectangle method known as the *Riemann sum*. In much the same way that area can be interpreted as a "limit" using the rectangle method, we will define the *definite integral* as a "limit" of Riemann sums.

The definite integral and antidifferentiation are the twin pillars on which integral calculus rests. Both are important. The definite integral is generally the means by which problems in integral calculus are recognized and formulated. For example, in addition to the area problem, the problems of computing the volume of a solid, finding the arc length of a curve, and determining the work done in pumping water out of a tank are all examples of problems that may be solved by means of a definite integral. On the other hand, it can be difficult to obtain exact solutions to such problems by direct computation of a definite integral. Fortunately, in many cases of interest, the Fundamental Theorem of Calculus will allow us to evaluate a definite integral by means of antidifferentiation. Much of the power of integral calculus lies in the two-pronged approach of the definite integral and antidifferentiation.

EXERCISE SET 5.1

In Exercises 1–8, estimate the area between the graph of the function f and the interval $[a, b]$. Use an approximation scheme with n rectangles similar to our treatment of $f(x) = x^2$ in this section. If your calculating utility will perform automatic summations, estimate the specified area using $n = 10, 50,$ and 100 rectangles. Otherwise, estimate this area using $n = 2, 5,$ and 10 rectangles.

1. $f(x) = \sqrt{x}$; $[a, b] = [0, 1]$

2. $f(x) = \dfrac{1}{x+1}$; $[a, b] = [0, 1]$

3. $f(x) = \sin x$; $[a, b] = [0, \pi]$

4. $f(x) = \cos x$; $[a, b] = [0, \pi/2]$

5. $f(x) = \dfrac{1}{x}$; $[a, b] = [1, 2]$

6. $f(x) = \cos x$; $[a, b] = [-\pi/2, \pi/2]$

7. $f(x) = \sqrt{1 - x^2}$; $[a, b] = [0, 1]$

8. $f(x) = \sqrt{1 - x^2}$; $[a, b] = [-1, 1]$

In Exercises 9–14, use simple area formulas from geometry to find the area function $A(x)$ that gives the area between the graph of the specified function f and the interval $[a, x]$. Confirm that $A'(x) = f(x)$ in every case.

9. $f(x) = 3$; $[a, x] = [1, x]$

10. $f(x) = 5$; $[a, x] = [2, x]$

11. $f(x) = 2x + 2$; $[a, x] = [0, x]$

12. $f(x) = 3x - 3$; $[a, x] = [1, x]$

13. $f(x) = 2x + 2$; $[a, x] = [1, x]$

14. $f(x) = 3x - 3$; $[a, x] = [2, x]$

15. How do the area functions in Exercises 11 and 13 compare? Explain.

16. Let $f(x)$ denote a *linear function* that is nonnegative on the interval $[a, b]$. For each value of x in $[a, b]$, define $A(x)$ to be the area between the graph of f and the interval $[a, x]$.
(a) Prove that $A(x) = \frac{1}{2}[f(a) + f(x)](x - a)$.
(b) Use part (a) to verify that $A'(x) = f(x)$.

17. Let A denote the area between the graph of $f(x) = \sqrt{x}$ and the interval $[0, 1]$, and let B denote the area between the graph of $f(x) = x^2$ and the interval $[0, 1]$. Explain geometrically why $A + B = 1$.

18. Let A denote the area between the graph of $f(x) = 1/x$ and the interval $[1, 2]$, and let B denote the area between the graph of f and the interval $[\frac{1}{2}, 1]$. Explain geometrically why $A = B$.

5.2 THE INDEFINITE INTEGRAL; INTEGRAL CURVES AND DIRECTION FIELDS

In the last section we saw the potential for antidifferentiation to play an important role in finding exact areas. In this section we will develop some fundamental results about antidifferentiation that will ultimately lead us to systematic procedures for solving many antiderivative problems.

5.2.1 DEFINITION. A function F is called an ***antiderivative*** of a function f on a given interval I if $F'(x) = f(x)$ for all x in the interval.

For example, the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ on the interval $(-\infty, +\infty)$ because for each x in this interval

$$F'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2 = f(x)$$

However, $F(x) = \frac{1}{3}x^3$ is not the only antiderivative of f on this interval. If we add any constant C to $\frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f on $(-\infty, +\infty)$, since

$$G'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3 + C\right] = x^2 + 0 = f(x)$$

In general, once any single antiderivative is known, other antiderivatives can be obtained by adding constants to the known antiderivative. Thus,

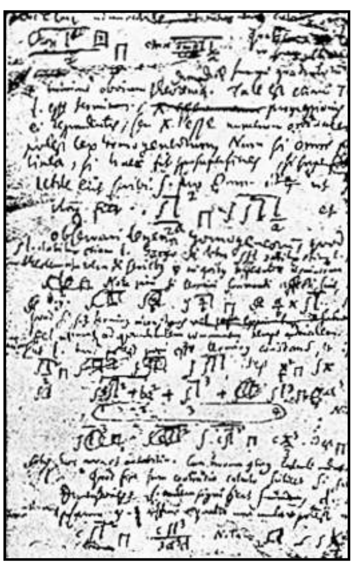
$$\tfrac{1}{3}x^3, \quad \tfrac{1}{3}x^3 + 2, \quad \tfrac{1}{3}x^3 - 5, \quad \tfrac{1}{3}x^3 + \sqrt{2}$$

are all antiderivatives of $f(x) = x^2$.

It is reasonable to ask if there are antiderivatives of a function f that cannot be obtained by adding some constant to a known antiderivative F. The answer is *no*—once a single antiderivative of f on an interval I is known, all other antiderivatives on that interval are obtainable by adding constants to the known antiderivative. This is so because Theorem 4.8.3 tells us that if two functions are differentiable on an open interval I such that their derivatives are equal on I, then the functions differ by a constant on I. The following theorem summarizes these observations.

5.2.2 THEOREM. *If $F(x)$ is any antiderivative of $f(x)$ on an interval I, then for any constant C the function $F(x) + C$ is also an antiderivative on that interval. Moreover, each antiderivative of $f(x)$ on the interval I can be expressed in the form $F(x) + C$ by choosing the constant C appropriately.*

THE INDEFINITE INTEGRAL



Extract from the manuscript of Leibniz dated October 29, 1675 in which the integral sign first appeared.

The process of finding antiderivatives is called ***antidifferentiation*** or ***integration***. Thus, if

$$\frac{d}{dx}[F(x)] = f(x) \tag{1}$$

then ***integrating*** (or ***antidifferentiating***) the function $f(x)$ produces an antiderivative of the form $F(x) + C$. To emphasize this process, Equation (1) is recast using ***integral notation***,

$$\int f(x)\,dx = F(x) + C \tag{2}$$

where C is understood to represent an arbitrary constant. It is important to note that (1) and (2) are just different notations to express the same fact. For example,

$$\int x^2\,dx = \tfrac{1}{3}x^3 + C \quad \text{is equivalent to} \quad \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2$$

Note that if we differentiate an antiderivative of $f(x)$, we obtain $f(x)$ back again. Thus,

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x) \tag{3}$$

The expression $\int f(x)\,dx$ is called an ***indefinite integral***. The adjective "indefinite" emphasizes that the result of antidifferentiation is a "generic" function, descibed only up to a constant summand. The "elongated s" that appears on the left side of (2) is called an ***integral sign***,* the function $f(x)$ is called the ***integrand***, and the constant C is called the ***constant of integration***. Equation (2) should be read as:

*This notation was devised by Leibniz. In his early papers Leibniz used the notation "omn." (an abbreviation for the Latin word "omnes") to denote integration. Then on October 29, 1675 he wrote, "It will be useful to write $\int$ for omn., thus $\int \ell$ for omn. $\ell \ldots$." Two or three weeks later he refined the notation further and wrote $\int[\ ]\,dx$ rather than $\int$ alone. This notation is so useful and so powerful that its development by Leibniz must be regarded as a major milestone in the history of mathematics and science.

The integral of $f(x)$ with respect to x is equal to $F(x)$ plus a constant.

The differential symbol, dx, in the differentiation and antidifferentiation operations

$$\frac{d}{dx}[\ ] \quad \text{and} \quad \int [\ ]\, dx$$

serves to identify the independent variable. If an independent variable other than x is used, say t, then the notation must be adjusted appropriately. Thus,

$$\frac{d}{dt}[F(t)] = f(t) \quad \text{and} \quad \int f(t)\, dt = F(t) + C$$

are equivalent statements.

Example 1

DERIVATIVE FORMULA	EQUIVALENT INTEGRATION FORMULA
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2\, dx = x^3 + C$
$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}}\, dx = \sqrt{x} + C$
$\frac{d}{dt}[\tan t] = \sec^2 t$	$\int \sec^2 t\, dt = \tan t + C$
$\frac{d}{du}[u^{3/2}] = \frac{3}{2}u^{1/2}$	$\int \frac{3}{2}u^{1/2}\, du = u^{3/2} + C$

◀

For simplicity, the dx is sometimes absorbed into the integrand. For example,

$$\int 1\, dx \quad \text{can be written as} \quad \int dx$$

$$\int \frac{1}{x^2}\, dx \quad \text{can be written as} \quad \int \frac{dx}{x^2}$$

The integral sign and differential serve as delimiters, flanking the integrand on the left and right, respectively. In particular, we do *not* write $\int dx f(x)$ when we intend $\int f(x)\, dx$.

INTEGRATION FORMULAS

Integration is essentially educated guesswork—given the derivative f of a function F, one tries to guess what the function F is. However, many basic integration formulas can be obtained directly from their companion differentiation formulas. Some of the most important are given in Table 5.2.1.

Example 2 The second integration formula in Table 5.2.1 will be easier to remember if you express it in words:

To integrate a power of x (other than -1), add 1 to the exponent and divide by the new exponent.

Here are some examples:

$$\int x^2\, dx = \frac{x^3}{3} + C \qquad \boxed{r = 2}$$

$$\int x^3\, dx = \frac{x^4}{4} + C \qquad \boxed{r = 3}$$

$$\int \frac{1}{x^5}\, dx = \int x^{-5}\, dx = \frac{x^{-5+1}}{-5+1} + C = -\frac{1}{4x^4} + C \qquad \boxed{r = -5}$$

$$\int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\, dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \tfrac{2}{3}x^{\frac{3}{2}} + C = \tfrac{2}{3}(\sqrt{x})^3 + C \qquad \boxed{r = \tfrac{1}{2}}$$

Table 5.2.1

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\,dx = -\cos x + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

It is clear that this pattern does not fit the case of

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx$$

since blind adherence to the pattern formula with $r = -1$ would lead to division by zero. We will resolve this missing case in Chapter 7. ◀

PROPERTIES OF THE INDEFINITE INTEGRAL

Our first properties of antiderivatives follow directly from the simple constant factor, sum, and difference rules for derivatives.

5.2.3 THEOREM. *Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, and that c is a constant. Then:*

(*a*) *A constant factor can be moved through an integral sign; that is,*

$$\int cf(x)\,dx = cF(x) + C$$

(*b*) *An antiderivative of a sum is the sum of the antiderivatives; that is,*

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + C$$

(*c*) *An antiderivative of a difference is the difference of the antiderivatives; that is,*

$$\int [f(x) - g(x)]\,dx = F(x) - G(x) + C$$

Proof. In general, to establish the validity of an equation of the form

$$\int h(x)\,dx = H(x) + C$$

one must show that

$$\frac{d}{dx}[H(x)] = h(x)$$

We are given that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, so we know that

$$\frac{d}{dx}[F(x)] = f(x) \quad \text{and} \quad \frac{d}{dx}[G(x)] = g(x)$$

Thus,

$$\frac{d}{dx}[cF(x)] = c\frac{d}{dx}[F(x)] = cf(x)$$

$$\frac{d}{dx}[F(x) + G(x)] = \frac{d}{dx}[F(x)] + \frac{d}{dx}[G(x)] = f(x) + g(x)$$

$$\frac{d}{dx}[F(x) - G(x)] = \frac{d}{dx}[F(x)] - \frac{d}{dx}[G(x)] = f(x) - g(x)$$

which proves the three statements of the theorem. ∎

In practice, the results of Theorem 5.2.3 are summarized by the following formulas:

$$\int cf(x)\,dx = c\int f(x)\,dx \tag{4}$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \tag{5}$$

$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx \tag{6}$$

However, these equations must be applied carefully to avoid errors and unnecessary complexities arising from the constants of integration. For example, if you were to use (4) to integrate $0x$ by writing

$$\int 0x\,dx = 0\int x\,dx = 0\left(\frac{x^2}{2} + C\right) = 0$$

then you will have erroneously lost the constant of integration, and if you use (4) to integrate $2x$ by writing

$$\int 2x\,dx = 2\int x\,dx = 2\left(\frac{x^2}{2} + C\right) = x^2 + 2C$$

then you will have an unnecessarily complicated form of the arbitrary constant. Similarly, if you use (5) to integrate $1 + x$ by writing

$$\int (1 + x)\,dx = \int 1\,dx + \int x\,dx = (x + C_1) + \left(\frac{x^2}{2} + C_2\right) = x + \frac{x^2}{2} + C_1 + C_2$$

then you will have two arbitrary constants when one will suffice. These three kinds of problems are caused by introducing constants of integration too soon and can be avoided by inserting the constant of integration in the final result, rather than in intermediate compuations.

Example 3 Evaluate

(a) $\displaystyle\int 4\cos x\,dx$ (b) $\displaystyle\int (x + x^2)\,dx$

Solution (a). Since $F(x) = \sin x$ is an antiderivative for $f(x) = \cos x$ (Table 5.2.1), we obtain

$$\int 4\cos x\,dx \underset{(4)}{=} 4\int \cos x\,dx = 4\sin x + C$$

Solution (b). From Table 5.2.1 we obtain

$$\int (x + x^2)\,dx \underset{(5)}{=} \int x\,dx + \int x^2\,dx = \frac{x^2}{2} + \frac{x^3}{3} + C$$ ◀

Parts (b) and (c) of Theorem 5.2.3 can be extended to more than two functions, which in combination with part (a) results in the following general formula:

$$\begin{aligned}&\int [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]\,dx \\ &\qquad = c_1 \int f_1(x)\,dx + c_2 \int f_2(x)\,dx + \cdots + c_n \int f_n(x)\,dx\end{aligned} \tag{7}$$

Example 4

$$\begin{aligned}\int (3x^6 - 2x^2 + 7x + 1)\,dx &= 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx + \int 1\,dx \\ &= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C\end{aligned}$$ ◀

Sometimes it is useful to rewrite an integrand in a different form before performing the integration.

Example 5 Evaluate

(a) $\displaystyle\int \frac{\cos x}{\sin^2 x}\,dx$ (b) $\displaystyle\int \frac{t^2 - 2t^4}{t^4}\,dt$

Solution (a).

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\,\frac{\cos x}{\sin x}\,dx = \int \csc x \cot x\,dx \underset{\text{Formula 8 in Table 5.2.1}}{=} -\csc x + C$$

Solution (b).

$$\begin{aligned}\int \frac{t^2 - 2t^4}{t^4}\,dt &= \int \left(\frac{1}{t^2} - 2\right) dt = \int (t^{-2} - 2)\,dt \\ &= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C\end{aligned}$$ ◀

INTEGRAL CURVES

Graphs of antiderivatives of a function f are called ***integral curves*** of f. We know from Theorem 5.2.2 that if $y = F(x)$ is any integral curve of $f(x)$, then all other integral curves are vertical translations of this curve, since they have equations of the form $y = F(x) + C$. For example, $y = \frac{1}{3}x^3$ is one integral curve for $f(x) = x^2$, so all the other integral curves have equations of the form $y = \frac{1}{3}x^3 + C$; conversely, the graph of any equation of this form is an integral curve (Figure 5.2.1).

In many problems one is interested in finding a function whose derivative satisfies specified conditions. The following example illustrates a geometric problem of this type.

Example 6 Suppose that a point moves along some unknown curve $y = f(x)$ in the xy-plane in such a way that at each point (x, y) on the curve, the tangent line has slope x^2. Find an equation for the curve given that it passes through the point (2, 1).

Solution. We know that $dy/dx = x^2$, so

$$y = \int x^2\,dx = \tfrac{1}{3}x^3 + C$$

Since the curve passes through (2, 1), a specific value for C can be found by using the fact that $y = 1$ if $x = 2$. Substituting these values in the above equation yields

$$1 = \tfrac{1}{3}(2^3) + C \quad \text{or} \quad C = -\tfrac{5}{3}$$

so the curve is $y = \frac{1}{3}x^3 - \frac{5}{3}$. ◀

Observe that in this example the requirement that the unknown curve pass through the point (2, 1) enabled us to determine a specific value for the constant of integration, thereby isolating the single integral curve $y = \frac{1}{3}x^3 - \frac{5}{3}$ from the family $y = \frac{1}{3}x^3 + C$ (Figure 5.2.2).

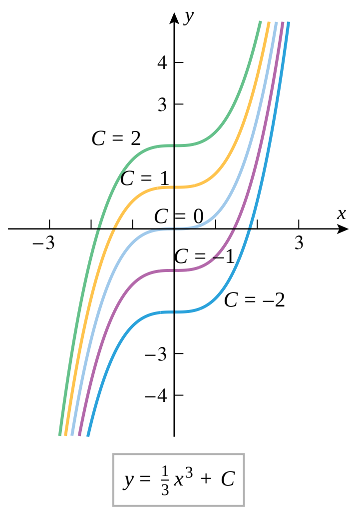


Figure 5.2.1

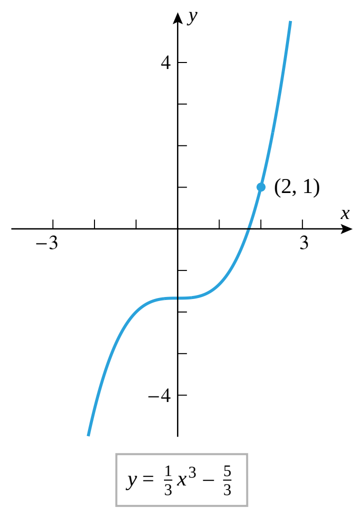


Figure 5.2.2

INTEGRATION FROM THE VIEWPOINT OF DIFFERENTIAL EQUATIONS

We will now consider another way of looking at integration that will be useful in our later work. Suppose that $f(x)$ is a known function and we are interested in finding a function $F(x)$ such that $y = F(x)$ satisfies the equation

$$\frac{dy}{dx} = f(x) \tag{8}$$

The solutions of this equation are the antiderivatives of $f(x)$, and we know that these can be obtained by integrating $f(x)$. For example, the solutions of the equation

$$\frac{dy}{dx} = x^2 \tag{9}$$

are

$$y = \int x^2\, dx = \frac{x^3}{3} + C$$

Equation (8) is called a ***differential equation*** because it involves a derivative of an unknown function. Differential equations are different from the kinds of equations we have encountered so far in that the unknown is a *function* and not a *number* as in an equation such as $x^2 + 5x - 6 = 0$.

Sometimes we will not be interested in finding all of the solutions of (8), but rather we will want only the solution whose integral curve passes through a specified point (x_0, y_0). For example, in Example 6 we solved (9) for the integral curve that passed through the point (2, 1).

For simplicity, it is common in the study of differential equations to denote a solution of $dy/dx = f(x)$ as $y(x)$ rather than $F(x)$, as earlier. With this notation, the problem of finding a function $y(x)$ whose derivative is $f(x)$ and whose integral curve passes through the point (x_0, y_0) is expressed as

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0 \tag{10}$$

This is called an ***initial-value problem***, and the requirement that $y(x_0) = y_0$ is called the ***initial condition*** for the problem.

Example 7 Solve the initial-value problem

$$\frac{dy}{dx} = \cos x, \quad y(0) = 1$$

Solution. The solution of the differential equation is

$$y = \int \cos x\, dx = \sin x + C \tag{11}$$

The initial condition $y(0) = 1$ implies that $y = 1$ if $x = 0$; substituting these values in (11) yields

$$1 = \sin(0) + C \quad \text{or} \quad C = 1$$

Thus, the solution of the initial-value problem is $y = \sin x + 1$. ◀

DIRECTION FIELDS

If we interpret dy/dx as the slope of a tangent line, then at a point (x, y) on an integral curve of the equation $dy/dx = f(x)$, the slope of the tangent line is $f(x)$. What is interesting about this is that the slopes of the tangent lines to the integral curves can be obtained without actually solving the differential equation. For example, if

$$\frac{dy}{dx} = \sqrt{x^2 + 1}$$

then we know without solving the equation that at the point where $x = 1$ the tangent line to an integral curve has slope $\sqrt{1^2 + 1} = \sqrt{2}$; and more generally, at a point where $x = a$, the tangent line to an integral curve has slope $\sqrt{a^2 + 1}$.

A geometric description of the integral curves of a differential equation $dy/dx = f(x)$ can be obtained by choosing a rectangular grid of points in the xy-plane, calculating the slopes of the tangent lines to the integral curves at the gridpoints, and drawing small portions of the tangent lines at those points. The resulting picture, which is called a ***direction field*** or ***slope field*** for the equation, shows the "direction" of the integral curves at the gridpoints. With sufficiently many gridpoints it is often possible to visualize the integral curves themselves; for example, Figure 5.2.3*a* shows a direction field for the differential equation $dy/dx = x^2$, and Figure 5.2.3*b* shows that same field with the integral curves

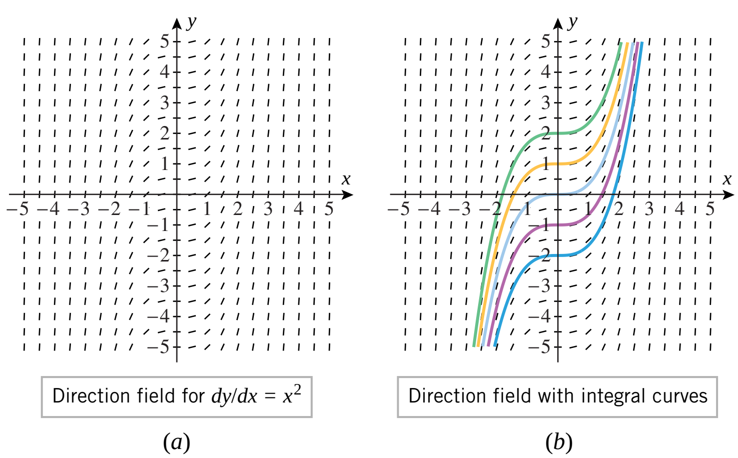


Figure 5.2.3

imposed on it—the more gridpoints that are used, the more completely the direction field reveals the shape of the integral curves. However, the amount of computation can be considerable, so computers are usually used when direction fields with many gridpoints are needed.

EXERCISE SET 5.2 Graphing Calculator

1. In each part, confirm that the formula is correct, and state a corresponding integration formula.

(a) $\dfrac{d}{dx}[\sqrt{1+x^2}\,] = \dfrac{x}{\sqrt{1+x^2}}$

(b) $\dfrac{d}{dx}\left[\frac{1}{3}\sin(1+x^3)\right] = x^2\cos(1+x^3)$

2. In each part, confirm that the stated formula is correct by differentiating.

(a) $\displaystyle\int x\sin x\,dx = \sin x - x\cos x + C$

(b) $\displaystyle\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$

In Exercises 3–6, find the derivative and state a corresponding integration formula.

3. $\dfrac{d}{dx}[\sqrt{x^3+5}\,]$ **4.** $\dfrac{d}{dx}\left[\dfrac{x}{x^2+3}\right]$

5. $\dfrac{d}{dx}[\sin(2\sqrt{x}\,)]$ **6.** $\dfrac{d}{dx}[\sin x - x\cos x]$

In Exercises 7 and 8, evaluate the integral by rewriting the integrand appropriately, if required, and then apply Formula 2 in Table 5.2.1.

7. (a) $\displaystyle\int x^8\,dx$ (b) $\displaystyle\int x^{5/7}\,dx$ (c) $\displaystyle\int x^3\sqrt{x}\,dx$

8. (a) $\displaystyle\int \sqrt[3]{x^2}\,dx$ (b) $\displaystyle\int \frac{1}{x^6}\,dx$ (c) $\displaystyle\int x^{-7/8}\,dx$

In Exercises 9–12, evaluate each integral by applying Theorem 5.2.3 and Formula 2 in Table 5.2.1 appropriately.

9. (a) $\displaystyle\int \frac{1}{2x^3}\,dx$ (b) $\displaystyle\int (u^3-2u+7)\,du$

10. $\displaystyle\int (x^{2/3}-4x^{-1/5}+4)\,dx$

11. $\displaystyle\int (x^{-3}+\sqrt{x}-3x^{1/4}+x^2)\,dx$

12. $\displaystyle\int \left(\frac{7}{y^{3/4}} - \sqrt[3]{y} + 4\sqrt{y}\right)dy$

In Exercises 13–28, evaluate the integral, and check your answer by differentiating.

13. $\displaystyle\int x(1+x^3)\,dx$ **14.** $\displaystyle\int \left(2+y^2\right)^2 dy$

15. $\displaystyle\int x^{1/3}(2-x)^2\,dx$ **16.** $\displaystyle\int (1+x^2)(2-x)\,dx$

17. $\displaystyle\int \frac{x^5+2x^2-1}{x^4}\,dx$ **18.** $\displaystyle\int \frac{1-2t^3}{t^3}\,dt$

19. $\displaystyle\int [4\sin x + 2\cos x]\,dx$

20. $\displaystyle\int [4\sec^2 x + \csc x\cot x]\,dx$

21. $\displaystyle\int \sec x(\sec x + \tan x)\,dx$ **22.** $\displaystyle\int \sec x(\tan x + \cos x)\,dx$

23. $\displaystyle\int \frac{\sec\theta}{\cos\theta}\,d\theta$ **24.** $\displaystyle\int \frac{dy}{\csc y}$

25. $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx$ **26.** $\displaystyle\int \left[\phi + \frac{2}{\sin^2\phi}\right] d\phi$

27. $\displaystyle\int [1+\sin^2\theta\csc\theta]\,d\theta$ **28.** $\displaystyle\int \frac{\sin 2x}{\cos x}\,dx$

29. Evaluate the integral
$$\int \frac{1}{1+\sin x}\,dx$$
by multiplying the numerator and denominator by an appropriate expression.

30. Use the double-angle formula $\cos 2x = 2\cos^2 x - 1$ to evaluate the integral
$$\int \frac{1}{1+\cos 2x}\,dx$$

31. (a) Use a graphing utility to generate a slope field for the differential equation $dy/dx = x$ in the region $-5 \le x \le 5$ and $-5 \le y \le 5$.
(b) Graph some representative integral curves of the function $f(x) = x$.
(c) Find an equation for the integral curve that passes through the point $(4, 7)$.

32. (a) Use a graphing utility to generate a slope field for the differential equation $dy/dx = \sqrt{x}$ in the region $0 \le x \le 10$ and $-5 \le y \le 5$.
(b) Graph some representative integral curves of the function $f(x) = \sqrt{x}$ for $x > 0$.
(c) Find an equation for the integral curve that passes through the point $\left(4, \frac{10}{3}\right)$.

33. Use a graphing utility to generate some representative integral curves of the function $f(x) = 5x^4 - \sec^2 x$ over the interval $(-\pi/2, \pi/2)$.

34. Use a graphing utility to generate some representative integral curves of the function $f(x) = (x^3 - 1)/x^2$ over the interval $(0, 5)$.

35. Suppose that a point moves along a curve $y = f(x)$ in the xy-plane in such a way that at each point (x, y) on the curve the tangent line has slope $-\sin x$. Find an equation for the curve, given that it passes through the point $(0, 2)$.

36. Suppose that a point moves along a curve $y = f(x)$ in the xy-plane in such a way that at each point (x, y) on the curve the tangent line has slope $(x + 1)^2$. Find an equation for the curve, given that it passes through the point $(-2, 8)$.

In Exercises 37 and 38, solve the initial-value problems.

37. (a) $\dfrac{dy}{dx} = \sqrt[3]{x},\ y(1) = 2$

(b) $\dfrac{dy}{dt} = \sin t + 1,\ y\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$

(c) $\dfrac{dy}{dx} = \dfrac{x + 1}{\sqrt{x}},\ y(1) = 0$

38. (a) $\dfrac{dy}{dx} = \dfrac{1}{(2x)^3},\ y(1) = 0$

(b) $\dfrac{dy}{dt} = \sec^2 t - \sin t,\ y\left(\dfrac{\pi}{4}\right) = 1$

(c) $\dfrac{dy}{dx} = x^2\sqrt{x^3},\ y(0) = 0$

39. Find the general form of a function whose second derivative is $\sqrt{x}$. [*Hint:* Solve the equation $f''(x) = \sqrt{x}$ for $f(x)$ by integrating both sides twice.]

40. Find a function f such that $f''(x) = x + \cos x$ and such that $f(0) = 1$ and $f'(0) = 2$. [*Hint:* Integrate both sides of the equation twice.]

In Exercises 41–43, find an equation of the curve that satisfies the given conditions.

41. At each point (x, y) on the curve the slope is $2x + 1$; the curve passes through the point $(-3, 0)$.

42. At each point (x, y) on the curve the slope equals the square of the distance between the point and the y-axis; the point $(-1, 2)$ is on the curve.

43. At each point (x, y) on the curve, y satisfies the condition $d^2y/dx^2 = 6x$; the line $y = 5 - 3x$ is tangent to the curve at the point where $x = 1$.

44. Suppose that a uniform metal rod 50 cm long is insulated laterally, and the temperatures at the exposed ends are maintained at 25°C and 85°C, respectively. Assume that an x-axis is chosen as in the accompanying figure and that the temperature $T(x)$ satisfies the equation

$$\frac{d^2T}{dx^2} = 0$$

Find $T(x)$ for $0 \le x \le 50$.

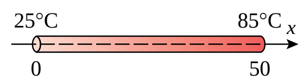


Figure Ex-44

45. (a) Show that

$$F(x) = \tfrac{1}{6}(3x + 4)^2 \quad \text{and} \quad G(x) = \tfrac{3}{2}x^2 + 4x$$

differ by a constant by showing that they are antiderivatives of the same function.

(b) Find the constant C such that $F(x) - G(x) = C$ by evaluating $F(x)$ and $G(x)$ at some point x_0.

(c) Check your answer in part (b) by simplifying the expression $F(x) - G(x)$ algebraically.

46. Follow the directions of Exercise 45 with

$$F(x) = \frac{x^2}{x^2 + 5} \quad \text{and} \quad G(x) = -\frac{5}{x^2 + 5}$$

In Exercises 47 and 48, use a trigonometric identity to help evaluate the integral.

47. $\displaystyle\int \tan^2 x\, dx$

48. $\displaystyle\int \cot^2 x\, dx$

49. Use the identities $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$ to help evaluate the integrals

(a) $\displaystyle\int \sin^2(x/2)\, dx$

(b) $\displaystyle\int \cos^2(x/2)\, dx$

50. Let F and G be the functions defined piecewise by

$$F(x) = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases} \quad \text{and} \quad G(x) = \begin{cases} x + 2, & x > 0 \\ -x + 3, & x < 0 \end{cases}$$

(a) Show that F and G have the same derivative.

(b) Show that $G(x) \ne F(x) + C$ for any constant C.

(c) Do parts (a) and (b) violate Theorem 5.2.2? Explain.

51. The speed of sound in air at 0°C (or 273 K on the Kelvin scale) is 1087 ft/s, but the speed v increases as the temperature T rises. Experimentation has shown that the rate of change of v with respect to T is

$$\frac{dv}{dT} = \frac{1087}{2\sqrt{273}}T^{-1/2}$$

where v is in feet per second and T is in kelvins (K). Find a formula that expresses v as a function of T.

5.3 INTEGRATION BY SUBSTITUTION

In this section we will study a technique, called ***substitution****, that can often be used to transform complicated integration problems into simpler ones.*

u-SUBSTITUTION

The method of substitution can be motivated by examining the chain rule from the viewpoint of antidifferentiation. For this purpose, suppose that F is an antiderivative of f and that g is a differentiable function. The chain rule implies that the derivative of $F(g(x))$ can be expressed as

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

which we can write in integral form as

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C \tag{1}$$

or since F is an antiderivative of f,

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C \tag{2}$$

For our purposes it will be useful to let $u = g(x)$ and to write $du/dx = g'(x)$ in the differential form $du = g'(x)\,dx$. With this notation (1) can be expressed as

$$\int f(u)\,du = F(u) + C \tag{3}$$

The process of evaluating an integral of form (2) by converting it into form (3) with the substitution

$$u = g(x) \quad \text{and} \quad du = g'(x)\,dx$$

is called the ***method of u-substitution***. Here our emphasis is *not* on the interpretation of the expression $du = g'(x)\,dx$ as a function of dx as was done in Section 3.8. Instead, the differential notation serves primarily as a useful "bookkeeping" device for the method of u-substitution. The following example illustrates how the method works.

Example 1 Evaluate $\int (x^2 + 1)^{50} \cdot 2x\,dx$.

Solution. If we let $u = x^2 + 1$, then $du/dx = 2x$, which implies that $du = 2x\,dx$. Thus, the given integral can be written as

$$\int (x^2 + 1)^{50} \cdot 2x\,dx = \int u^{50}\,du = \frac{u^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C$$ ◀

It is important to realize that in the method of u-substitution you have control over the choice of u, but once you make that choice you have no control over the resulting expression for du. Thus, in the last example we *chose* $u = x^2 + 1$ but $du = 2x\,dx$ was *computed*. Fortunately, our choice of u, combined with the computed du, worked out perfectly to produce an integral involving u that was easy to evaluate. However, in general, the method of u-substitution will fail if the chosen u and the computed du cannot be used to produce an integrand in which no expressions involving x remain, or if you cannot evaluate the resulting integral. Thus, for example, the substitution $u = x^2$, $du = 2x\,dx$ will not work for the integral

$$\int 2x \sin x^4\,dx$$

because this substitution results in the integral

$$\int \sin u^2\,du$$

which still cannot be evaluated in terms of familiar functions.

In general, there are no hard and fast rules for choosing u, and in some problems no choice of u will work. In such cases other methods need to be used, some of which will be discussed later. Making appropriate choices for u will come with experience, but you may find the following guidelines, combined with a mastery of the basic integrals in Table 5.2.1, helpful.

Step 1. Look for some composition $f(g(x))$ within the integrand for which the substitution

$$u = g(x), \quad du = g'(x)$$

produces an integral that is expressed entirely in terms of u and du. This may or may not be possible.

Step 2. If you are successful in Step 1, then try to evaluate the resulting integral in terms of u. Again, this may or may not be possible.

Step 3. If you are successful in Step 2, then replace u by $g(x)$ to express your final answer in terms of x.

Step 4. If you are unsuccessful in reaching or completing Step 2, then it may either be that the integrand in the original integral has no antiderivative that can be expressed in terms of familiar functions, or that the integral is tricky and you have not yet hit on the right substitution. This state of affairs makes integration challenging, since one often does not know whether a failure of the substitution method is due to the nature of the integral or to a lack of cleverness on the part of the problem solver.

EASY TO RECOGNIZE SUBSTITUTIONS

In the simplest cases, it is unnecessary to consider Step 1(b) or 1(c). The easiest substitutions occur when the integrand is the derivative of a known function, except for a constant added to or subtracted from the independent variable.

Example 2

$$\int \sin(x+9)\,dx = \int \sin u\,du = -\cos u + C = -\cos(x+9) + C$$

$$\boxed{\begin{array}{c} u = x+9 \\ du = 1 \cdot dx = dx \end{array}}$$

$$\int (x-8)^{23}\,dx = \int u^{23}\,du = \frac{u^{24}}{24} + C = \frac{(x-8)^{24}}{24} + C \qquad \blacktriangleleft$$

$$\boxed{\begin{array}{c} u = x-8 \\ du = 1 \cdot dx = dx \end{array}}$$

Another easy u-substitution occurs when the integrand is the derivative of a known function, except for a constant that multiplies or divides the independent variable. The following example illustrates two ways to evaluate such integrals.

Example 3 Evaluate $\displaystyle\int \cos 5x\,dx$.

Solution.

$$\int \cos 5x\,dx = \int (\cos u)\cdot\frac{1}{5}du = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C$$

$$\boxed{\begin{array}{c} u = 5x \\ du = 5\,dx \text{ or } dx = \frac{1}{5}\,du \end{array}}$$

Alternative Solution. There is a variation of the preceding method that some people prefer. The substitution $u = 5x$ requires $du = 5\,dx$. If there were a factor of 5 in the integrand, then we could group the 5 and dx together to form the du required by the substitution. Since there is no factor of 5, we will insert one and compensate by putting a factor of $\frac{1}{5}$ in front of the integral. The computations are as follows:

$$\int \cos 5x\,dx = \frac{1}{5}\int \cos 5x \cdot 5\,dx = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C \quad ◀$$

$$\boxed{\begin{array}{c} u = 5x \\ du = 5\,dx \end{array}}$$

More generally, if the integrand is a composition of the form $f(ax + b)$, where $f(x)$ is an easy to integrate function, then the substitution $u = ax + b$, $du = a\,dx$ will work.

Example 4

$$\int \frac{dx}{\left(\frac{1}{3}x - 8\right)^5} = \int \frac{3\,du}{u^5} = 3\int u^{-5}\,du = -\frac{3}{4}u^{-4} + C = -\frac{3}{4}\left(\frac{1}{3}x - 8\right)^{-4} + C \quad ◀$$

$$\boxed{\begin{array}{c} u = \frac{1}{3}x - 8 \\ du = \frac{1}{3}\,dx \text{ or } dx = 3\,du \end{array}}$$

With the help of Theorem 5.2.3, a complicated integral can sometimes be computed by expressing it as a sum of simpler integrals.

Example 5

$$\begin{aligned} \int \left(\frac{1}{x^2} + \sec^2 \pi x\right) dx &= \int \frac{dx}{x^2} + \int \sec^2 \pi x\,dx = -\frac{1}{x} + \int \sec^2 \pi x\,dx \\ &= -\frac{1}{x} + \frac{1}{\pi}\int \sec^2 u\,du \\ &= -\frac{1}{x} + \frac{1}{\pi}\tan u + C = -\frac{1}{x} + \frac{1}{\pi}\tan \pi x + C \quad ◀ \end{aligned}$$

$$\boxed{\begin{array}{c} u = \pi x \\ du = \pi\,dx \text{ or } dx = \frac{1}{\pi}\,du \end{array}}$$

The next three examples illustrate Step 1(a) when the composition involves nonlinear functions.

Example 6 Evaluate $\int \sin^2 x \cos x\,dx$.

Solution. If we let $u = \sin x$, then

$$\frac{du}{dx} = \cos x, \quad \text{so} \quad du = \cos x\,dx$$

Thus,

$$\int \sin^2 x \cos x\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C \quad ◀$$

Example 7 Evaluate $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$.

Solution. If we let $u = \sqrt{x}$, then

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad du = \frac{1}{2\sqrt{x}}\,dx \quad \text{or} \quad 2\,du = \frac{1}{\sqrt{x}}\,dx$$

Thus,

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int 2\cos u\,du = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C \quad \blacktriangleleft$$

Example 8 Evaluate $\int t^4 \sqrt[3]{3-5t^5}\,dt$.

Solution.

$$\int t^4 \sqrt[3]{3-5t^5}\,dt = -\frac{1}{25}\int \sqrt[3]{u}\,du = -\frac{1}{25}\int u^{1/3}\,du$$

$u = 3 - 5t^5$
$du = -25t^4\,dt$ or $-\frac{1}{25}\,du = t^4\,dt$

$$= -\frac{1}{25}\frac{u^{4/3}}{4/3} + C = -\frac{3}{100}\left(3-5t^5\right)^{4/3} + C \quad \blacktriangleleft$$

LESS APPARENT SUBSTITUTIONS

The next two examples illustrate Steps 1(b) and 1(c), respectively.

Example 9 Evaluate $\int x^2\sqrt{x-1}\,dx$.

Solution. Let

$$u = x - 1 \quad \text{so that} \quad du = dx \tag{4}$$

From the first equality in (4)

$$x^2 = (u+1)^2 = u^2 + 2u + 1$$

so that

$$\int x^2\sqrt{x-1}\,dx = \int (u^2+2u+1)\sqrt{u}\,du = \int (u^{5/2} + 2u^{3/2} + u^{1/2})\,du$$

$$= \tfrac{2}{7}u^{7/2} + \tfrac{4}{5}u^{5/2} + \tfrac{2}{3}u^{3/2} + C$$

$$= \tfrac{2}{7}(x-1)^{7/2} + \tfrac{4}{5}(x-1)^{5/2} + \tfrac{2}{3}(x-1)^{3/2} + C \quad \blacktriangleleft$$

Example 10 Evaluate $\int \cos^3 x\,dx$.

Solution. The only compositions in the integrand that suggest themselves are

$$\cos^3 x = (\cos x)^3 \quad \text{and} \quad \cos^2 x = (\cos x)^2$$

However, neither the substitution $u = \cos x$ nor the substitution $u = \cos^2 x$ work (verify). Following the suggestion in Step 1(c), we write

$$\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx$$

and solve the equation $du = \cos x\,dx$ for $u = \sin x$. Since $\sin^2 x + \cos^2 x = 1$, we then have

$$\int \cos^3 x\,dx = \int \cos^2 x\cos x\,dx = \int (1-\sin^2 x)\cos x\,dx = \int (1-u^2)\,du$$

$$= u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C \quad \blacktriangleleft$$

INTEGRATION USING COMPUTER ALGEBRA SYSTEMS

The advent of computer algebra systems has made it possible to evaluate many kinds of integrals that would be laborious to evaluate by hand. For example, *Derive*, running on a handheld calculator, evaluated the integral

$$\int \frac{5x^2}{(1+x)^{1/3}}\,dx = \frac{3(x+1)^{2/3}(5x^2-6x+9)}{8} + C$$

in about a second. The computer algebra system *Mathematica*, running on a personal computer, required even less time to evaluate this same integral. However, just as one would not want to rely on a calculator to compute $2+2$, so one would not want to use a CAS to integrate a simple function such as $f(x) = x^2$. Thus, even if you have a CAS, you will want to develop a reasonable level of competence in evaluating basic integrals. Moreover, the mathematical techniques that we will introduce for evaluating basic integrals are precisely the techniques that computer algebra systems use to evaluate more complicated integrals.

FOR THE READER. If you have a CAS, use it to calculate the integrals in the examples of this section. If your CAS produces a form of the answer that is different from the one in the text, then confirm algebraically that the two answers agree. Your CAS has various commands for simplifying answers. Explore the effect of using the CAS to simplify the expressions it produces for the integrals.

EXERCISE SET 5.3 Graphing Calculator C CAS

In Exercises 1–4, evaluate the integrals by making the indicated substitutions.

1. (a) $\int 2x\left(x^2+1\right)^{23}\,dx;\ u = x^2+1$

(b) $\int \cos^3 x \sin x\,dx;\ u = \cos x$

(c) $\int \frac{1}{\sqrt{x}} \sin\sqrt{x}\,dx;\ u = \sqrt{x}$

(d) $\int \frac{3x\,dx}{\sqrt{4x^2+5}};\ u = 4x^2+5$

2. (a) $\int \sec^2(4x+1)\,dx;\ u = 4x+1$

(b) $\int y\sqrt{1+2y^2}\,dy;\ u = 1+2y^2$

(c) $\int \sqrt{\sin\pi\theta}\cos\pi\theta\,d\theta;\ u = \sin\pi\theta$

(d) $\int (2x+7)(x^2+7x+3)^{4/5}\,dx;\ u = x^2+7x+3$

3. (a) $\int \cot x \csc^2 x\,dx;\ u = \cot x$

(b) $\int (1+\sin t)^9 \cos t\,dt;\ u = 1+\sin t$

(c) $\int \cos 2x\,dx;\ u = 2x$

(d) $\int x\sec^2 x^2\,dx;\ u = x^2$

4. (a) $\int x^2\sqrt{1+x}\,dx;\ u = 1+x$

(b) $\int [\csc(\sin x)]^2 \cos x\,dx;\ u = \sin x$

(c) $\int \sin(x-\pi)\,dx;\ u = x-\pi$

(d) $\int \frac{5x^4}{(x^5+1)^2}\,dx;\ u = x^5+1$

In Exercises 5–30, evaluate the integrals by making appropriate substitutions.

5. $\int x\left(2-x^2\right)^3\,dx$

6. $\int (3x-1)^5\,dx$

7. $\int \cos 8x\,dx$

8. $\int \sin 3x\,dx$

9. $\int \sec 4x\tan 4x\,dx$

10. $\int \sec^2 5x\,dx$

11. $\int t\sqrt{7t^2+12}\,dt$

12. $\int \frac{x}{\sqrt{4-5x^2}}\,dx$

13. $\int \frac{x^2}{\sqrt{x^3+1}}\,dx$

14. $\int \frac{1}{(1-3x)^2}\,dx$

15. $\int \frac{x}{(4x^2+1)^3}\,dx$

16. $\int x\cos(3x^2)\,dx$

17. $\int \frac{\sin(5/x)}{x^2}\,dx$

18. $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx$

19. $\int x^2\sec^2(x^3)\,dx$

20. $\int \cos^3 2t \sin 2t\,dt$

21. $\displaystyle\int \sin^5 3t \cos 3t\, dt$

22. $\displaystyle\int \frac{\sin 2\theta}{(5+\cos 2\theta)^3}\, d\theta$

23. $\displaystyle\int \cos 4\theta \sqrt{2-\sin 4\theta}\, d\theta$

24. $\displaystyle\int \tan^3 5x \sec^2 5x\, dx$

25. $\displaystyle\int \sec^3 2x \tan 2x\, dx$

26. $\displaystyle\int [\sin(\sin\theta)]\cos\theta\, d\theta$

27. $\displaystyle\int x\sqrt{x-3}\, dx$

28. $\displaystyle\int \frac{y\, dy}{\sqrt{y+1}}$

29. $\displaystyle\int \sin^3 2\theta\, d\theta$

30. $\displaystyle\int \sec^4 3\theta\, d\theta$
[*Hint:* Apply Step 1(c) and a trigonometric identity.]

In Exercises 31–33, evaluate the integrals assuming that n is a positive integer and $b \neq 0$.

31. $\displaystyle\int (a+bx)^n\, dx$

32. $\displaystyle\int \sqrt[n]{a+bx}\, dx$

33. $\displaystyle\int \sin^n(a+bx)\cos(a+bx)\, dx$

34. Use a CAS to check the answers you obtained in Exercises 31–33. If the answer produced by the CAS does not match yours, show that the two answers are equivalent. [*Suggestion: Mathematica* users may find it helpful to apply the Simplify command to the answer.]

35. (a) Evaluate the integral $\int \sin x \cos x\, dx$ by two methods: first by letting $u = \sin x$, then by letting $u = \cos x$.
(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.

36. (a) Evaluate $\int (5x-1)^2\, dx$ by two methods: first square and integrate, then let $u = 5x - 1$.
(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.

In Exercises 37 and 38, solve the initial-value problems.

37. $\dfrac{dy}{dx} = \sqrt{3x+1};\ \ y(1) = 5$

38. $\dfrac{dy}{dx} = 6 - 5\sin 2x;\ \ y(0) = 3$

39. Find a function f such that the slope of the tangent line at a point (x, y) on the curve $y = f(x)$ is $\sqrt{3x+1}$, and the curve passes through the point $(0, 1)$.

40. Use a graphing utility to generate some typical integral curves of $f(x) = x/\sqrt{x^2+1}$ over the interval $(-5, 5)$.

41. A population of frogs is estimated to be 100,000 at the beginning of the year 2000. Suppose that the rate of growth of the population $p(t)$ (in thousands) after t years is $p'(t) = (4+0.15t)^{3/2}$. Estimate the projected population at the beginning of the year 2005.

5.4 SIGMA NOTATION; AREA AS A LIMIT

Recall from the informal discussion in Section 5.1 that if a function f is continuous and nonnegative on an interval $[a, b]$, then the "rectangle method" provides us with one approach to computing the area between the graph of f and the interval $[a, b]$. We begin this section with a discussion of a notation to represent lengthy sums in a concise form. Then we will discuss the rectangle method in more detail, both as a means for defining and for computing the area under a curve. In particular, we will show that such an area may be interpreted as a limit.

SIGMA NOTATION

The notation we will discuss is called ***sigma notation*** or ***summation notation*** because it uses the uppercase Greek letter Σ (sigma) to denote various kinds of sums. To illustrate how this notation works, consider the sum

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

in which each term is of the form k^2, where k is one of the integers from 1 to 5. In sigma notation this sum can be written as

$$\sum_{k=1}^{5} k^2$$

which is read "the summation of k^2, where k runs from 1 to 5." The notation tells us to form the sum of the terms that result when we substitute successive integers for k in the expression k^2, starting with $k = 1$ and ending with $k = 5$.

More generally, if $f(k)$ is a function of k, and if m and n are integers such that $m \leq n$, then

$$\sum_{k=m}^{n} f(k) \tag{1}$$

denotes the sum of the terms that result when we substitute successive integers for k, starting with $k = m$ and ending with $k = n$ (Figure 5.4.1).

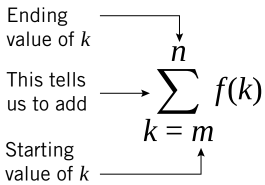


Figure 5.4.1

Example 1

$$\sum_{k=4}^{8} k^3 = 4^3 + 5^3 + 6^3 + 7^3 + 8^3$$

$$\sum_{k=1}^{5} 2k = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 = 2 + 4 + 6 + 8 + 10$$

$$\sum_{k=0}^{5} (2k + 1) = 1 + 3 + 5 + 7 + 9 + 11$$

$$\sum_{k=0}^{5} (-1)^k (2k + 1) = 1 - 3 + 5 - 7 + 9 - 11$$

$$\sum_{k=-3}^{1} k^3 = (-3)^3 + (-2)^3 + (-1)^3 + 0^3 + 1^3 = -27 - 8 - 1 + 0 + 1$$

$$\sum_{k=1}^{3} k \sin\left(\frac{k\pi}{5}\right) = \sin\frac{\pi}{5} + 2\sin\frac{2\pi}{5} + 3\sin\frac{3\pi}{5}$$ ◀

The numbers m and n in (1) are called, respectively, the ***lower*** and ***upper limits of summation***; and the letter k is called the ***index of summation***. It is not essential to use k as the index of summation; any letter not reserved for another purpose will do. For example,

$$\sum_{i=1}^{6} \frac{1}{i}, \quad \sum_{j=1}^{6} \frac{1}{j}, \quad \text{and} \quad \sum_{n=1}^{6} \frac{1}{n}$$

all denote the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

If the upper and lower limits of summation are the same, then the "sum" in (1) reduces to a single term. For example,

$$\sum_{k=2}^{2} k^3 = 2^3 \quad \text{and} \quad \sum_{i=1}^{1} \frac{1}{i+2} = \frac{1}{1+2} = \frac{1}{3}$$

In the sums

$$\sum_{i=1}^{5} 2 \quad \text{and} \quad \sum_{j=0}^{2} x^3$$

the expression to the right of the Σ sign does not involve the index of summation. In such cases, we take all the terms in the sum to be the same, with one term for each allowable value of the summation index. Thus,

$$\sum_{i=1}^{5} 2 = 2 + 2 + 2 + 2 + 2 \quad \text{and} \quad \sum_{j=0}^{2} x^3 = x^3 + x^3 + x^3$$

CHANGING THE LIMITS OF SUMMATION

A sum can be written in more than one way using sigma notation with different limits of summation and correspondingly different summands. For example,

$$\sum_{i=1}^{5} 2i = 2 + 4 + 6 + 8 + 10 = \sum_{j=0}^{4}(2j+2) = \sum_{k=3}^{7}(2k-4)$$

On occasion we will want to change the sigma notation for a given sum to a sigma notation with different limits of summation.

Example 2 Express

$$\sum_{k=3}^{7} 5^{k-2}$$

in sigma notation so that the lower limit of summation is 0 rather than 3.

Solution.

$$\begin{aligned}\sum_{k=3}^{7} 5^{k-2} &= 5^1 + 5^2 + 5^3 + 5^4 + 5^5 \\ &= 5^{0+1} + 5^{1+1} + 5^{2+1} + 5^{3+1} + 5^{4+1} \\ &= \sum_{j=0}^{4} = \sum_{k=0}^{4} 5^{k+1}\end{aligned}$$

◀

PROPERTIES OF SUMS

When stating general properties of sums it is often convenient to use a subscripted letter such as a_k in place of the function notation $f(k)$. For example,

$$\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = \sum_{j=1}^{5} a_j = \sum_{k=-1}^{3} a_{k+2}$$

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j = \sum_{k=-1}^{n-2} a_{k+2}$$

Our first properties provide some basic rules for manipulating sums.

5.4.1 THEOREM.

(*a*) $$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k \qquad (\text{if } c \text{ does not depend on } k)$$

(*b*) $$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

(*c*) $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

We will prove parts (*a*) and (*b*) and leave part (*c*) as an exercise.

Proof (a).

$$\sum_{k=1}^{n} ca_k = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k$$

Proof (b).

$$\begin{aligned}\sum_{k=1}^{n}(a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) \\ &= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k\end{aligned}$$

Restating Theorem 5.4.1 in words:

(*a*) *A constant factor can be moved through a sigma sign.*
(*b*) *Sigma distributes across sums.*
(*c*) *Sigma distributes across differences.*

SUMMATION FORMULAS

5.4.2 THEOREM.

(*a*) $$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(*b*) $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(*c*) $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

We will prove parts (*a*) and (*b*) and leave part (*c*) as an exercise.

Proof (a). Writing

$$\sum_{k=1}^{n} k$$

two ways, with summands in increasing order and in decreasing order, and then adding, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} k &= 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\
\sum_{k=1}^{n} k &= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\
\hline
2\sum_{k=1}^{n} k &= (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) \\
&= n(n+1)
\end{aligned}$$

Thus,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Proof (b). Note that

$$(k+1)^3 - k^3 = k^3 + 3k^2 + 3k + 1 - k^3 = 3k^2 + 3k + 1$$

So,

$$\sum_{k=1}^{n} [(k+1)^3 - k^3] = \sum_{k=1}^{n} (3k^2 + 3k + 1) \tag{2}$$

Writing out the left side of (2) with the index running *down* from $k = n$ to $k = 1$, we have

$$\begin{aligned}
\sum_{k=1}^{n} [(k+1)^3 - k^3] &= [(n+1)^3 - n^3] + \cdots + [4^3 - 3^3] + [3^3 - 2^3] + [2^3 - 1^3] \\
&= (n+1)^3 - 1
\end{aligned} \tag{3}$$

Combining (3) and (2), and expanding the right side of (2) by using Theorem 5.4.1 and part

(a) of this theorem yields

$$(n+1)^3 - 1 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$
$$= 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n$$

So,

$$3\sum_{k=1}^{n} k^2 = [(n+1)^3 - 1] - 3\frac{n(n+1)}{2} - n$$
$$= (n+1)^3 - 3(n+1)\left(\frac{n}{2}\right) - (n+1)$$
$$= \frac{n+1}{2}[2(n+1)^2 - 3n - 2]$$
$$= \frac{n+1}{2}[2n^2 + n] = \frac{n(n+1)(2n+1)}{2}$$

Thus,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

REMARK. The sum in (3) is an example of a ***telescoping sum***, since the cancellation of each of the two parts of an interior summand with parts of its neighboring summands allows the entire sum to collapse like a telescope.

Example 3 Evaluate $\sum_{k=1}^{30} k(k+1)$.

Solution.

$$\sum_{k=1}^{30} k(k+1) = \sum_{k=1}^{30} (k^2 + k) = \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k$$
$$= \frac{30(31)(61)}{6} + \frac{30(31)}{2} = 9920$$

Theorem 5.4.2(a), (b)

◀

In formulas such as

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{or} \quad 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

the left side of the equality is said to express the sum in ***open form*** and the right side is said to express it in ***closed form***. The open form indicates the summands and the closed form is an explicit formula for the sum.

Example 4 Express $\sum_{k=1}^{n} (3+k)^2$ in closed form.

Solution.

$$\sum_{k=1}^{n} (3+k)^2 = 4^2 + 5^2 + \cdots + (3+n)^2$$
$$= [1^2 + 2^2 + 3^3 + 4^2 + 5^2 + \cdots + (3+n)^2] - [1^2 + 2^2 + 3^2]$$
$$= \left(\sum_{k=1}^{3+n} k^2\right) - 14$$
$$= \frac{(3+n)(4+n)(7+2n)}{6} - 14 = \frac{1}{6}(73n + 21n^2 + 2n^3)$$

◀

FOR THE READER. Your numerical calculating utility probably provides some way of evaluating sums that can be expressed in sigma notation. Check your documentation to find out how to do this, and then use your utility to confirm that the numerical result obtained in Example 3 is correct. If you have access to a CAS, it provides some method for finding closed forms for sums such as those in Theorem 5.4.2. Use your CAS to confirm the formulas in that theorem, and then find closed forms for

$$\sum_{k=1}^{n} k^4 \quad \text{and} \quad \sum_{k=1}^{n} k^5$$

A DEFINITION OF AREA

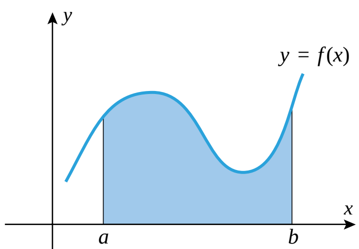


Figure 5.4.2

Suppose that f is a continuous function that is nonnegative on an interval $[a, b]$, and let R denote the region that is bounded below by the x-axis, bounded on the sides by the vertical lines $x = a$ and $x = b$, and bounded above by the curve $y = f(x)$ (Figure 5.4.2). Recall from the informal discussion in Section 5.1 that the "rectangle method" provides us with one approach to computing the area between the graph of f and the interval $[a, b]$. Our goal now is to define formally what we mean by the area of R. We will work from the definition of the area of a rectangle as the product of its length and width. Define the area of a region decomposed into a finite collection of rectangles to be the sum of the areas of those rectangles. To define the area of the region R, we will use these definitions and the rectangle method of Section 5.1. The basic idea is as follows:

- Divide the interval $[a, b]$ into n equal subintervals.
- Over each subinterval construct a rectangle whose height is the value of f at any point in the subinterval.
- The union of these rectangles forms a region R_n whose area can be regarded as an approximation to the "area" A of the region R.
- Repeat the process using more and more subdivisions.
- Define the area of R to be the "limit" of the areas of the approximating regions R_n, as n is made larger and larger without bound. We can express this idea symbolically as

$$A = \text{area}(R) = \lim_{n \to +\infty} [\text{area}(R_n)] \tag{4}$$

REMARK. There is a difference in interpretation between writing $\lim_{n\to+\infty}$ and writing $\lim_{x\to+\infty}$, where n represents a positive integer and x has no such restriction. Equation (4) should be interpreted to mean that by choosing the positive integer n sufficiently large, we can make area(R_n) as close to A as desired. Later we will study limits of the type $\lim_{n\to+\infty}$ in detail, but for now suffice it to say that the computational techniques we have used for limits of type $\lim_{x\to+\infty}$ will also work for $\lim_{n\to+\infty}$.

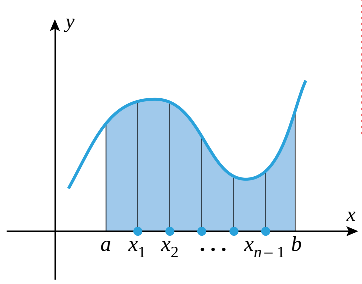


Figure 5.4.3

To make all of this more precise, it will be helpful to capture this procedure in mathematical notation. For this purpose, suppose that we divide the interval $[a, b]$ into n subintervals by inserting $n - 1$ equally spaced points between a and b, say

$$x_1, x_2, \ldots, x_{n-1}$$

(Figure 5.4.3). Each of these subintervals has width $(b - a)/n$, which it is customary to denote by

$$\Delta x = \frac{b-a}{n}$$

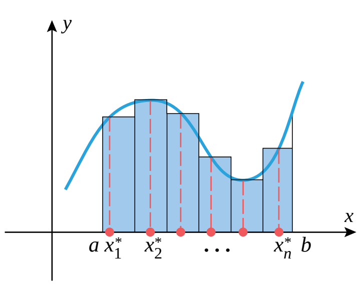


Figure 5.4.4

In each subinterval we need to choose an x-value at which to evaluate the function f to determine the height of a rectangle over the interval. If we denote those x-values by

$$x_1^*, x_2^*, \ldots, x_n^*$$

(Figure 5.4.4), then the areas of the rectangles constructed over these intervals will be

$$f(x_1^*)\Delta x, \quad f(x_2^*)\Delta x, \ldots, \quad f(x_n^*)\Delta x$$

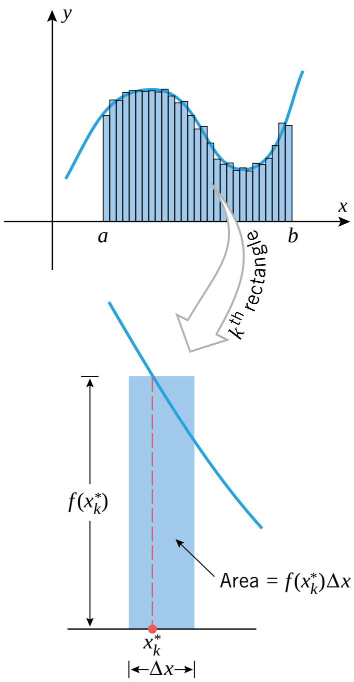


Figure 5.4.5

(Figure 5.4.5), and the total area of the region R_n will be

$$\text{area}(R_n) = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

With this notation (4) can be expressed as

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

which suggests the following definition of the area of the region R.

5.4.3 DEFINITION (*Area Under a Curve*). If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the ***area*** under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{5}$$

In (5) the values of $x_1^*, x_2^*, \ldots, x_n^*$ may be chosen in many different ways, so it is conceivable that different choices of these values might produce different values of A. Were this to happen, then Definition 5.4.3 would not be an acceptable definition of area. Fortunately, this does not happen; it is proved in advanced courses that when f is continuous (as we have assumed), the same value of A results no matter how the x_k^* are chosen. In practice they are chosen in some systematic fashion, some common choices being:

- The left endpoint of each subinterval.
- The right endpoint of each subinterval.
- The midpoint of each subinterval.

If, as shown in Figure 5.4.6, the subinterval $[a, b]$ is divided by $x_1, x_2, x_3, \ldots, x_{n-1}$ into n equal parts each of length $\Delta x = (b - a)/n$, and if we let $x_0 = a$ and $x_n = b$, then

$$x_k = a + k\Delta x \quad \text{for } k = 0, 1, 2, \ldots, n$$

Thus,

$$x_k^* = x_{k-1} = a + (k-1)\Delta x \qquad \boxed{\text{Left endpoint}} \tag{6}$$

$$x_k^* = x_k = a + k\Delta x \qquad \boxed{\text{Right endpoint}} \tag{7}$$

$$x_k^* = \tfrac{1}{2}(x_{k-1} + x_k) = a + \left(k - \tfrac{1}{2}\right)\Delta x \qquad \boxed{\text{Midpoint}} \tag{8}$$

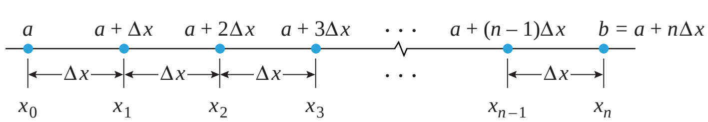


Figure 5.4.6

NUMERICAL APPROXIMATIONS OF AREA

We would expect from Definition 5.4.3 that for each of the choices (6), (7), and (8), the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \Delta x \sum_{k=1}^{n} f(x_k^*) = \Delta x[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)] \tag{9}$$

would yield a good approximation to the area A, provided n is a large positive integer. According to which of these three options is used in choosing the x_k^*, we refer to Formula (9) as the ***left endpoint approximation***, the ***right endpoint approximation***, or the ***midpoint approximation*** of the exact area (Figure 5.4.7).

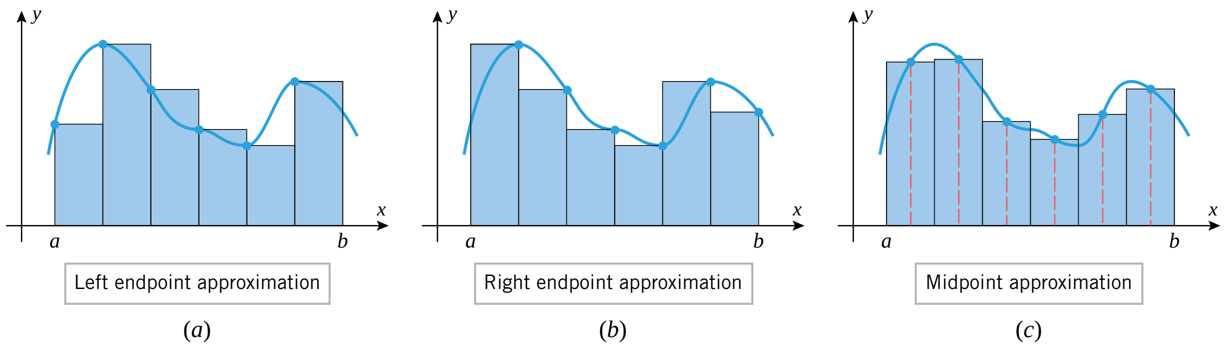


Figure 5.4.7

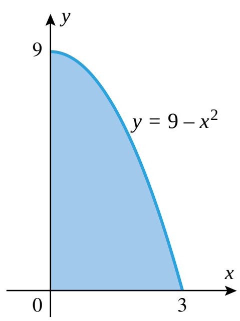


Figure 5.4.8

Example 5 Find the left endpoint, right endpoint, and midpoint approximations of the area under the curve $y = 9 - x^2$ over the interval $[0, 3]$ with $n = 10$, $n = 20$, and $n = 50$ (Figure 5.4.8).

Solution. Details of the computations for the case $n = 10$ are shown to six decimal places in Table 5.4.1 and the results of all computations are given in Table 5.4.2. ◀

Table 5.4.1

$n = 10,\ \Delta x = (b - a)/n = (3 - 0)/10 = 0.3$

	LEFT ENDPOINT APPROXIMATION		RIGHT ENDPOINT APPROXIMATION		MIDPOINT APPROXIMATION	
k	x_k^*	$9 - (x_k^*)^2$	x_k^*	$9 - (x_k^*)^2$	x_k^*	$9 - (x_k^*)^2$
1	0.0	9.000000	0.3	8.910000	0.15	8.977500
2	0.3	8.910000	0.6	8.640000	0.45	8.797500
3	0.6	8.640000	0.9	8.190000	0.75	8.437500
4	0.9	8.190000	1.2	7.560000	1.05	7.897500
5	1.2	7.560000	1.5	6.750000	1.35	7.177500
6	1.5	6.750000	1.8	5.760000	1.65	6.277500
7	1.8	5.760000	2.1	4.590000	1.95	5.197500
8	2.1	4.590000	2.4	3.240000	2.25	3.937500
9	2.4	3.240000	2.7	1.710000	2.55	2.497500
10	2.7	1.710000	3.0	0.000000	2.85	0.877500
		64.350000		55.350000		60.075000
$\Delta x \sum_{k=1}^{n} f(x_k^*)$		(0.3)(64.350000) = 19.305000		(0.3)(55.350000) = 16.605000		(0.3)(60.075000) = 18.022500

Table 5.4.2

n	LEFT ENDPOINT APPROXIMATION	RIGHT ENDPOINT APPROXIMATION	MIDPOINT APPROXIMATION
10	19.305000	16.605000	18.022500
20	18.663750	17.313750	18.005625
50	18.268200	17.728200	18.000900

REMARK. We will show below that the exact area under $y = 9 - x^2$ over the interval [0, 3] is 18 (i.e., 18 square units), so that in the preceding example the midpoint approximation is more accurate than either of the endpoint approximations. This can also be seen geometrically from the approximating rectangles: Since the graph of $y = 9 - x^2$ is decreasing over the interval [0, 3], each left endpoint approximation overestimates the area, each right endpoint approximation underestimates the area, and each midpoint approximation falls between the overestimate and the underestimate (Figure 5.4.9). This is consistent with the values in Table 5.4.2. Later in the text we will investigate the error that results when an area is approximated by the midpoint rule.

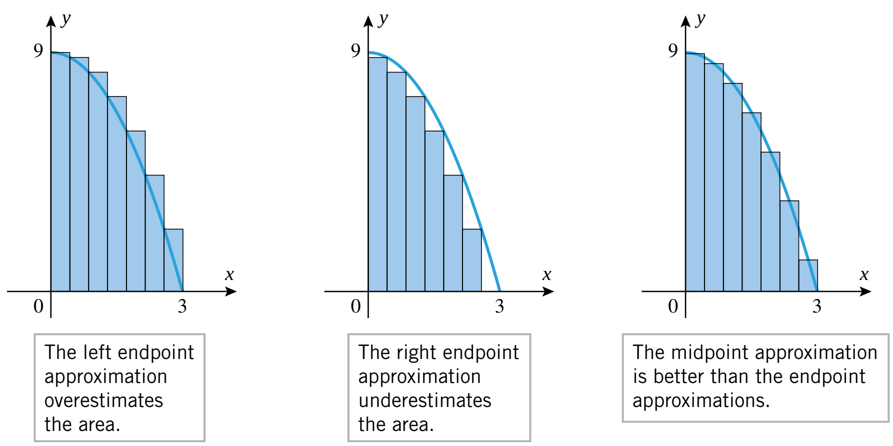


Figure 5.4.9

COMPUTING THE EXACT VALUE OF AN AREA

Although numerical approximations of area are useful, we will often wish to compute the *exact* value of some area. In certain cases this can be done by explicitly evaluating the limit in Definition 5.4.3.

Example 6 Use Definition 5.4.3 with x_k^* as the right endpoint of each subinterval to find the area between the graph of $f(x) = x^2$ and the interval [0, 1].

Solution. We have

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

and from (7)

$$x_k^* = a + k\Delta x = \frac{k}{n}$$

so that

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} (x_k^*)^2 \Delta x = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

Therefore,

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

(Note that this conclusion agrees with the numerical evidence we collected in Table 5.1.2.) ◀

In the solution to Example 6 we made use of one of the "closed form" summation formulas from Theorem 5.4.2. The next result collects some consequences of Theorem 5.4.2 that can facilitate computations of area using Definition 5.4.3.

5.4.4 THEOREM.

(*a*) $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} 1 = 1$ (*b*) $\lim_{n\to+\infty} \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{2}$

(*c*) $\lim_{n\to+\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^3 = \frac{1}{3}$ (*d*) $\lim_{n\to+\infty} \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{1}{4}$

The proof of Theorem 5.4.4 is left as an exercise for the reader.

Example 7 Use Definition 5.4.3 with x_k^* as the midpoint of each subinterval to find the area under the parabola $y = f(x) = 9 - x^2$ and over the interval $[0, 3]$.

Solution. Each subinterval will have length

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

and from (8)

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = \left(k - \frac{1}{2}\right)\left(\frac{3}{n}\right)$$

Thus,

$$\begin{aligned}
f(x_k^*)\Delta x = [9 - (x_k^*)^2]\Delta x &= \left[9 - \left(k - \frac{1}{2}\right)^2\left(\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) \\
&= \left[9 - \left(k^2 - k + \frac{1}{4}\right)\left(\frac{9}{n^2}\right)\right]\left(\frac{3}{n}\right) \\
&= \frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^{n} f(x_k^*)\Delta x &= \sum_{k=1}^{n}\left(\frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}\right) \\
&= 27\left[\frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n}\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{1}{4n^2}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
A &= \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x \\
&= \lim_{n\to+\infty} 27\left[\frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n}\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{1}{4n^2}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right] \\
&= 27\left[1 - \frac{1}{3} + 0\cdot\frac{1}{2} - 0\cdot 1\right] = 18
\end{aligned}$$

where we used Theorem 5.4.4 to compute the limits as $n\to+\infty$ of the expressions

$$\frac{1}{n^j}\sum_{k=1}^{n} k^{j-1} \quad \text{for } j = 1, 2, 3$$

◀

NET SIGNED AREA

In Definition 5.4.3 we assumed that f is continuous and nonnegative on the interval $[a, b]$. If f is continuous and attains both positive and negative values on $[a, b]$, then the limit

$$\lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{10}$$

no longer represents the area between the curve $y = f(x)$ and the interval $[a, b]$ on the x-axis; rather, it represents a difference of areas—the area of the region that is above the interval $[a, b]$ and below the curve $y = f(x)$ minus the area of the region that is below the interval $[a, b]$ and above the curve $y = f(x)$. We call this the ***net signed area*** between the graph of $y = f(x)$ and the interval $[a, b]$. For example, in Figure 5.4.10*a*, the net signed area between the curve $y = f(x)$ and the interval $[a, b]$ is

$$(A_I + A_{III}) - A_{II} = \big[\text{area above } [a, b]\big] - \big[\text{area below } [a, b]\big]$$

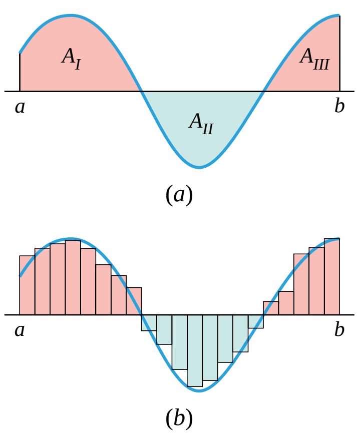


Figure 5.4.10

To explain why the limit in (10) represents this net signed area, let us subdivide the interval $[a, b]$ in Figure 5.4.10*a* into n equal subintervals and examine the terms in the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x \tag{11}$$

If $f(x_k^*)$ is positive, then the product $f(x_k^*)\Delta x$ represents the area of the rectangle with height $f(x_k^*)$ and base Δx (the biege rectangles in Figure 5.4.10*b*). However, if $f(x_k^*)$ is negative, then the product $f(x_k^*)\Delta x$ is the *negative* of the area of the rectangle with height $|f(x_k^*)|$ and base Δx (the green rectangles in Figure 5.4.10*b*). Thus, (11) represents the total area of the beige rectangles minus the total area of the green rectangles. As n increases, the pink rectangles fill out the regions with areas A_I and A_{III} and the green rectangles fill out the region with area A_{II}, which explains why the limit in (10) represents the signed area between $y = f(x)$ and the interval $[a, b]$. We formalize this in the following definition.

5.4.5 DEFINITION (*Net Signed Area*). If the function f is continuous on $[a, b]$, then the ***net signed area*** A between $y = f(x)$ and the interval $[a, b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

As with Definition 5.4.3, it can be shown that for a continuous function this limit always exists (independently of the choice of the numbers x_k^*). The net signed area between the curve $y = f(x)$ and $[a, b]$ can be positive, negative, or zero; it is positive when there is more area above the interval than below, negative when there is more area below than above, and zero when the areas above and below are equal.

Example 8 Use Definition 5.4.5 with x_k^* as the left endpoint of each subinterval to find the net signed area between the graph of $y = f(x) = x - 1$ and the interval $[0, 2]$.

Solution. Each subinterval will have length

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

and from (6)

$$x_k^* = a + (k-1)\Delta x = (k-1)\left(\frac{2}{n}\right)$$

Thus,

$$f(x_k^*)\Delta x = (x_k^* - 1)\Delta x = \left[(k-1)\left(\frac{2}{n}\right) - 1\right]\left(\frac{2}{n}\right) = \left(\frac{4}{n^2}\right)k - \frac{4}{n^2} - \frac{2}{n}$$

and

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left[\left(\frac{4}{n^2}\right)k - \frac{4}{n^2} - \frac{2}{n}\right]$$

$$= 4\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{4}{n}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) - 2\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)$$

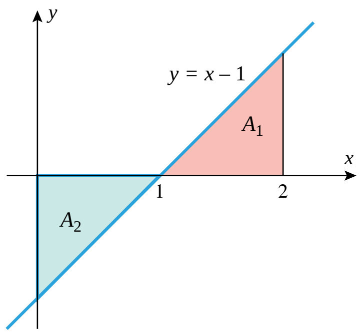


Figure 5.4.11

Therefore,

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty}\left[4\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{4}{n}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) - 2\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right]$$

$$= 4\left(\frac{1}{2}\right) - 0\cdot 1 - 2(1) = 0$$

Since the net signed area is zero, the area A_1 below the graph of f and above the interval $[0, 2]$ must equal the area A_2 above the graph of f and below the interval $[0, 2]$. This conclusion agrees with the graph of f shown in Figure 5.4.11. ◀

EXERCISE SET 5.4 C CAS

1. Evaluate

(a) $\sum_{k=1}^{3} k^3$ (b) $\sum_{j=2}^{6}(3j-1)$ (c) $\sum_{i=-4}^{1}(i^2-i)$

(d) $\sum_{n=0}^{5} 1$ (e) $\sum_{k=0}^{4}(-2)^k$ (f) $\sum_{n=1}^{6}\sin n\pi$.

2. Evaluate

(a) $\sum_{k=1}^{4} k\sin\frac{k\pi}{2}$ (b) $\sum_{j=0}^{5}(-1)^j$ (c) $\sum_{i=7}^{20}\pi^2$

(d) $\sum_{m=3}^{5} 2^{m+1}$ (e) $\sum_{n=1}^{6}\sqrt{n}$ (f) $\sum_{k=0}^{10}\cos k\pi$.

In Exercises 3–8, write each expression in sigma notation, but do not evaluate.

3. $1+2+3+\cdots+10$

4. $3\cdot 1+3\cdot 2+3\cdot 3+\cdots+3\cdot 20$

5. $2+4+6+8+\cdots+20$ **6.** $1+3+5+7+\cdots+15$

7. $1-3+5-7+9-11$ **8.** $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$

9. (a) Express the sum of the even integers from 2 to 100 in sigma notation.
(b) Express the sum of the odd integers from 1 to 99 in sigma notation.

10. Express in sigma notation.
(a) $a_1-a_2+a_3-a_4+a_5$
(b) $-b_0+b_1-b_2+b_3-b_4+b_5$
(c) $a_0+a_1x+a_2x^2+\cdots+a_nx^n$
(d) $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$

In Exercises 11–16, use Theorem 5.4.2 to evaluate the sums, and check your answers using the summation feature of a calculating utility.

11. $\sum_{k=1}^{100} k$ **12.** $\sum_{k=1}^{100}(7k+1)$ **13.** $\sum_{k=1}^{20} k^2$

14. $\sum_{k=4}^{20} k^2$ **15.** $\sum_{k=1}^{30} k(k-2)(k+2)$ **16.** $\sum_{k=1}^{6}(k-k^3)$

In Exercises 17–20, express the sums in closed form.

17. $\sum_{k=1}^{n}\frac{3k}{n}$ **18.** $\sum_{k=1}^{n-1}\frac{k^2}{n}$ **19.** $\sum_{k=1}^{n-1}\frac{k^3}{n^2}$

20. $\sum_{k=1}^{n}\left(\frac{5}{n}-\frac{2k}{n}\right)$

C **21.** For each of the sums that you obtained in Exercises 17–20, use a CAS to check your answer. If the answer produced by the CAS does not match your own, show that the two answers are equivalent.

22. Solve the equation $\sum_{k=1}^{n} k = 465$.

In Exercises 23–26, express the function of n in closed form and then find the limit.

23. $\lim_{n\to+\infty}\frac{1+2+3+\cdots+n}{n^2}$

24. $\lim_{n\to+\infty}\frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$

25. $\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{5k}{n^2}$

26. $\lim_{n\to+\infty}\sum_{k=1}^{n-1}\frac{2k^2}{n^3}$

27. Express $1+2+2^2+2^3+2^4+2^5$ in sigma notation with
(a) $j=0$ as the lower limit of summation
(b) $j=1$ as the lower limit of summation
(c) $j=2$ as the lower limit of summation.

28. Express

$$\sum_{k=5}^{9} k2^{k+4}$$

in sigma notation with
(a) $k=1$ as the lower limit of summation
(b) $k=13$ as the upper limit of summation.

In Exercises 29–32, divide the interval $[a,b]$ into $n=4$ subintervals of equal length, and then compute

$$\sum_{k=1}^{4} f(x_k^*)\Delta x$$

with x_k^* as (a) the left endpoint of each subinterval, (b) the midpoint of each subinterval, and (c) the right endpoint of each subinterval.

29. $f(x)=3x+1;\ a=2, b=6$

30. $f(x)=1/x;\ a=1, b=9$

31. $f(x)=\cos x;\ a=0, b=\pi$

32. $f(x)=2x-x^2;\ a=-1, b=3$

In Exercises 33–36, use a calculating utility with summation capabilities or a CAS to obtain an approximate value for the area between the curve and the specified interval with $n=10, 20,$ and 50 subintervals by using the (a) left endpoint, (b) right endpoint, and (c) midpoint approximations. (If you do not have access to such a utility, then just do the case $n=10$.)

C **33.** $y=1/x;\ [1,2]$

C **34.** $y=1/x^2;\ [1,3]$

C **35.** $y=\sqrt{x};\ [0,4]$

C **36.** $y=\sin x;\ [0,\pi/2]$

In Exercises 37–42, use Definition 5.4.3 with x_k^* as the *right* endpoint of each subinterval to find the area under the curve $y=f(x)$ over the interval $[a,b]$.

37. $y=\frac{1}{2}x;\ a=1, b=4$

38. $y=5-x;\ a=0, b=5$

39. $y=9-x^2;\ a=0, b=3$

40. $y=4-\frac{1}{4}x^2;\ a=0, b=3$

41. $y=x^3;\ a=2, b=6$

42. $y=1-x^3;\ a=-3, b=-1$

In Exercises 43–46, use Definition 5.4.5 with x_k^* as the *left* endpoint of each subinterval to find the area under the curve $y=f(x)$ over the interval $[a,b]$.

43. The function f and interval $[a,b]$ of Exercise 37.

44. The function f and interval $[a,b]$ of Exercise 38.

45. The function f and interval $[a,b]$ of Exercise 39.

46. The function f and interval $[a,b]$ of Exercise 40.

In Exercises 47 and 48, use Definition 5.4.3 with x_k^* as the *midpoint* of each subinterval to find the area under the curve $y=f(x)$ over the interval $[a,b]$.

47. The function $f(x)=x^2;\ a=0, b=1$

48. The function $f(x)=x^2;\ a=-1, b=1$

In Exercises 49–52, use Definition 5.4.5 with x_k^* as the *right* endpoint of each subinterval to find the net signed area between the curve $y=f(x)$ and the interval $[a,b]$.

49. $y=x;\ a=-1, b=1$. Verify your answer with a simple geometric argument.

50. $y=x;\ a=-1, b=2$. Verify your answer with a simple geometric argument.

51. $y=x^2-1;\ a=0, b=2$

52. $y=x^3;\ a=-1, b=1$

53. Use Definition 5.4.3 with x_k^* as the left endpoint of each subinterval to find the area under the graph of $y=mx$ and over the interval $[a,b]$, where $m>0$ and $a\geq 0$.

54. Use Definition 5.4.5 with x_k^* as the right endpoint of each subinterval to find the net signed area between the graph of $y=mx$ and the interval $[a,b]$.

55. (a) Show that the area under the graph of $y=x^3$ and over the interval $[0,b]$ is $b^4/4$.
(b) Find a formula for the area under $y=x^3$ over the interval $[a,b]$, where $a\geq 0$.

56. Find the area between the graph of $y=\sqrt{x}$ and the interval $[0,1]$. [*Hint:* Use the result of Exercise 17 of Section 5.1.]

57. An artist wants to create a rough triangular design using uniform square tiles glued edge to edge. She places n tiles in a row to form the base of the triangle and then makes each successive row two tiles shorter than the preceding row. Find a formula for the number of tiles used in the design. [*Hint:* Your answer will depend on whether n is even or odd.]

58. An artist wants to create a sculpture by gluing together uniform spheres. She creates a rough rectangular base that has 50 spheres along one edge and 30 spheres along the other. She then creates successive layers by gluing spheres in the grooves of the preceding layer. How many spheres will there be in the sculpture?

59. By writing out the sums, determine whether the following are valid identities.

(a) $$\int\left[\sum_{i=1}^{n} f_i(x)\right]dx=\sum_{i=1}^{n}\left[\int f_i(x)\,dx\right]$$

(b) $$\frac{d}{dx}\left[\sum_{i=1}^{n} f_i(x)\right]=\sum_{i=1}^{n}\left[\frac{d}{dx}[f_i(x)]\right]$$

60. Which of the following are valid identities?

(a) $\displaystyle\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$ (b) $\displaystyle\sum_{i=1}^{n} \frac{a_i}{b_i} = \sum_{i=1}^{n} a_i \Big/ \sum_{i=1}^{n} b_i$

(c) $\displaystyle\sum_{i=1}^{n} a_i^2 = \left(\sum_{i=1}^{n} a_i\right)^2$

61. Prove part (c) of Theorem 5.4.1.

62. Prove part (c) of Theorem 5.4.2. [*Hint:* Begin with the difference $(k+1)^4 - k^4$ and follow the steps used to prove part (b) of the theorem.]

63. Prove Theorem 5.4.4.

5.5 THE DEFINITE INTEGRAL

In this section we will introduce the concept of a "definite integral," which will link the concept of area to other important concepts such as length, volume, density, probability, and work.

RIEMANN SUMS AND THE DEFINITE INTEGRAL

In our definition of net signed area (Definition 5.4.5), we assumed that for each positive number n, the interval $[a, b]$ was subdivided into n subintervals of equal length to create bases for the approximating rectangles. For some functions it may be more convenient to use rectangles with different widths (see Exercise 33); however, if we are to "exhaust" an area with rectangles of different widths, then it is important that successive subdivisions be constructed in such a way that the widths of the rectangles approach zero as n increases (Figure 5.5.1). Thus, we must preclude the kind of situation that occurs in Figure 5.5.2 in which the right half of the interval is never subdivided. If this kind of subdivision were allowed, the error in the approximation would not approach zero as n increased.

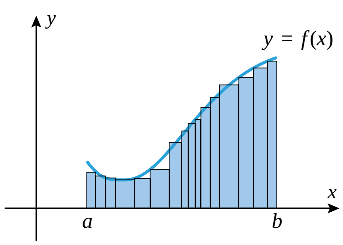


Figure 5.5.1

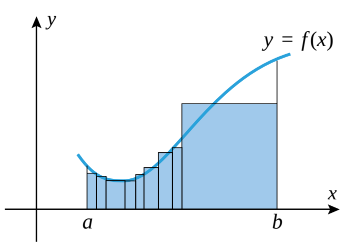


Figure 5.5.2

A ***partition*** of the interval $[a, b]$ is a collection of numbers

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

that divides $[a, b]$ into n subintervals of lengths

$$\Delta x_1 = x_1 - x_0, \quad \Delta x_2 = x_2 - x_1, \quad \Delta x_3 = x_3 - x_2, \ldots, \quad \Delta x_n = x_n - x_{n-1}$$

The partition is said to be ***regular*** provided the subintervals all have the same length

$$\Delta x_k = \Delta x = \frac{b-a}{n}$$

For a regular partition, the widths of the approximating rectangles approach zero as n is made large. Since this need not be the case for a general partition, we need some way to measure the "size" of these widths. One approach is to let max Δx_k denote the largest of the subinterval widths. The magnitude max Δx_k is called the ***mesh size*** of the partition. For example, Figure 5.5.3 shows a partition of the interval [0, 6] into four subintervals with a mesh size of 2.

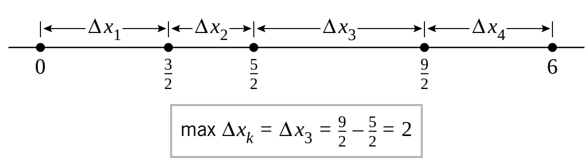


Figure 5.5.3

If we are to generalize Definition 5.4.5 so that it allows for unequal subinterval widths, we must replace the constant length Δx by the variable length Δx_k. When this is done the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x \quad \text{is replaced by} \quad \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

We also need to replace the expression $n \to +\infty$ by an expression that guarantees us that the lengths of all subintervals approach zero. We will use the expression $\max \Delta x_k \to 0$ for this purpose. (Some writers use the symbol $\|\Delta\|$ rather than $\max \Delta x_k$ for the mesh size of the partition, in which case $\max \Delta x_k \to 0$ would be replaced by $\|\Delta\| \to 0$.) Based on our initiutive concept of area, we would then expect the net signed area A between the graph of f and the interval $[a, b]$ to satisfy the equation

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

(We will see in a moment that this is the case.) The limit that appears in this expression is one of the fundamental concepts of integral calculus and forms the basis for the following definition.

5.5.1 DEFINITION. A function f is said to be ***integrable*** on a finite closed interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the numbers x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

which is called the ***definite integral*** of f from a to b. The numbers a and b are called the ***lower limit of integration*** and the ***upper limit of integration***, respectively, and $f(x)$ is called the ***integrand***.

The notation used for the definite integral deserves some comment. Historically, the expression "$f(x)\,dx$" was interpreted to be the "infinitesimal area" of a rectangle with height $f(x)$ and "infinitesimal" width dx. By "summing" these infinitesimal areas, the entire area under the curve was obtained. The integral symbol "$\int$" is an "elongated s" that was used to indicate this summation. For us, the integral symbol "$\int$" and the symbol "dx" can serve as reminders that the definite integral is actually a limit of a *summation* as $\Delta x_k \to 0$. The sum that appears in Definition 5.5.1 is called a ***Riemann**** ***sum***, and the definite integral

*GEORG FRIEDRICH BERNHARD RIEMANN (1826–1866). German mathematician. Bernhard Riemann, as he is commonly known, was the son of a Protestant minister. He received his elementary education from his father and showed brilliance in arithmetic at an early age. In 1846 he enrolled at Göttingen University to study theology and philology, but he soon transferred to mathematics. He studied physics under W. E. Weber and mathematics under Karl Friedrich Gauss, whom some people consider to be the greatest mathematician who ever lived. In 1851 Riemann received his Ph.D. under Gauss, after which he remained at Göttingen to teach. In 1862, one month after his marriage, Riemann suffered an attack of pleuritis, and for the remainder of his life was an extremely sick man. He finally succumbed to tuberculosis in 1866 at age 39.

An interesting story surrounds Riemann's work in geometry. For his introductory lecture prior to becoming an associate professor, Riemann submitted three possible topics to Gauss. Gauss surprised Riemann by choosing the topic Riemann liked the least, the foundations of geometry. The lecture was like a scene from a movie. The old and failing Gauss, a giant in his day, watching intently as his brilliant and youthful protégé skillfully pieced together portions of the old man's own work into a complete and beautiful system. Gauss is said to have gasped with delight as the lecture neared its end, and on the way home he marveled at his student's brilliance. Gauss died shortly thereafter. The results presented by Riemann that day eventually evolved into a fundamental tool that Einstein used some 50 years later to develop relativity theory.

In addition to his work in geometry, Riemann made major contributions to the theory of complex functions and mathematical physics. The notion of the definite integral, as it is presented in most basic calculus courses, is due to him. Riemann's early death was a great loss to mathematics, for his mathematical work was brilliant and of fundamental importance.

is sometimes called the ***Riemann integral*** in honor of the German mathematician Bernhard Riemann who formulated many of the basic concepts of integral calculus. (The reason for the similarity in notation between the definite integral and the indefinite integral will become clear in the next section, where we will establish a link between the two types of "integration.")

The limit that appears in Definition 5.5.1 is somewhat different from the kinds of limits discussed in Chapter 2. Loosely phrased, the expression

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = L$$

is intended to convey the idea that we can force the Riemann sums to be as close as we please to L, regardless of how the x_k^* are chosen, by making the mesh size of the partition sufficiently small. Although it is possible to give a more formal definition of this limit, we will simply rely on intuitive arguments when applying Definition 5.5.1.

Example 1 Use Definition 5.5.1 to show that if $f(x) = C$ is a constant function, then

$$\int_a^b f(x)\,dx = C(b-a)$$

Solution. Since $f(x) = C$ is constant, it follows that no matter how the x_k^* are chosen,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \sum_{k=1}^{n} C\Delta x_k = C\sum_{k=1}^{n} \Delta x_k = C(b-a)$$

Since every Riemann sum has the same value $C(b-a)$, it follows that

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{\max \Delta x_k \to 0} C(b-a) = C(b-a)$$ ◀

Note that in Definition 5.5.1, we do *not* assume that the function f is necessarily continuous on the interval $[a, b]$.

Example 2 Define a function f on the interval $[0, 1]$ by $f(x) = 1$ if $0 < x \le 1$ and $f(0) = 0$. Use Definition 5.5.1 to show that

$$\int_0^1 f(x)\,dx = 1$$

Solution. We first note that since

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1 \ne 0 = f(0)$$

f is *not* continuous on the interval $[0, 1]$. Consider any partition of $[0, 1]$ and any choice of the x_k^* corresponding to this partition. Then either $x_1^* = 0$ or it does not. If not, then

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \sum_{k=1}^{n} \Delta x_k = 1$$

On the other hand, if $x_1^* = 0$, then $f(x_1^*) = f(0) = 0$ and

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \sum_{k=2}^{n} \Delta x_k = -\Delta x_1 + \sum_{k=1}^{n} \Delta x_k = 1 - \Delta x_1$$

In either case we see that the *difference* between the Riemann sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

and 1 is at most Δx_1. Since Δx_1 approaches zero as $\max \Delta x_k \to 0$, it follows that

$$\int_0^1 f(x)\,dx = 1$$ ◀

Although Example 2 shows that a function does not have to be continuous on an interval to be integrable on that interval, we will be interested primarily in the definite integrals of continuous functions. Our earlier discussion of net signed area suggests that a function that is continuous on an interval should also be integrable on that interval. This is the content of the next result, which we state without proof.

5.5.2 THEOREM. *If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$.*

We can use Theorem 5.5.2 to clarify the connection between the definite integral and net signed area. Suppose that f is a continuous function on an interval $[a, b]$. Recall that in Section 5.4 we defined the net signed area A between the graph of f and the interval $[a, b]$ to be given by the limit

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

On the other hand, it follows from Theorem 5.5.2 and Definition 5.5.1 that we can use *regular* partitions of $[a, b]$ to compute the definite integral of f over $[a, b]$ as the limit

$$\int_a^b f(x)\,dx = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Since the two limits are the same, we conclude that

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

In other words, the definite integral of a continuous function f from a to b may always be interpreted as the net signed area between the graph of f and the interval $[a, b]$. Of course, if f is nonnegative, this is simply the area beneath the graph of f and above the interval $[a, b]$. It follows that our area computations in Section 5.4 may be reformulated as computations of particular definite integrals. For example, we showed that the area between the graph of $f(x) = 9 - x^2$ and the interval $[0, 3]$ is 18 square units. Equivalently, this computation shows us that

$$\int_0^3 (9 - x^2)\,dx = 18$$

Fortunately, there are often effective and efficient methods for evaluating definite integrals that do not require the explicit evaluation of limits. (We will have more to say about this in Section 5.6.) In the simplest cases, definite integrals can be calculated using formulas from plane geometry to compute signed areas.

Example 3 Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

(a) $\displaystyle\int_1^4 2\,dx$ (b) $\displaystyle\int_{-1}^2 (x + 2)\,dx$ (c) $\displaystyle\int_0^1 \sqrt{1 - x^2}\,dx$

Solution (a). The graph of the integrand is the horizontal line $y = 2$, so the region is a rectangle of height 2 extending over the interval from 1 to 4 (Figure 5.5.4*a*). Thus,

$$\int_1^4 2\,dx = (\text{area of rectangle}) = 2(3) = 6$$

Solution (b). The graph of the integrand is the line $y = x + 2$, so the region is a trapezoid whose base extends from $x = -1$ to $x = 2$ (Figure 5.5.4*b*). Thus,

$$\int_{-1}^2 (x + 2)\,dx = (\text{area of trapezoid}) = \tfrac{1}{2}(1 + 4)(3) = \tfrac{15}{2}$$

Solution (c). The graph of $y = \sqrt{1 - x^2}$ is the upper semicircle of radius 1, centered at the origin, so the region is the right quarter-circle extending from $x = 0$ to $x = 1$ (Figure 5.5.4*c*). Thus,

$$\int_0^1 \sqrt{1 - x^2}\, dx = (\text{area of quarter-circle}) = \tfrac{1}{4}\pi(1^2) = \frac{\pi}{4} \qquad \blacktriangleleft$$

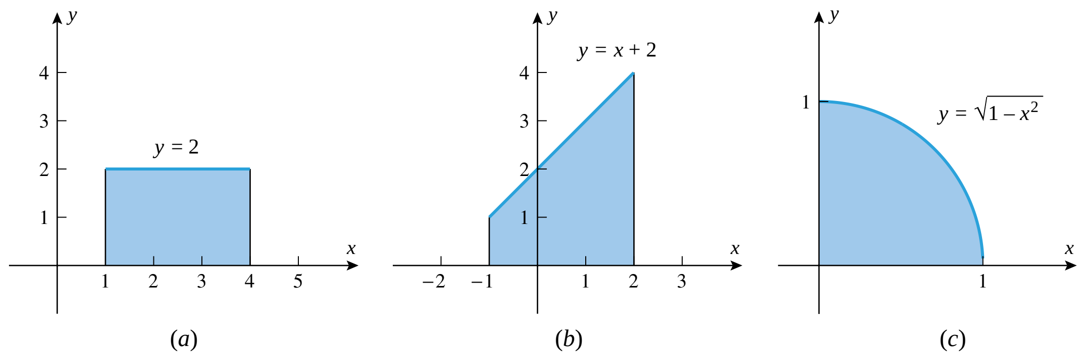


Figure 5.5.4

Example 4 Evaluate

$$\text{(a)} \int_0^2 (x - 1)\, dx \qquad \text{(b)} \int_0^1 (x - 1)\, dx$$

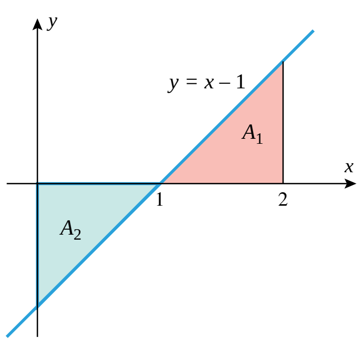


Figure 5.5.5

Solution. The graph of $y = x - 1$ is shown in Figure 5.5.5, and we leave it for you to verify that the shaded triangular regions both have area $\frac{1}{2}$. Over the interval $[0, 2]$ the net signed area is $A_1 - A_2 = \frac{1}{2} - \frac{1}{2} = 0$, and over the interval $[0, 1]$ the net signed area is $-A_2 = -\frac{1}{2}$. Thus,

$$\int_0^2 (x - 1)\, dx = 0 \quad \text{and} \quad \int_0^1 (x - 1)\, dx = -\tfrac{1}{2}$$

(Recall that in Example 8 of Section 5.4, we used Definition 5.4.5 to show that the net signed area between the graph of $y = x - 1$ and the interval $[0, 2]$ is 0.) $\blacktriangleleft$

PROPERTIES OF THE DEFINITE INTEGRAL

It is assumed in Definition 5.5.1 that $[a, b]$ is a finite closed interval with $a < b$, and hence the upper limit of integration in the definite integral is greater than the lower limit of integration. However, it will be convenient to extend this definition to allow for cases in which the upper and lower limits of integration are equal or the lower limit of integration is greater than the upper limit of integration. For this purpose we make the following special definitions.

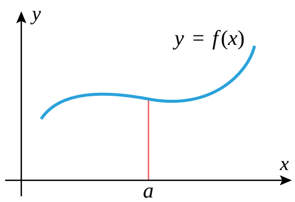
5.5.3 DEFINITION.

(a) If a is in the domain of f, we define

$$\int_a^a f(x)\, dx = 0$$

(b) If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$$



The area between $y = f(x)$ and a is zero.

Figure 5.5.6

REMARK. Part (a) of this definition is consistent with the intuitive idea that the area between a point on the x-axis and a curve $y = f(x)$ should be zero (Figure 5.5.6). Part (b) of the definition is simply a useful convention; it states that interchanging the limits of integration reverses the sign of the integral.

Example 5

(a) $\int_1^1 x^2\,dx = 0$

(b) $\int_1^0 \sqrt{1-x^2}\,dx = -\int_0^1 \sqrt{1-x^2}\,dx = -\dfrac{\pi}{4}$ ◀

Example 3(c)

Because definite integrals are defined as limits, they inherit many of the properties of limits. For example, we know that constants can be moved through limit signs and that the limit of a sum or difference is the sum or difference of the limits. Thus, you should not be surprised by the following theorem, which we state without formal proof.

5.5.4 THEOREM. *If f and g are integrable on $[a, b]$ and if c is a constant, then cf, $f+g$, and $f-g$ are integrable on $[a, b]$ and*

(a) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(b) $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(c) $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Part (b) of this theorem can be extended to more than two functions. More precisely,

$$\int_a^b [f_1(x)+f_2(x)+\cdots+f_n(x)]\,dx$$
$$= \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx + \cdots + \int_a^b f_n(x)\,dx$$

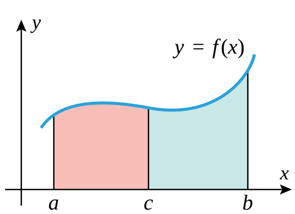


Figure 5.5.7

Some properties of definite integrals can be motivated by interpreting the integral as an area. For example, if f is continuous and nonnegative on the interval $[a, b]$, and if c is a point between a and b, then the area under $y = f(x)$ over the interval $[a, b]$ can be split into two parts and expressed as the area under the graph from a to c plus the area under the graph from c to b (Figure 5.5.7), that is,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

This is a special case of the following theorem about definite integrals, which we state without proof.

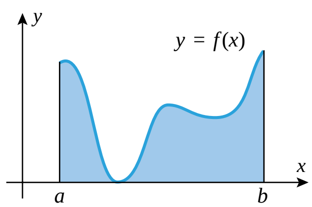
5.5.5 THEOREM. *If f is integrable on a closed interval containing the three numbers a, b, and c, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

no matter how the numbers are ordered.

The following theorem, which we state without formal proof, can also be motivated by interpreting definite integrals as areas.

> **5.5.6 THEOREM.**
> (*a*) *If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then*
> $$\int_a^b f(x)\,dx \geq 0$$
> (*b*) *If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then*
> $$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

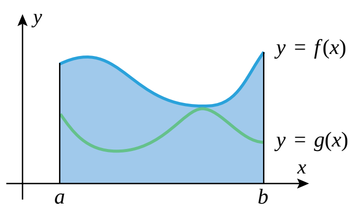


Net signed area ≥ 0

Figure 5.5.8

Geometrically, part (*a*) of this theorem states the obvious fact that if f is nonnegative on $[a, b]$, then the net signed area between the graph of f and the interval $[a, b]$ is also nonnegative (Figure 5.5.8). Part (*b*) has its simplest interpretation when f and g are nonnegative on $[a, b]$, in which case the theorem states that if the graph of f does not go below the graph of g, then the area under the graph of f is at least as large as the area under the graph of g (Figure 5.5.9).

REMARK. Part (*b*) of this theorem states that one can integrate both sides of the inequality $f(x) \geq g(x)$ without altering the sense of the inequality. We also note that in the case where $b > a$, both parts of the theorem remain true if $\geq$ is replaced by $\leq$, $>$, or $<$ throughout.



Area under f ≥ area under g

Figure 5.5.9

Example 6 Evaluate

$$\int_0^1 (5 - 3\sqrt{1 - x^2}\,)\,dx$$

Solution. From parts (*a*) and (*c*) of Theorem 5.5.4 we can write

$$\int_0^1 (5 - 3\sqrt{1 - x^2}\,)\,dx = \int_0^1 5\,dx - \int_0^1 3\sqrt{1 - x^2}\,dx = \int_0^1 5\,dx - 3\int_0^1 \sqrt{1 - x^2}\,dx$$

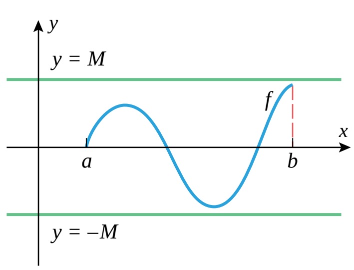
The first integral can be interpreted as the area of a rectangle of height 5 and base 1, so its value is 5, and from Example 3 the value of the second integral is $\pi/4$. Thus,

$$\int_0^1 (5 - 3\sqrt{1 - x^2}\,)\,dx = 5 - 3\left(\frac{\pi}{4}\right) = 5 - \frac{3\pi}{4}$$ ◀

DISCONTINUITIES AND INTEGRABILITY

The problem of determining when functions with discontinuities are integrable is quite complex and beyond the scope of this text. However, there are a few basic results about integrability that are important to know; we begin with a definition.

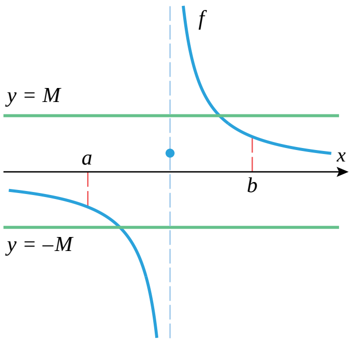
> **5.5.7 DEFINITION.** A function f that is defined on an interval I is said to be ***bounded*** on I if there is a positive number M such that
> $$-M \leq f(x) \leq M$$
> for all x in the interval I. Geometrically, this means that the graph of f over the interval I lies between the lines $y = -M$ and $y = M$.



f is bounded on $[a, b]$.

Figure 5.5.10

For example, a continuous function f is bounded on *every* finite closed interval because the Extreme-Value Theorem (4.5.3) implies that f has an absolute maximum and an absolute minimum on the interval; hence, its graph will lie between the line $y = -M$ and $y = M$, provided we make M large enough (Figure 5.5.10). In contrast, a function that has a vertical asymptote inside of an interval is not bounded on that interval because its graph



f is not bounded on $[a, b]$.

Figure 5.5.11

over the interval cannot be made to lie between the lines $y = -M$ and $y = M$, no matter how large we make the value of M (Figure 5.5.11).

The following theorem, which we state without proof, provides some facts about integrability for functions with discontinuities.

5.5.8 THEOREM. *Let f be a function that is defined on the finite closed interval $[a, b]$.*
(*a*) *If f has finitely many discontinuities in $[a, b]$ but is bounded on $[a, b]$, then f is integrable on $[a, b]$.*
(*b*) *If f is not bounded on $[a, b]$, then f is not integrable on $[a, b]$.*

FOR THE READER. Sketch the graph of a function over the interval $[0, 1]$ that has the properties stated in part (a) of this theorem.

EXERCISE SET 5.5

In Exercises 1–4, find the value of

(a) $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$ (b) $\max \Delta x_k$.

1. $f(x) = x + 1;\ a = 0, b = 4;\ n = 3;$
$\Delta x_1 = 1, \Delta x_2 = 1,\ \Delta x_3 = 2;$
$x_1^* = \frac{1}{3}, x_2^* = \frac{3}{2}, x_3^* = 3$

2. $f(x) = \cos x;\ a = 0, b = 2\pi;\ n = 4;$
$\Delta x_1 = \pi/2, \Delta x_2 = 3\pi/4,\ \Delta x_3 = \pi/2, \Delta x_4 = \pi/4;$
$x_1^* = \pi/4, x_2^* = \pi, x_3^* = 3\pi/2. x_4^* = 7\pi/4$

3. $f(x) = 4 - x^2;\ a = -3, b = 4;\ n = 4;$
$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 1, \Delta x_4 = 3;$
$x_1^* = -\frac{5}{2}, x_2^* = -1, x_3^* = \frac{1}{4}, x_4^* = 3$

4. $f(x) = x^3;\ a = -3, b = 3;\ n = 4;$
$\Delta x_1 = 2, \Delta x_2 = 1, \Delta x_3 = 1, \Delta x_4 = 2;$
$x_1^* = -2, x_2^* = 0, x_3^* = 0, x_4^* = 2$

In Exercises 5–8, use the given values of a and b to express the following limits as definite integrals. (Do not evaluate the integrals.)

5. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^2 \Delta x_k;\ a = -1, b = 2$

6. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^3 \Delta x_k;\ a = 1, b = 2$

7. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 4x_k^*(1 - 3x_k^*)\Delta x_k;\ a = -3, b = 3$

8. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (\sin^2 x_k^*)\Delta x_k;\ a = 0, b = \pi/2$

In Exercises 9 and 10, use Definition 5.5.1 to express the integrals as limits of Riemann sums. Do not try to evaluate the integrals.

9. (a) $\int_1^2 2x\,dx$ (b) $\int_0^1 \frac{x}{x+1}\,dx$

10. (a) $\int_1^2 \sqrt{x}\,dx$ (b) $\int_{-\pi/2}^{\pi/2} (1 + \cos x)\,dx$

In Exercises 11–14, sketch the region whose signed area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry, where needed.

11. (a) $\int_0^3 x\,dx$ (b) $\int_{-2}^{-1} x\,dx$
(c) $\int_{-1}^4 x\,dx$ (d) $\int_{-5}^5 x\,dx$

12. (a) $\int_0^2 \left(1 - \frac{1}{2}x\right)dx$ (b) $\int_{-1}^1 \left(1 - \frac{1}{2}x\right)dx$
(c) $\int_2^3 \left(1 - \frac{1}{2}x\right)dx$ (d) $\int_0^3 \left(1 - \frac{1}{2}x\right)dx$

13. (a) $\int_0^5 2\,dx$ (b) $\int_0^\pi \cos x\,dx$
(c) $\int_{-1}^2 |2x - 3|\,dx$ (d) $\int_{-1}^1 \sqrt{1 - x^2}\,dx$

14. (a) $\int_{-10}^{-5} 6\,dx$ (b) $\int_{-\pi/3}^{\pi/3} \sin x\,dx$
(c) $\int_0^3 |x - 2|\,dx$ (d) $\int_0^2 \sqrt{4 - x^2}\,dx$

15. Use the areas shown in the accompanying figure to find

(a) $\int_a^b f(x)\,dx$ (b) $\int_b^c f(x)\,dx$

(c) $\int_a^c f(x)\,dx$ (d) $\int_a^d f(x)\,dx$.

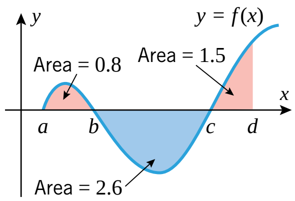


Figure Ex-15

16. In each part, evaluate the integral, given that

$$f(x) = \begin{cases} 2x, & x \le 1 \\ 2, & x > 1 \end{cases}$$

(a) $\int_0^1 f(x)\,dx$ (b) $\int_{-1}^1 f(x)\,dx$

(c) $\int_1^{10} f(x)\,dx$ (d) $\int_{1/2}^5 f(x)\,dx$

17. Find $\int_{-1}^2 [f(x) + 2g(x)]\,dx$ if

$$\int_{-1}^2 f(x)\,dx = 5 \quad \text{and} \quad \int_{-1}^2 g(x)\,dx = -3$$

18. Find $\int_1^4 [3f(x) - g(x)]\,dx$ if

$$\int_1^4 f(x)\,dx = 2 \quad \text{and} \quad \int_1^4 g(x)\,dx = 10$$

19. Find $\int_1^5 f(x)\,dx$ if

$$\int_0^1 f(x)\,dx = -2 \quad \text{and} \quad \int_0^5 f(x)\,dx = 1$$

20. Find $\int_3^{-2} f(x)\,dx$ if

$$\int_{-2}^1 f(x)\,dx = 2 \quad \text{and} \quad \int_1^3 f(x)\,dx = -6$$

In Exercises 21 and 22, use Theorem 5.5.4 and appropriate formulas from geometry to evaluate the integrals.

21. (a) $\int_0^1 (x + 2\sqrt{1 - x^2})\,dx$ (b) $\int_{-1}^3 (4 - 5x)\,dx$

22. (a) $\int_{-3}^0 (2 + \sqrt{9 - x^2})\,dx$ (b) $\int_{-2}^2 (1 - 3|x|)\,dx$

In Exercises 23 and 24, use Theorem 5.5.6 to determine whether the value of the integral is positive or negative.

23. (a) $\int_2^3 \frac{\sqrt{x}}{1 - x}\,dx$ (b) $\int_0^4 \frac{x^2}{3 - \cos x}\,dx$

24. (a) $\int_{-3}^{-1} \frac{x^4}{\sqrt{3 - x}}\,dx$ (b) $\int_{-2}^2 \frac{x^3 - 9}{|x| + 1}\,dx$

In Exercises 25 and 26, evaluate the integrals by completing the square and applying appropriate formulas from geometry.

25. $\int_0^{10} \sqrt{10x - x^2}\,dx$ **26.** $\int_0^3 \sqrt{6x - x^2}\,dx$

In Exercises 27 and 28, evaluate the limit over the interval $[a, b]$ by expressing it as a definite integral and applying an appropriate formula from geometry.

27. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (3x_k^* + 1)\Delta x_k;\ a = 0, b = 1$

28. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{4 - (x_k^*)^2}\,\Delta x_k;\ a = -2, b = 2$

29. In each part, use Theorems 5.5.2 and 5.5.8 to determine whether the function f is integrable on the interval $[-1, 1]$.

(a) $f(x) = \cos x$

(b) $f(x) = \begin{cases} x/|x|, & x \ne 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 1/x^2, & x \ne 0 \\ 0, & x = 0 \end{cases}$

(d) $f(x) = \begin{cases} \sin 1/x, & x \ne 0 \\ 0, & x = 0 \end{cases}$

30. It can be shown that every interval contains both rational and irrational numbers. Accepting this to be so, do you believe that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is integrable on a closed interval $[a, b]$? Explain your reasoning.

31. It can be shown that the limit in Definition 5.5.1 has all of the limit properties stated in Theorem 2.2.2. Accepting this to be so, show that

(a) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(b) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

32. Find the smallest and largest values that the Riemann sum

$$\sum_{k=1}^{3} f(x_k^*)\Delta x_k$$

can have on the interval $[0, 4]$ if $f(x) = x^2 - 3x + 4$ and $\Delta x_1 = 1$, $\Delta x_2 = 2$, $\Delta x_3 = 1$.

33. The function $f(x) = \sqrt{x}$ is continuous on [0, 4] and therefore integrable on this interval. Evaluate

$$\int_0^4 \sqrt{x}\, dx$$

by using Definition 5.5.1. Use subintervals of unequal length given by the partition

$$0 < 4(1)^2/n^2 < 4(2)^2/n^2 < \cdots < 4(n-1)^2/n^2 < 4$$

and let x_k^* be the right endpoint of the kth subinterval.

34. Suppose that f is defined on the interval $[a, b]$ and that $f(x) = 0$ for $a < x \le b$. Use Definition 5.5.1 to prove that

$$\int_a^b f(x)\, dx = 0$$

35. Suppose that g is a continuous function on the interval $[a, b]$ and that f is a function defined on $[a, b]$ with $f(x) = g(x)$ for $a < x \le b$. Prove that

$$\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$$

[*Hint:* Write

$$\int_a^b f(x)\, dx = \int_a^b [(f(x) - g(x)) + g(x)]\, dx$$

and use the result of Exercise 34 along with Theorem 5.5.4(*b*).]

36. Define the function f by $f(x) = 1/x$, $x \ne 0$ and $f(0) = 0$. It follows from Theorem 5.5.8(*b*) that f is not integrable on the interval [0, 1]. Prove this to be the case by applying Definition 5.5.1. [*Hint:* Argue that no matter how small the mesh size is for a partition of [0, 1], there will always be a choice of x_1^* that will make the Riemann sum in Definition 5.5.1 as large as we like.]

5.6 THE FUNDAMENTAL THEOREM OF CALCULUS

In this section we will establish two basic relationships between definite and indefinite integrals that together constitute a result called the Fundamental Theorem of Calculus. One part of this theorem will relate the rectangle and antiderivative methods for calculating areas, and the second part will provide a powerful method for evaluating definite integrals using antiderivatives.

THE FUNDAMENTAL THEOREM OF CALCULUS

As in earlier sections, let us begin by assuming that f is nonnegative and continuous on an interval $[a, b]$, in which case the area A under the graph of f over the interval $[a, b]$ is represented by the definite integral

$$A = \int_a^b f(x)\, dx \tag{1}$$

(Figure 5.6.1).

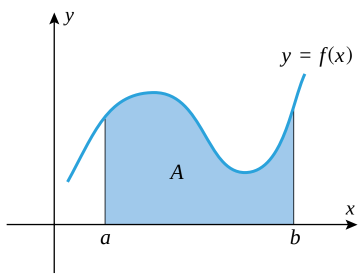


Figure 5.6.1

Recall that our discussion of the antiderivative method in Section 5.1 suggested that if $A(x)$ is the area under the graph of f from a to x (Figure 5.6.2), then:

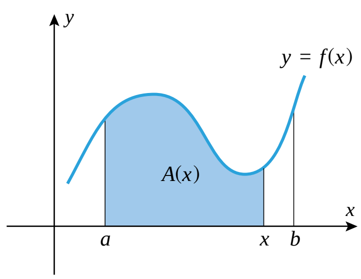


Figure 5.6.2

- $A'(x) = f(x)$
- $A(a) = 0$ The area under the curve from a to a is the area above the single point a, and hence is zero.
- $A(b) = A$ The area under the curve from a to b is A.

The formula $A'(x) = f(x)$ states that $A(x)$ is an antiderivative of $f(x)$, which implies that every other antiderivative of $f(x)$ on $[a, b]$ can be obtained by adding a constant to $A(x)$. Accordingly, let

$$F(x) = A(x) + C$$

be any antiderivative of $f(x)$, and consider what happens when we subtract $F(a)$ from $F(b)$:

$$F(b) - F(a) = [A(b) + C] - [A(a) + C] = A(b) - A(a) = A - 0 = A$$

Hence (1) can be expressed as

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

In words, this equation states:

The definite integral can be evaluated by finding any antiderivative of the integrand and then subtracting the value of this antiderivative at the lower limit of integration from its value at the upper limit of integration.

Although our evidence for this result assumed that f is nonnegative on $[a, b]$, this assumption is not essential.

5.6.1 THEOREM **(*The Fundamental Theorem of Calculus, Part 1*).** *If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{2}$$

Proof. Let $x_1, x_2, \ldots, x_{n-1}$ be any numbers in $[a, b]$ such that

$$a < x_1 < x_2 < \cdots < x_{n-1} < b$$

These values divide $[a, b]$ into n subintervals

$$[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b] \tag{3}$$

whose lengths, as usual, we denote by

$$\Delta x_1, \Delta x_2, \ldots, \Delta x_n$$

By hypothesis, $F'(x) = f(x)$ for all x in $[a, b]$, so F satisfies the hypotheses of the Mean-Value Theorem (4.8.2) on each subinterval in (3). Hence, we can find numbers $x_1^*, x_2^*, \ldots, x_n^*$ in the respective subintervals in (3) such that

$$\begin{aligned}
F(x_1) - F(a) &= F'(x_1^*)(x_1 - a) = f(x_1^*)\Delta x_1\\
F(x_2) - F(x_1) &= F'(x_2^*)(x_2 - x_1) = f(x_2^*)\Delta x_2\\
F(x_3) - F(x_2) &= F'(x_3^*)(x_3 - x_2) = f(x_3^*)\Delta x_3\\
&\ \ \vdots\\
F(b) - F(x_{n-1}) &= F'(x_n^*)(b - x_{n-1}) = f(x_n^*)\Delta x_n
\end{aligned}$$

Adding the preceding equations yields

$$F(b) - F(a) = \sum_{k=1}^{n} f(x_k^*)\Delta x_k \tag{4}$$

Let us now increase n in such a way that $\max \Delta x_k \to 0$. Since f is assumed to be continuous, the right side of (4) approaches $\int_a^b f(x)\,dx$ by Theorem 5.5.2 and Definition 5.5.1. However, the left side of (4) is independent of n; that is, the left side of (4) remains constant as n increases. Thus,

$$F(b) - F(a) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx \qquad \blacksquare$$

It is standard to denote the difference $F(b) - F(a)$ as

$$F(x)\Big]_a^b = F(b) - F(a) \quad \text{or} \quad \big[F(x)\big]_a^b = F(b) - F(a)$$

For example, using the first of these notations we can express (2) as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \tag{5}$$

Example 1 Evaluate $\int_1^2 x\,dx$.

Solution. The function $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$; thus, from (2)

$$\int_1^2 x\,dx = \frac{1}{2}x^2\Big]_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = 2 - \frac{1}{2} = \frac{3}{2} \qquad ◀$$

Example 2 In Example 5 of Section 5.4 we approximated the area under the graph of $y = 9 - x^2$ over the interval $[0, 3]$ using left endpoint, right endpoint, and midpoint approximations, all of which produced an approximation of roughly 18 (square units). In Example 7 of that section we used Definition 5.4.3 to prove that the exact area A is indeed 18. We can now solve this problem more quickly using the Fundamental Theorem of Calculus:

$$A = \int_0^3 (9 - x^2)\,dx = 9x - \frac{x^3}{3}\Big]_0^3 = \left(27 - \frac{27}{3}\right) - 0 = 18 \qquad ◀$$

Example 3

(a) Find the area under the curve $y = \cos x$ over the interval $[0, \pi/2]$ (Figure 5.6.3).

(b) Make a conjecture about the value of the integral

$$\int_0^\pi \cos x\,dx$$

and confirm your conjecture using the Fundamental Theorem of Calculus.

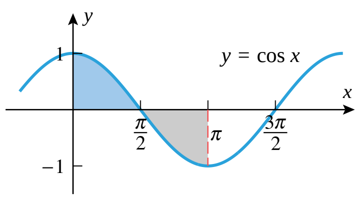


Figure 5.6.3

Solution (a). Since $\cos x \geq 0$ over the interval $[0, \pi/2]$, the area A under the curve is

$$A = \int_0^{\pi/2} \cos x\,dx = \sin x\Big]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$$

Solution (b). The given integral can be interpreted as the signed area between the graph of $y = \cos x$ and the interval $[0, \pi]$. The graph in Figure 5.6.3 suggests that over the interval $[0, \pi]$ the portion of area above the x-axis is the same as the portion of area below the x-axis, so we conjecture that the signed area is zero; this implies that the value of the integral is zero. This is confirmed by the computations

$$\int_0^\pi \cos x\,dx = \sin x\Big]_0^\pi = \sin\pi - \sin 0 = 0 \qquad ◀$$

THE RELATIONSHIP BETWEEN DEFINITE AND INDEFINITE INTEGRALS

Observe that in the preceding examples we did not include a constant of integration in the antiderivatives. In general, when applying the Fundamental Theorem of Calculus there is no need to include a constant of integration because it will drop out anyhow. To see that this is so, let F be any antiderivative of the integrand on $[a, b]$, and let C be any constant; then

$$\int_a^b f(x)\,dx = F(x) + C\Big]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

Thus, for purposes of evaluating a definite integral we can omit the constant of integration in

$$\int_a^b f(x)\,dx = F(x) + C\Big]_a^b$$

and express (5) as

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b \tag{6}$$

which relates the definite and indefinite integrals.

Example 4

$$\int_1^9 \sqrt{x}\,dx = \int \sqrt{x}\,dx\bigg]_1^9 = \int x^{1/2}\,dx\bigg]_1^9 = \frac{2}{3}x^{3/2}\bigg]_1^9 = \frac{2}{3}(27-1) = \frac{52}{3} \quad \blacktriangleleft$$

REMARK. Usually, we will dispense with the step of displaying the indefinite integral explicitly and write the antiderivative immediately, as in our first three examples.

Example 5 Table 5.2.1 will be helpful for the following computations.

Solution.

$$\int_4^9 x^2\sqrt{x}\,dx = \int_4^9 x^{5/2}\,dx = \tfrac{2}{7}x^{7/2}\bigg]_4^9 = \tfrac{2}{7}(2187-128) = \tfrac{4118}{7} = 588\tfrac{2}{7}$$

$$\int_0^{\pi/2} \frac{\sin x}{5}\,dx = -\frac{\cos x}{5}\bigg]_0^{\pi/2} = -\frac{1}{5}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = -\frac{1}{5}[0-1] = \frac{1}{5}$$

$$\int_0^{\pi/3} \sec^2 x\,dx = \tan x\bigg]_0^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \tan 0 = \sqrt{3} - 0 = \sqrt{3}$$

$$\int_{-\pi/4}^{\pi/4} \sec x \tan x\,dx = \sec x\bigg]_{-\pi/4}^{\pi/4} = \sec\left(\frac{\pi}{4}\right) - \sec\left(-\frac{\pi}{4}\right) = \sqrt{2} - \sqrt{2} = 0 \quad \blacktriangleleft$$

WARNING. The requirements in the Fundamental Theorem of Calculus that f be continuous on $[a, b]$ and that F be an antiderivative for f over the entire interval $[a, b]$ are important to keep in mind. Disregarding these assumptions will likely lead to incorrect results. For example, the function $f(x) = 1/x^2$ fails on two counts to be continuous at $x = 0$: $f(x)$ is not defined at $x = 0$ and $\lim_{x\to 0} f(x)$ does not exist. Thus, the Fundamental Theorem of Calculus should not be used to integrate f on any interval that contains $x = 0$. However, if we ignore this and blindly apply Formula (2) over the interval $[-1, 1]$, we might think that

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg]_{-1}^{1} = -[1-(-1)] = -2$$

This answer is clearly ridiculous, since $f(x) = 1/x^2$ is a nonnegative function and hence cannot possibly produce a negative definite integral. Indeed, even if we were to extend f to be defined at 0, say by setting

$$f(x) = \begin{cases} 1/x^2, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f would still be unbounded on any interval containing $x = 0$, so Theorem 5.5.8(b) tells us that f is not even integrable across any such interval.

FOR THE READER. If you have a CAS, read the documentation on evaluating definite integrals, and then check the results in the preceding examples.

The Fundamental Theorem of Calculus can be applied without modification to definite integrals in which the lower limit of integration is greater than or equal to the upper limit of integration.

Example 6

$$\int_1^1 x^2\,dx = \frac{x^3}{3}\Big]_1^1 = \frac{1}{3} - \frac{1}{3} = 0$$

$$\int_4^0 x\,dx = \frac{x^2}{2}\Big]_4^0 = \frac{0}{2} - \frac{16}{2} = -8$$

The latter result is consistent with the result that would be obtained by first reversing the limits of integration in accordance with Definition 5.5.3(b):

$$\int_4^0 x\,dx = -\int_0^4 x\,dx = -\frac{x^2}{2}\Big]_0^4 = -\left[\frac{16}{2} - \frac{0}{2}\right] = -8 \quad \blacktriangleleft$$

To integrate a continuous function that is defined piecewise on an interval $[a, b]$, split this interval into subintervals at the breakpoints of the function, and integrate separately over each subinterval in accordance with Theorem 5.5.5.

Example 7 Evaluate $\displaystyle\int_0^6 f(x)\,dx$ if

$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

Solution. From Theorem 5.5.5

$$\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^6 (3x-2)\,dx$$

$$= \frac{x^3}{3}\Big]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^6 = \left(\frac{8}{3} - 0\right) + (42 - 2) = \frac{128}{3} \quad \blacktriangleleft$$

Example 8 Evaluate $\displaystyle\int_{-1}^2 |x|\,dx$.

Solution. Since $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x \leq 0$,

$$\int_{-1}^2 |x|\,dx = \int_{-1}^0 |x|\,dx + \int_0^2 |x|\,dx$$

$$= \int_{-1}^0 (-x)\,dx + \int_0^2 x\,dx$$

$$= -\frac{x^2}{2}\Big]_{-1}^0 + \frac{x^2}{2}\Big]_0^2 = \frac{1}{2} + 2 = \frac{5}{2} \quad \blacktriangleleft$$

DUMMY VARIABLES

To evaluate a definite integral using the Fundamental Theorem of Calculus, one needs to be able to find an antiderivative of the integrand; thus, it is important to know what kinds of functions have antiderivatives. It is our next objective to show that all continuous functions have antiderivatives, but to do this we will need some preliminary results.

Formula (6) shows that there is a close relationship between the integrals

$$\int_a^b f(x)\,dx \quad \text{and} \quad \int f(x)\,dx$$

However, the definite and indefinite integrals differ in some important ways. For one thing, the two integrals are different kinds of objects—the definite integral is a *number* (the net signed area between the graph of $y = f(x)$ and the interval $[a, b]$), whereas the indefinite integral is a *function*, or more accurately a set of functions [the antiderivatives of $f(x)$].

However, the two types of integrals also differ in the role played by the variable of integration. In an indefinite integral, the variable of integration is "passed through" to the antiderivative in the sense that integrating a function of x produces a function of x, integrating a function of t produces a function of t, and so forth. For example,

$$\int x^2\,dx = \frac{x^3}{3} + C \quad \text{and} \quad \int t^2\,dt = \frac{t^3}{3} + C$$

In contrast, the variable of integration in a definite integral is not passed through to the end result, since the end result is a number. Thus, integrating a function of x over an interval and integrating the same function of t over the same interval of integration produce the same value for the integral. For example,

$$\int_1^3 x^2\,dx = \frac{x^3}{3}\bigg]_{x=1}^{3} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3} \quad \text{and} \quad \int_1^3 t^2\,dt = \frac{t^3}{3}\bigg]_{t=1}^{3} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

However, this latter result should not be surprising, since the area under the graph of the curve $y = f(x)$ over an interval $[a, b]$ on the x-axis is the same as the area under the graph of the curve $y = f(t)$ over the interval $[a, b]$ on the t-axis (Figure 5.6.4).

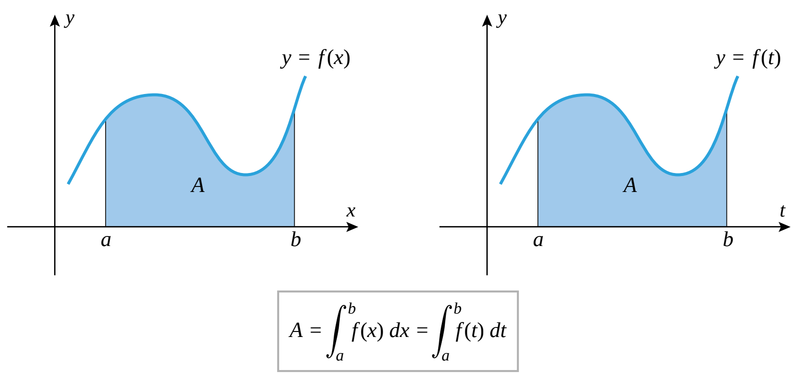


Figure 5.6.4

Because the variable of integration in a definite integral plays no role in the end result, it is often referred to as a ***dummy variable***. In summary:

> *Whenever you find it convenient to change the letter used for the variable of integration in a definite integral, you can do so without changing the value of the integral.*

THE MEAN-VALUE THEOREM FOR INTEGRALS

To reach our goal of showing that continuous functions have antiderivatives, we will need to develop a basic property of definite integrals, known as the *Mean-Value Theorem for Integrals*. In the next section we will use this theorem to extend the familiar idea of "average value" so that it applies to continuous functions, but here we will need it as a tool for developing other results.

Let f be a continuous nonnegative function on $[a, b]$, and let m and M be the minimum and maximum values of $f(x)$ on this interval. Consider the rectangles of heights m and M over the interval $[a, b]$ (Figure 5.6.5). It is clear geometrically from this figure that the area

$$A = \int_a^b f(x)\,dx$$

under $y = f(x)$ is at least as large as the area of the rectangle of height m and no larger than the area of the rectangle of height M. It seems reasonable, therefore, that there is a rectangle over the interval $[a, b]$ of some appropriate height $f(x^*)$ between m and M whose area is

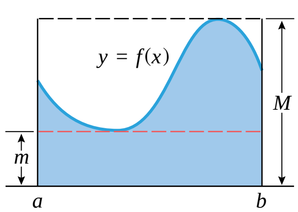


Figure 5.6.5

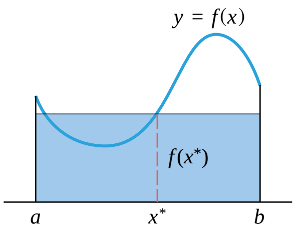


Figure 5.6.6

precisely A; that is,

$$\int_a^b f(x)\,dx = f(x^*)(b-a)$$

(Figure 5.6.6). This is a special case of the following result.

5.6.2 THEOREM (*The Mean-Value Theorem for Integrals*). *If f is continuous on a closed interval $[a,b]$, then there is at least one number x^* in $[a,b]$ such that*

$$\int_a^b f(x)\,dx = f(x^*)(b-a) \tag{7}$$

Proof. By the Extreme-Value Theorem (4.5.3), f assumes a maximum value M and a minimum value m on $[a,b]$. Thus, for all x in $[a,b]$,

$$m \le f(x) \le M$$

and from Theorem 5.5.6(b)

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

or

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \tag{8}$$

or

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

This implies that

$$\frac{1}{b-a}\int_a^b f(x)\,dx \tag{9}$$

is a number between m and M, and since $f(x)$ assumes the values m and M on $[a,b]$, it follows from the Intermediate-Value Theorem (2.5.8) that $f(x)$ must assume the value (9) at some x^* in $[a,b]$; that is,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(x^*) \quad \text{or} \quad \int_a^b f(x)\,dx = f(x^*)(b-a)$$

Example 9 Since $f(x) = x^2$ is continuous on the interval $[1,4]$, the Mean-Value Theorem for Integrals guarantees that there is a number x^* in $[1,4]$ such that

$$\int_1^4 x^2\,dx = f(x^*)(4-1) = (x^*)^2(4-1) = 3(x^*)^2$$

But

$$\int_1^4 x^2\,dx = \frac{x^3}{3}\bigg]_1^4 = 21$$

so that

$$3(x^*)^2 = 21 \quad \text{or} \quad (x^*)^2 = 7 \quad \text{or} \quad x^* = \pm\sqrt{7}$$

Thus, $x^* = \sqrt{7} \approx 2.65$ is the number in the interval $[1,4]$ whose existence is guaranteed by the Mean-Value Theorem for Integrals. ◀

PART 2 OF THE FUNDAMENTAL THEOREM OF CALCULUS

In Section 5.1 we suggested that if f is continuous and nonnegative on $[a, b]$, and if $A(x)$ is the area under the graph of $y = f(x)$ over the interval $[a, x]$ (Figure 5.6.2), then $A'(x) = f(x)$. But $A(x)$ can be expressed as the definite integral

$$A(x) = \int_a^x f(t)\,dt$$

(where we have used t rather than x as the variable of integration to avoid confusion with the x that appears as the upper limit of integration). Thus, the relationship $A'(x) = f(x)$ can be expressed as

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

This is a special case of the following more general result, which applies even if f has negative values.

5.6.3 THEOREM (*The Fundamental Theorem of Calculus, Part 2*). *If f is continuous on an interval I, then f has an antiderivative on I. In particular, if a is any number in I, then the function F defined by*

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f on I; that is, $F'(x) = f(x)$ for each x in I, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \tag{10}$$

Proof. We will show first that $F(x)$ is defined at each x in the interval I. If $x > a$ and x is in the interval I, then Theorem 5.5.2 applied to the interval $[a, x]$ and the continuity of f on I ensure that $F(x)$ is defined; and if x is in the interval I and $x \le a$, then Definition 5.5.3 combined with Theorem 5.5.2 ensures that $F(x)$ is defined. Thus, $F(x)$ is defined for all x in I.

Next we will show that $F'(x) = f(x)$ for each x in the interval I. If x is not an endpoint of I, then it follows from the definition of a derivative that

$$\begin{aligned}
F'(x) &= \lim_{w\to x} \frac{F(w) - F(x)}{w - x} \\
&= \lim_{w\to x}\left(\frac{1}{w-x}\left[\int_a^w f(t)\,dt - \int_a^x f(t)\,dt\right]\right) \\
&= \lim_{w\to x}\left(\frac{1}{w-x}\left[\int_a^w f(t)\,dt + \int_x^a f(t)\,dt\right]\right) \\
&= \lim_{w\to x}\left(\frac{1}{w-x}\int_x^w f(t)\,dt\right) \qquad (11)
\end{aligned}$$

Applying the Mean-Value Theorem for Integrals (5.6.2) to $\int_x^w f(t)\,dt$, we obtain

$$\frac{1}{w-x}\int_x^w f(t)\,dt = \frac{1}{w-x}[f(t^*)\cdot(w-x)] = f(t^*) \tag{12}$$

where t^* is some number between x and w. Because t^* is between x and w, it follows that $t^* \to x$ as $w \to x$. Thus $f(t^*) \to f(x)$ as $w \to x$, since f is assumed continuous at x. Therefore, it follows from (11) and (12) that

$$F'(x) = \lim_{w\to x}\left(\frac{1}{w-x}\int_x^w f(t)\,dt\right) = \lim_{w\to x} f(t^*) = f(x)$$

If x is an endpoint of the interval I, then the two-sided limits in the proof must be replaced by the appropriate one-sided limits, but otherwise the arguments are identical.

In words, Formula (10) states:

If a definite integral has a variable upper limit of integration, a constant lower limit of integration, and a continuous integrand, then the derivative of the integral with respect to its upper limit is equal to the integrand evaluated at the upper limit.

Example 10 Find

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$$

by applying Part 2 of the Fundamental Theorem of Calculus, and then confirm the result by performing the integration and then differentiating.

Solution. The integrand is a continuous function, so from (10)

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = x^3$$

Alternatively, evaluating the integral and then differentiating yields

$$\int_1^x t^3\,dt = \frac{t^4}{4}\bigg]_{t=1}^{x} = \frac{x^4}{4} - \frac{1}{4}, \quad \frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right] = x^3$$

so the two methods for differentiating the integral agree. ◀

Example 11 Since

$$f(x) = \frac{\sin x}{x}$$

is continuous on any interval that does not contain the origin, it follows from (10) that on the interval $(0, +\infty)$ we have

$$\frac{d}{dx}\left[\int_1^x \frac{\sin t}{t}\,dt\right] = \frac{\sin x}{x}$$

Unlike the preceding example, there is no way to evaluate the integral in terms of familiar functions, so Formula (10) provides the only simple method for finding the derivative. ◀

DIFFERENTIATION AND INTEGRATION ARE INVERSE PROCESSES

The two parts of the Fundamental Theorem of Calculus, when taken together, tell us that differentiation and integration are inverse processes in the sense that each undoes the effect of the other. To see why this is so, note that Part 1 of the Fundamental Theorem of Calculus (5.6.1) implies that

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

which tells us that if the value of $f(a)$ is known, then the function f can be recovered from its derivative f' by integrating. Conversely, Part 2 of the Fundamental Theorem of Calculus (5.6.3) states that

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

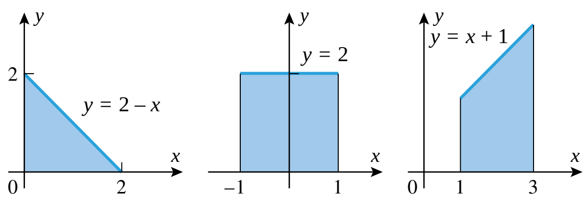
which tells us that the function f can be recovered from its integral by differentiating. Thus, differentiation and integration can be viewed as inverse processes.

It is common to treat parts 1 and 2 of the Fundamental Theorem of Calculus as a single theorem, and refer to it simply as the *Fundamental Theorem of Calculus*. This theorem ranks as one of the greatest discoveries in the history of science, and its formulation by Newton and Leibniz is generally regarded to be the “discovery of calculus.”

EXERCISE SET 5.6 Graphing Calculator C CAS

1. In each part, use a definite integral to find the area of the region, and check your answer using an appropriate formula from geometry.

(a) (b) (c)



2. In each part, use a definite integral to find the area under the curve $y = f(x)$ over the stated interval, and check your answer using an appropriate formula from geometry.
(a) $f(x) = x$; $[0, 5]$
(b) $f(x) = 5$; $[3, 9]$
(c) $f(x) = x + 3$; $[-1, 2]$

In Exercises 3–6, find the area under the curve $y = f(x)$ over the stated interval.

3. $f(x) = x^3$; $[2, 3]$ **4.** $f(x) = x^4$; $[-1, 1]$

5. $f(x) = \sqrt{x}$; $[1, 9]$ **6.** $f(x) = x^{-3/5}$; $[1, 4]$

In Exercises 7–19, evaluate the integrals using Part 1 of the Fundamental Theorem of Calculus.

7. $\displaystyle\int_{-3}^{0} (x^2 - 4x + 7)\, dx$ **8.** $\displaystyle\int_{-1}^{2} x(1 + x^3)\, dx$

9. $\displaystyle\int_{1}^{3} \frac{1}{x^2}\, dx$ **10.** $\displaystyle\int_{1}^{2} \frac{1}{x^6}\, dx$

11. $\displaystyle\int_{4}^{9} 2x\sqrt{x}\, dx$ **12.** $\displaystyle\int_{1}^{8} (5x^{2/3} - 4x^{-2})\, dx$

13. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin\theta\, d\theta$ **14.** $\displaystyle\int_{0}^{\pi/4} \sec^2\theta\, d\theta$

15. $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\, dx$ **16.** $\displaystyle\int_{0}^{1} (x - \sec x \tan x)\, dx$

17. $\displaystyle\int_{1}^{4} \left(\frac{3}{\sqrt{t}} - 5\sqrt{t} - t^{-3/2}\right) dt$

18. $\displaystyle\int_{4}^{9} (4y^{-1/2} + 2y^{1/2} + y^{-5/2})\, dy$

19. $\displaystyle\int_{\pi/6}^{\pi/2} \left(x + \frac{2}{\sin^2 x}\right) dx$

C **20.** Use a CAS to evaluate the integral
$$\int_{a}^{4a} (a^{1/2} - x^{1/2})\, dx$$
and check the answer by hand.

In Exercises 21 and 22, use Theorem 5.5.5 to evaluate the given integrals.

21. (a) $\displaystyle\int_{0}^{2} |2x - 3|\, dx$ (b) $\displaystyle\int_{0}^{3\pi/4} |\cos x|\, dx$

22. (a) $\displaystyle\int_{-1}^{2} \sqrt{2 + |x|}\, dx$ (b) $\displaystyle\int_{0}^{\pi/2} \left|\tfrac{1}{2} - \sin x\right|\, dx$

C **23.** (a) CAS programs provide methods for entering functions that are defined piecewise. Check your documentation to see how this is done, and then use the CAS to evaluate
$$\int_{0}^{2} f(x)\, dx, \quad \text{where} \quad f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$$
Use Theorem 5.5.5 to check the answer by hand.
(b) Find a formula for an antiderivative F of f on the interval $[0, 4]$ and verify that
$$\int_{0}^{2} f(x)\, dx = F(2) - F(0)$$

C **24.** (a) Use a CAS to evaluate
$$\int_{0}^{4} f(x)\, dx, \quad \text{where} \quad f(x) = \begin{cases} \sqrt{x}, & 0 \le x < 1 \\ 1/x^2, & x \ge 1 \end{cases}$$
Use Theorem 5.5.5 to check the answer by hand.
(b) Find a formula for an antiderivative F of f on the interval $[0, 4]$ and verify that
$$\int_{0}^{4} f(x)\, dx = F(4) - F(0)$$

In Exercises 25–27, use a calculating utility to find the midpoint approximation of the integral using $n = 20$ subintervals, and then find the exact value of the integral using Part 1 of the Fundamental Theorem of Calculus.

25. $\displaystyle\int_{1}^{3} \frac{1}{x^2}\, dx$ **26.** $\displaystyle\int_{0}^{\pi/2} \sin x\, dx$ **27.** $\displaystyle\int_{-1}^{1} \sec^2 x\, dx$

C **28.** Compare the answers obtained by the midpoint rule in Exercises 25–27 to those obtained using the built-in numerical (approximate) integration command of a calculating utility or a CAS.

29. Find the area under the curve $y = x^2 + 1$ over the interval $[0, 3]$. Make a sketch of the region.

30. Find the area that is above the x-axis, but below the curve $y = (1 - x)(x - 2)$. Make a sketch of the region.

31. Find the area under the curve $y = 3\sin x$ over the interval $[0, 2\pi/3]$. Sketch the region.

32. Find the area below the interval $[-2, -1]$, but above the curve $y = x^3$. Make a sketch of the region.

33. Find the total area between the curve $y = x^2 - 3x - 10$ and the interval $[-3, 8]$. Make a sketch of the region. [*Hint:* Find the portion of area above the interval and the portion of area below the interval separately.]

34. (a) Use a graphing utility to generate the graph of

$$f(x) = \frac{1}{100}(x+2)(x+1)(x-3)(x-5)$$

and use the graph to make a conjecture about the sign of the integral

$$\int_{-2}^{5} f(x)\,dx$$

(b) Check your conjecture by evaluating the integral.

35. (a) Let f be an odd function; that is, $f(-x) = -f(x)$. Invent a theorem that makes a statement about the value of an integral of the form

$$\int_{-a}^{a} f(x)\,dx$$

(b) Confirm that your theorem works for the integrals

$$\int_{-1}^{1} x^3\,dx \quad \text{and} \quad \int_{-\pi/2}^{\pi/2} \sin x\,dx$$

(c) Let f be an even function; that is, $f(-x) = f(x)$. Invent a theorem that makes a statement about the relationship between the integrals

$$\int_{-a}^{a} f(x)\,dx \quad \text{and} \quad \int_{0}^{a} f(x)\,dx$$

(d) Confirm that your theorem works for the integrals

$$\int_{-1}^{1} x^2\,dx \quad \text{and} \quad \int_{-\pi/2}^{\pi/2} \cos x\,dx$$

36. Use the theorem you invented in Exercise 35(a) to evaluate the integral

$$\int_{-5}^{5} \frac{x^7 - x^5 + x}{x^4 + x^2 + 7}\,dx$$

and check your answer with a CAS.

37. Define $F(x)$ by

$$F(x) = \int_{1}^{x} (t^3 + 1)\,dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check the result in part (a) by first integrating and then differentiating.

38. Define $F(x)$ by

$$F(x) = \int_{\pi/4}^{x} \cos 2t\,dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check the result in part (a) by first integrating and then differentiating.

In Exercises 39–42, use Part 2 of the Fundamental Theorem of Calculus to find the derivatives.

39. (a) $\dfrac{d}{dx}\displaystyle\int_{1}^{x} \sin(\sqrt{t}\,)\,dt$ (b) $\dfrac{d}{dx}\displaystyle\int_{1}^{x} \sqrt{1+\cos^2 t}\,dt$

40. (a) $\dfrac{d}{dx}\displaystyle\int_{0}^{x} \frac{dt}{1+\sqrt{t}}$ (b) $\dfrac{d}{dx}\displaystyle\int_{1}^{x} \frac{dt}{1+t+t^2}\,dt$

41. $\dfrac{d}{dx}\displaystyle\int_{x}^{0} \frac{t}{\cos t}\,dt$ [*Hint:* Use Definition 5.5.3(b).]

42. $\dfrac{d}{du}\displaystyle\int_{0}^{u} |x|\,dx$

43. Let $F(x) = \displaystyle\int_{2}^{x} \sqrt{3t^2+1}\,dt$. Find

(a) $F(2)$ (b) $F'(2)$ (c) $F''(2)$.

44. Let $F(x) = \displaystyle\int_{0}^{x} \frac{\cos t}{t^2+3}\,dt$. Find

(a) $F(0)$ (b) $F'(0)$ (c) $F''(0)$.

45. Let $F(x) = \displaystyle\int_{0}^{x} \frac{t-3}{t^2+7}\,dt$ for $-\infty < x < +\infty$.

(a) Find the value of x where F attains its minimum value.

(b) Find intervals over which F is only increasing or only decreasing.

(c) Find open intervals over which F is only concave up or only concave down.

46. Use the plotting and numerical integration commands of a CAS to generate the graph of the function F in Exercise 45 over the interval $-20 \le x \le 20$, and confirm that the graph is consistent with the results obtained in that exercise.

47. (a) Over what open interval does the formula

$$F(x) = \int_{1}^{x} \frac{dt}{t}$$

represent an antiderivative of $f(x) = 1/x$?

(b) Find a point where the graph of F crosses the x-axis.

48. (a) Over what open interval does the formula

$$F(x) = \int_{1}^{x} \frac{1}{t^2-9}\,dt$$

represent an antiderivative of

$$f(x) = \frac{1}{x^2-9}?$$

(b) Find a point where the graph of F crosses the x-axis.

In Exercises 49 and 50, find all values of x^* in the stated interval that satisfy Equation (7) in the Mean-Value Theorem for Integrals (5.6.2), and explain what these numbers represent.

49. (a) $f(x) = \sqrt{x}$; $[0, 9]$

(b) $f(x) = 3x^2 + 2x + 1$; $[-1, 2]$

50. (a) $f(x) = \sin x$; $[-\pi, \pi]$ (b) $f(x) = 1/x^2$; $[1, 3]$

It was shown in the proof of the Mean-Value Theorem for Integrals (5.6.2) that if f is continuous on $[a, b]$, and if $m \le f(x) \le M$ on $[a, b]$, then

$$m(b-a) \le \int_{a}^{b} f(x)\,dx \le M(b-a)$$

[see (8)]. These inequalities make it possible to obtain bounds on the size of a definite integral from bounds on the size of its integrand. This is illustrated in Exercises 51 and 52.

51. Find the maximum and minimum values of $\sqrt{x^3+2}$ for $0 \le x \le 3$, and use these values to find bounds on the value of the integral
$$\int_0^3 \sqrt{x^3+2}\,dx$$

52. Find values of m and M such that $m \le x\sin x \le M$ for $0 \le x \le \pi$, and use these values to find bounds on the value of the integral
$$\int_0^{\pi} x\sin x\,dx$$

53. Prove:
(a) $[cF(x)]_a^b = c[F(x)]_a^b$
(b) $[F(x)+G(x)]_a^b = F(x)]_a^b + G(x)]_a^b$
(c) $[F(x)-G(x)]_a^b = F(x)]_a^b - G(x)]_a^b$.

54. Prove the Mean-Value Theorem for Integrals (Theorem 5.6.2) by applying the Mean-Value Theorem (4.8.2) to an antiderivative F for f.

5.7 RECTILINEAR MOTION REVISITED; AVERAGE VALUE

In Section 4.4 we used the derivative to define the notions of instantaneous velocity and acceleration for a particle moving along a line. In this section we will resume the study of such motion using the tools of integration. We will also investigate the general problem of integrating a rate of change, and we will show how the definite integral can be used to define the average value of a continuous function. More applications of integration will be given in Chapter 6.

FINDING POSITION AND VELOCITY BY INTEGRATION

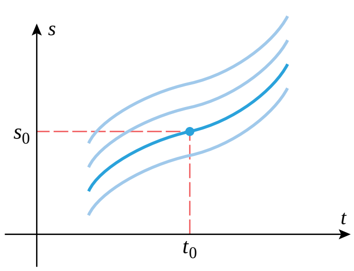
Recall from Definitions 4.4.1 and 4.4.2 that if $s(t)$ is the position function of a particle moving on a coordinate line, then the instantaneous velocity and acceleration of the particle are given by the formulas

$$v(t) = s'(t) = \frac{ds}{dt} \quad \text{and} \quad a(t) = v'(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

It follows from these formulas that $s(t)$ is an antiderivative of $v(t)$ and $v(t)$ is an antiderivative of $a(t)$; that is,

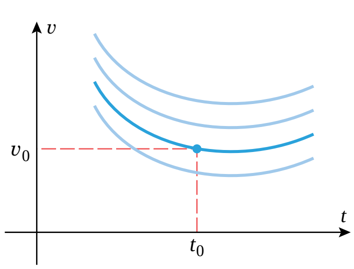
$$s(t) = \int v(t)\,dt \quad \text{and} \quad v(t) = \int a(t)\,dt \tag{1–2}$$

Thus, if the velocity of a particle is known, then its position function can be obtained from (1) by integration, provided there is sufficient additional information to determine the constant of integration. In particular, we can determine the constant of integration if we know the position s_0 of the particle at some time t_0, since this information determines a unique antiderivative $s(t)$ (Figure 5.7.1). Similarly, if the acceleration function of the particle is known, then its velocity function can be obtained from (2) by integration if we know the velocity v_0 of the particle at some time t_0 (Figure 5.7.2).



There is a unique position function such that $s(t_0) = s_0$.

Figure 5.7.1



There is a unique velocity function such that $v(t_0) = v_0$.

Figure 5.7.2

Example 1 Find the position function of a particle that moves with velocity $v(t) = \cos \pi t$ along a coordinate line, assuming that the particle has coordinate $s = 4$ at time $t = 0$.

Solution. The position function is

$$s(t) = \int v(t)\,dt = \int \cos \pi t\,dt = \frac{1}{\pi}\sin \pi t + C$$

Since $s = 4$ when $t = 0$, it follows that

$$4 = s(0) = \frac{1}{\pi}\sin 0 + C = C$$

Thus,

$$s(t) = \frac{1}{\pi}\sin \pi t + 4$$ ◀

UNIFORMLY ACCELERATED MOTION

One of the most important cases of rectilinear motion occurs when a particle has constant acceleration. We call this ***uniformly accelerated motion***.

We will show that if a particle moves with constant acceleration along an s-axis, and if the position and velocity of the particle are known at some point in time, say when $t = 0$, then it is possible to derive formulas for the position $s(t)$ and the velocity $v(t)$ at any time t. To see how this can be done, suppose that the particle has constant acceleration

$$a(t) = a \tag{3}$$

and

$$s = s_0 \quad \text{when} \quad t = 0 \tag{4}$$

$$v = v_0 \quad \text{when} \quad t = 0 \tag{5}$$

where s_0 and v_0 are known. We call (4) and (5) the ***initial conditions*** for the motion.

With (3) as a starting point, we can integrate $a(t)$ to obtain $v(t)$, and we can integrate $v(t)$ to obtain $s(t)$, using an initial condition in each case to determine the constant of integration. The computations are as follows:

$$v(t) = \int a(t)\,dt = \int a\,dt = at + C_1 \tag{6}$$

To determine the constant of integration C_1 we apply initial condition (5) to this equation to obtain

$$v_0 = v(0) = a \cdot 0 + C_1 = C_1$$

Substituting this in (6) and putting the constant term first yields

$$v(t) = v_0 + at$$

Since v_0 is constant, it follows that

$$s(t) = \int v(t)\,dt = \int (v_0 + at)\,dt = v_0 t + \tfrac{1}{2}at^2 + C_2 \tag{7}$$

To determine the constant C_2 we apply initial condition (4) to this equation to obtain

$$s_0 = s(0) = v_0 \cdot 0 + \tfrac{1}{2}a \cdot 0 + C_2 = C_2$$

Substituting this in (7) and putting the constant term first yields

$$s(t) = s_0 + v_0 t + \tfrac{1}{2}at^2$$

In summary, we have the following result.

5.7.1 UNIFORMLY ACCELERATED MOTION. *If a particle moves with constant acceleration a along an s-axis, and if the position and velocity at time $t = 0$ are s_0 and v_0, respectively, then the position and velocity functions of the particle are*

$$s(t) = s_0 + v_0 t + \tfrac{1}{2}at^2 \tag{8}$$

$$v(t) = v_0 + at \tag{9}$$

FOR THE READER. How can you tell from the velocity versus time curve whether a particle moving along a line has uniformly accelerated motion?

Example 2 Suppose that an intergalactic spacecraft uses a sail and the "solar wind" to produce a constant acceleration of 0.032 $\mathrm{m/s^2}$. Assuming that the spacecraft has a velocity of 10,000 m/s when the sail is first raised, how far will the spacecraft travel in 1 hour, and what will its velocity be at the end of this hour?

Solution. In this problem the choice of a coordinate axis is at our discretion, so we will choose it to make the computations as simple as possible. Accordingly, let us introduce an

s-axis whose positive direction is in the direction of motion, and let us take the origin to coincide with the position of the spacecraft at the time $t = 0$ when the sail is raised. Thus, the Formulas (8) and (9) for uniformly accelerated motion apply with

$$s_0 = s(0) = 0, \quad v_0 = v(0) = 10{,}000, \quad \text{and} \quad a = 0.032$$

Since 1 hour corresponds to $t = 3600$ s, it follows from (8) that in 1 hour the spacecraft travels a distance of

$$s(3600) = 10{,}000(3600) + \tfrac{1}{2}(0.032)(3600)^2 \approx 36{,}200{,}000 \text{ m}$$

and it follows from (9) that after 1 hour its velocity is

$$v(3600) = 10{,}000 + (0.032)(3600) \approx 10{,}100 \text{ m/s}$$ ◀

Example 3 A bus has stopped to pick up riders, and a woman is running at a constant velocity of 5 m/s to catch it. When she is 11 m behind the front door the bus pulls away with a constant acceleration of 1 m/s^2. From that point in time, how long will it take for the woman to reach the front door of the bus if she keeps running with a velocity of 5 m/s?

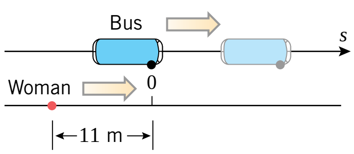


Figure 5.7.3

Solution. As shown in Figure 5.7.3, choose the s-axis so that the bus and the woman are moving in the positive direction, and the front door of the bus is at the origin at the time $t = 0$ when the bus begins to pull away. To catch the bus at some later time t, the woman will have to cover a distance $s_w(t)$ that is equal to 11 m plus the distance $s_b(t)$ traveled by the bus; that is, the woman will catch the bus when

$$s_w(t) = s_b(t) + 11 \tag{10}$$

Since the woman has a constant velocity of 5 m/s, the distance she travels in t seconds is $s_w(t) = 5t$. Thus, (10) can be written as

$$s_b(t) = 5t - 11 \tag{11}$$

Since the bus has a constant acceleration of $a = 1$ m/s^2, and since $s_0 = v_0 = 0$ at time $t = 0$ (why?), it follows from (8) that

$$s_b(t) = \tfrac{1}{2}t^2$$

Substituting this equation into (11) and reorganizing the terms yields the quadratic equation

$$\tfrac{1}{2}t^2 - 5t + 11 = 0 \quad \text{or} \quad t^2 - 10t + 22 = 0$$

Solving this equation for t using the quadratic formula yields two solutions:

$$t = 5 - \sqrt{3} \approx 3.3 \quad \text{and} \quad t = 5 + \sqrt{3} \approx 6.7$$

(verify). Thus, the woman can reach the door at two different times, $t = 3.3$ s and $t = 6.7$ s. The reason that there are two solutions can be explained as follows: When the woman first reaches the door, she is running faster than the bus and can run past it if the driver does not see her. However, as the bus speeds up, it eventually catches up to her, and she has another chance to flag it down. ◀

THE FREE-FALL MODEL

In Section 4.4 we discussed the free-fall model of motion near the surface of the Earth with the promise that we would derive Formula (5) of that section later in the text; we will now show how to do this. As stated in 4.4.4 and illustrated in Figure 4.4.8, we will assume that the object moves on an s-axis whose origin is at the surface of the Earth and whose positive direction is up; and we will assume that the position and velocity of the object at time $t = 0$ are s_0 and v_0, respectively.

It is a fact of physics that a particle moving on a vertical line near the Earth's surface and subject only to the force of the Earth's gravity moves with essentially constant acceleration. The magnitude of this constant, denoted by the letter g, is approximately 9.8 m/s^2 or 32 ft/s^2, depending on whether distance is measured in meters or feet.*

*Strictly speaking, the constant g varies with the latitude and the distance from the Earth's center. However, for motion at a fixed latitude and near the surface of the Earth, the assumption of a constant g is satisfactory for many applications.

Recall that a particle is speeding up when its velocity and acceleration have the same sign and is slowing down when they have opposite signs. Thus, because we have chosen the positive direction to be up, it follows that the acceleration $a(t)$ of a particle in free fall is negative for all values of t. To see that this is so, observe that an upward-moving particle (positive velocity) is slowing down, so its acceleration must be negative; and a downward-moving particle (negative velocity) is speeding up, so its acceleration must also be negative. Thus, we conclude that

$$a(t) = -g$$

and hence it follows from (8) and (9) that the position and velocity functions of an object in free fall are

$$s(t) = s_0 + v_0 t - \tfrac{1}{2}gt^2 \tag{12}$$

$$v(t) = v_0 - gt \tag{13}$$

FOR THE READER. Had we chosen the positive direction of the s-axis to be down, then the acceleration would have been $a(t) = g$ (why?). How would this have affected Formulas (12) and (13)?

Example 4 A ball is hit directly upward with an initial velocity of 49 m/s and is struck at a point that is 1 m above the ground. Assuming that the free-fall model applies, how high will the ball travel?

Solution. Since distance is in meters, we take $g = 9.8\ \text{m/s}^2$. Initially, we have $s_0 = 8$ and $v_0 = 49$, so from (12) and (13)

$$v(t) = -9.8t + 49$$
$$s(t) = -4.9t^2 + 49t + 1$$

The ball will rise until $v(t) = 0$, that is, until $-9.8t + 49 = 0$ or $t = 5$. At this instant the height above the ground will be

$$s(5) = -4.9(5)^2 + 49(5) + 1 = 123.5\ \text{m}$$ ◀

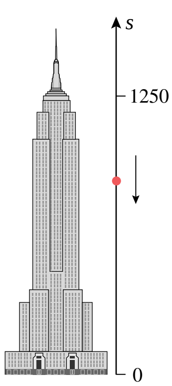


Figure 5.7.4

Example 5 A penny is released from rest near the top of the Empire State Building at a point that is 1250 ft above the ground (Figure 5.7.4). Assuming that the free-fall model applies, how long does it take for the penny to hit the ground, and what is its speed at the time of impact?

Solution. Since distance is in feet, we take $g = 32\ \text{ft/s}^2$. Initially, we have $s_0 = 1250$ and $v_0 = 0$, so from (12)

$$s(t) = -16t^2 + 1250 \tag{14}$$

Impact occurs when $s(t) = 0$. Solving this equation for t, we obtain

$$-16t^2 + 1250 = 0$$
$$t^2 = \frac{1250}{16} = \frac{625}{8}$$
$$t = \pm\frac{25}{\sqrt{8}} \approx \pm 8.8\ \text{s}$$

Since $t \geq 0$, we can discard the negative solution and conclude that it takes $25/\sqrt{8} \approx 8.8$ s for the penny to hit the ground. To obtain the velocity at the time of impact, we substitute

$t = 25/\sqrt{8}$, $v_0 = 0$, and $g = 32$ in (13) to obtain

$$v\left(\frac{25}{\sqrt{8}}\right) = 0 - 32\left(\frac{25}{\sqrt{8}}\right) = -200\sqrt{2} \approx -282.8 \text{ ft/s}$$

Thus, the speed at the time of impact is

$$\left|v\left(\frac{25}{\sqrt{8}}\right)\right| = 200\sqrt{2} \approx 282.8 \text{ ft/s}$$

which is more than 192 mi/h. ◀

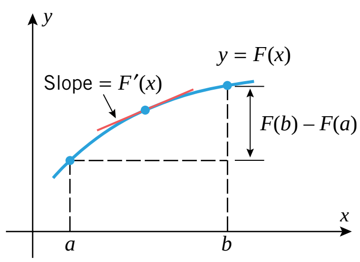
INTEGRATING RATES OF CHANGE

The Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{15}$$

has a useful interpretation that can be seen by rewriting it in a slightly different form. Since F is an antiderivative of f on the interval $[a, b]$, we can use the relationship $F'(x) = f(x)$ to rewrite (15) as

$$\int_a^b F'(x)\,dx = F(b) - F(a) \tag{16}$$



Integrating the slope of $y = F(x)$ over the interval $[a, b]$ produces the change $F(b) - F(a)$ in the value of $F(x)$.

Figure 5.7.5

In this formula we can view $F'(x)$ as the rate of change of $F(x)$ with respect to x, and we can view $F(b) - F(a)$ as the *change* in the value of $F(x)$ as x increases from a to b (Figure 5.7.5). Thus, we have the following useful principle.

> **5.7.2 INTEGRATING A RATE OF CHANGE.** Integrating the rate of change of $F(x)$ with respect to x over an interval $[a, b]$ produces the change in the value of $F(x)$ that occurs as x increases from a to b.

Here are some examples of this idea:

- If $P(t)$ is a population (e.g., plants, animals, or people) at time t, then $P'(t)$ is the rate at which the population is changing at time t, and

  $$\int_{t_1}^{t_2} P'(t)\,dt = P(t_2) - P(t_1)$$

  is the change in the population between times t_1 and t_2.
- If $A(t)$ is the area of an oil spill at time t, then $A'(t)$ is the rate at which the area of the spill is changing at time t, and

  $$\int_{t_1}^{t_2} A'(t)\,dt = A(t_2) - A(t_1)$$

  is the change in the area of the spill between times t_1 and t_2.
- If $P'(x)$ is the marginal profit that results from producing and selling x units of a product (see Section 4.6), then

  $$\int_{x_1}^{x_2} P'(x)\,dx = P(x_2) - P(x_1)$$

  is the change in the profit that results when the production level increases from x_1 units to x_2 units.

DISPLACEMENT IN RECTILINEAR MOTION

As another application of (16), suppose that $s(t)$ and $v(t)$ are the position and velocity functions of a particle moving on a coordinate line. Since $v(t)$ is the rate of change of $s(t)$ with respect to t, it follows from the principle in 5.7.2 that integrating $v(t)$ over an interval

$[t_0, t_1]$ will produce the change in the value of $s(t)$ as t increases from t_0 to t_1; that is,

$$\int_{t_0}^{t_1} v(t)\,dt = \int_{t_0}^{t_1} s'(t)\,dt = s(t_1) - s(t_0) \tag{17}$$

The expression $s(t_1) - s(t_0)$ in this formula is called the ***displacement*** or ***change in position*** of the particle over the time interval $[t_0, t_1]$. For a particle moving horizontally, the displacement is positive if the final position of the particle is to the right of its initial position, negative if it is to the left of its initial position, and zero if it coincides with the initial position (Figure 5.7.6).

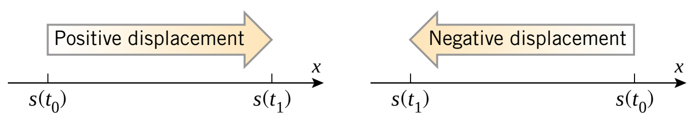


Figure 5.7.6

REMARK. In physical problems it is important to associate the correct units with definite integrals. In general, the units for the definite integral

$$\int_a^b f(x)\,dx$$

will be units of $f(x)$ times units of x. This is because the definite integral is a limit of Riemann sums each of whose terms is a product of the form $f(x) \cdot \Delta x$. For example, if time is measured in seconds (s) and velocity is measured in meters per second (m/s), then integrating velocity over a time interval will produce a result whose units are in meters, since m/s $\times$ s $=$ m. Note that this is consistent with Formula (17), since displacement has units of length.

DISTANCE TRAVELED IN RECTILINEAR MOTION

In general, the displacement of a particle is not the same as the distance traveled by the particle. For example, a particle that travels 100 units in the positive direction and then 100 units in the negative direction travels a distance of 200 units but has a displacement of zero, since it returns to its starting position. The only case in which the displacement and the distance traveled are the same occurs when the particle moves in the positive direction without reversing the direction of its motion.

FOR THE READER. What is the relationship between the displacement of a particle and the distance it travels if the particle moves in the negative direction without reversing the direction of motion?

From (17), integrating the velocity function of a particle over a time interval yields the displacement of a particle over that time interval. In contrast, to find the *total distance* traveled by the particle over the time interval (the distance traveled in the positive direction plus the distance traveled in the negative direction), we must integrate the *absolute value* of the velocity function; that is, we must integrate the speed:

$$\begin{bmatrix} \text{total distance} \\ \text{traveled during} \\ \text{time interval} \\ [t_0, t_1] \end{bmatrix} = \int_{t_0}^{t_1} |v(t)|\,dt \tag{18}$$

Example 6 Suppose that a particle moves on a coordinate line so that its velocity at time t is $v(t) = t^2 - 2t$ m/s.

(a) Find the displacement of the particle during the time interval $0 \le t \le 3$.

(b) Find the distance traveled by the particle during the time interval $0 \le t \le 3$.

Solution (a). From (17) the displacement is

$$\int_0^3 v(t)\,dt = \int_0^3 (t^2 - 2t)\,dt = \left[\frac{t^3}{3} - t^2\right]_0^3 = 0$$

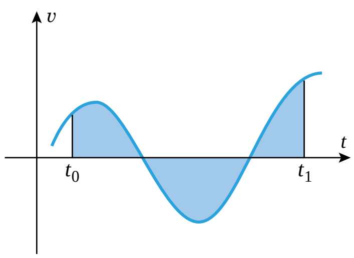
Thus, the particle is at the same position at time $t = 3$ as at $t = 0$.

Solution (b). The velocity can be written as $v(t) = t^2 - 2t = t(t-2)$, from which we see that $v(t) \le 0$ for $0 \le t \le 2$ and $v(t) \ge 0$ for $2 \le t \le 3$. Thus, it follows from (18) that the distance traveled is

$$\begin{aligned}\int_0^3 |v(t)|\,dt &= \int_0^2 -v(t)\,dt + \int_2^3 v(t)\,dt \\ &= \int_0^2 -(t^2 - 2t)\,dt + \int_2^3 (t^2 - 2t)\,dt \\ &= -\left[\frac{t^3}{3} - t^2\right]_0^2 + \left[\frac{t^3}{3} - t^2\right]_2^3 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}\text{ m}\end{aligned}$$ ◀

ANALYZING THE VELOCITY VERSUS TIME CURVE

In Section 4.4 we showed how to use the position versus time curve to obtain information about the behavior of a particle moving on a coordinate line (Table 4.4.1). Similarly, there is valuable information that can be obtained from the *velocity versus time curve*. For example, the integral in (17) can be interpreted geometrically as the net signed area between the graph of $v(t)$ and the interval $[t_0, t_1]$, and it can be interpreted physically as the displacement of the particle over this interval. Thus, we have the following result.



The net signed area is the displacement of the particle during the interval $[t_0, t_1]$.

Figure 5.7.7

5.7.3 FINDING DISPLACEMENT FROM THE VELOCITY VERSUS TIME CURVE. For a particle in rectilinear motion, the net signed area between the velocity versus time curve and an interval $[t_0, t_1]$ on the t-axis represents the displacement of the particle over that time interval (Figure 5.7.7).

Example 7 Figure 5.7.8 shows three velocity versus time curves for a particle in rectilinear motion along a horizontal line. In each case, find the displacement of the particle over the time interval $0 \le t \le 4$, and explain what it tells you about the motion of the particle.

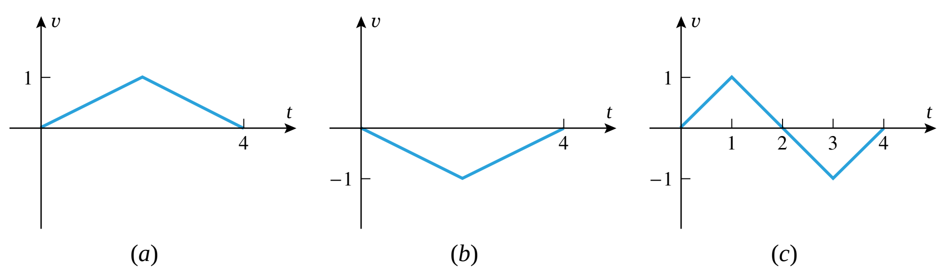


Figure 5.7.8

Solution. In part (*a*) of Figure 5.7.8 the net signed area under the curve is 2, so the particle is 2 units to the right of its starting point at the end of the time period. In part (*b*) the net signed area under the curve is -2, so the particle is 2 units to the left of its starting point at the end of the time period. In part (*c*) the net signed area under the curve is 0, so the particle is back at its starting point at the end of the time period. ◀

By replacing the concept of net signed area with that of "total area," we can also interpret geometrically the total distance traveled by a particle in rectilinear motion. If $f(x)$ is a continuous function on an interval $[a, b]$, we define the ***total area*** between the curve $y = f(x)$ and the interval to be the integral of $|f(x)|$ over the interval $[a, b]$. Geometrically, the total area is the area of the region that is between the graph of f and the x-axis.

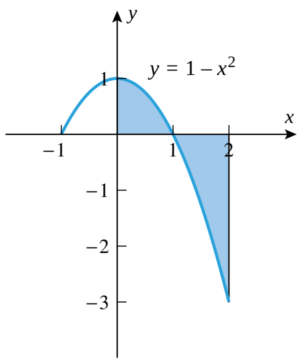


Figure 5.7.9

Example 8 Find the total area between the curve $y = 1 - x^2$ and the x-axis over the interval [0, 2] (Figure 5.7.9).

Solution. The area A is given by

$$A = \int_0^2 |1 - x^2|\, dx = \int_0^1 (1 - x^2)\, dx + \int_1^2 -(1 - x^2)\, dx$$

$$= \left[x - \frac{x^3}{3}\right]_0^1 - \left[x - \frac{x^3}{3}\right]_1^2$$

$$= \frac{2}{3} - \left(-\frac{4}{3}\right) = 2$$ ◀

From (18), integrating the speed $|v(t)|$ over a time interval $[t_0, t_1]$ produces the distance traveled by the particle during the time interval. However, we can also interpret the integral in (18) as the total area between the velocity versus time curve and the interval $[t_0, t_1]$ on the t-axis. Thus, we have the following result.

5.7.4 FINDING DISTANCE TRAVELED FROM THE VELOCITY VERSUS TIME CURVE. For a particle in rectilinear motion, the total area between the velocity versus time curve and an interval $[t_0, t_1]$ on the t-axis represents the distance traveled by the particle over that time interval.

Example 9 For each of the velocity versus time curves in Figure 5.7.8 find the total distance traveled by the particle over the time interval $0 \le t \le 4$.

Solution. In all three parts of Figure 5.7.8 the total area between the curve and the interval [0, 4] is 2, so the particle travels a distance of 2 units during the time period in all three cases, even though the displacement is different in each case, as discussed in Example 7. ◀

AVERAGE VALUE OF A CONTINUOUS FUNCTION

In scientific work, numerical information is often summarized by computing some sort of *average* or *mean* value of the observed data. There are various kinds of averages, but the most common is the ***arithmetic mean*** or ***arithmetic average***, which is formed by adding the data and dividing by the number of data points. Thus, the arithmetic average $\overline{a}$ of n numbers $a_1, a_2, \ldots, a_n$ is

$$\overline{a} = \frac{1}{n}(a_1 + a_2 + \cdots + a_n) = \frac{1}{n}\sum_{k=1}^{n} a_k$$

In the case where the a_k's are values of a function f, say,

$$a_1 = f(x_1), a_2 = f(x_2), \ldots, a_n = f(x_n)$$

then the arithmetic average $\overline{a}$ of these function values is

$$\overline{a} = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$$

We will now show how to extend this concept so that we can compute not only the arithmetic average of finitely many function values but an average of *all* values of $f(x)$ as x varies over a closed interval $[a, b]$. For this purpose recall the Mean-Value Theorem for Integrals (5.6.2), which states that if f is continuous on the interval $[a, b]$, then there is at least one number x^* in this interval such that

$$\int_a^b f(x)\, dx = f(x^*)(b - a)$$

The quantity

$$f(x^*) = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{19}$$

will be our candidate for the average value of f over the interval $[a, b]$. To explain what motivates this, divide the interval $[a, b]$ into n subintervals of equal length

$$\Delta x = \frac{b-a}{n} \tag{20}$$

and choose arbitrary numbers $x_1^*, x_2^*, \ldots, x_n^*$ in successive subintervals. Then the arithmetic average of the values $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$ is

$$\text{ave} = \frac{1}{n}[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]$$

or from (20)

$$\text{ave} = \frac{1}{b-a}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

Taking the limit as $n \to +\infty$ yields

$$\lim_{n\to+\infty}\frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Since this equation describes what happens when we compute the average of "more and more" values of $f(x)$, we are led to the following definition.

5.7.5 DEFINITION. If f is continuous on $[a, b]$, then the ***average value*** (or ***mean value***) of f on $[a, b]$ is defined to be

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{21}$$

REMARK. When f is nonnegative on $[a, b]$, the quantity f_{ave} has a simple geometric interpretation, which can be seen by writing (21) as

$$f_{\text{ave}} \cdot (b-a) = \int_a^b f(x)\,dx$$

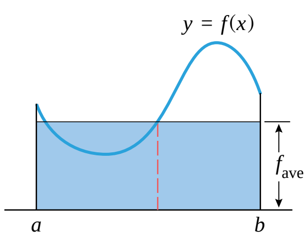


Figure 5.7.10

The left side of this equation is the area of a rectangle with a height of f_{ave} and base of length $b-a$, and the right side is the area under $y = f(x)$ over $[a, b]$. Thus, f_{ave} is the height of a rectangle constructed over the interval $[a, b]$, whose area is the same as the area under the graph of f over that interval (Figure 5.7.10). Note also that the Mean-Value Theorem, when expressed in form (21), ensures that there is always at least one number x^* in $[a, b]$ at which the value of f is equal to the average value of f over the interval.

Example 10 Find the average value of the function $f(x) = \sqrt{x}$ over the interval $[1, 4]$, and find all numbers in the interval at which the value of f is the same as the average.

Solution.

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-1}\int_1^4 \sqrt{x}\,dx = \frac{1}{3}\left[\frac{2x^{3/2}}{3}\right]_1^4$$

$$= \frac{1}{3}\left[\frac{16}{3} - \frac{2}{3}\right] = \frac{14}{9} \approx 1.6$$

The x-values at which $f(x) = \sqrt{x}$ is the same as the average satisfy $\sqrt{x} = 14/9$, from which we obtain $x = 196/81 \approx 2.4$ (Figure 5.7.11). ◀

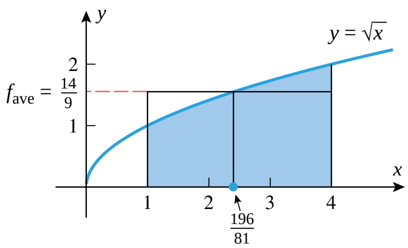


Figure 5.7.11

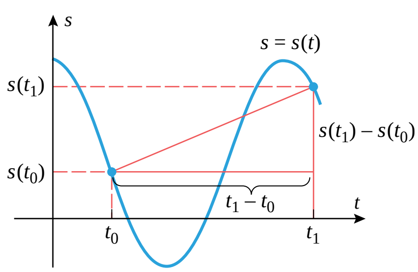
AVERAGE VELOCITY REVISITED

In Section 3.1 we considered the motion of a particle moving along a coordinate line, and we motivated the concept of instantaneous velocity by viewing it as the limit of average velocities over smaller and smaller time intervals. That discussion led us to conclude that the average velocity of the particle over a time interval could be interpreted as the slope of a secant line of the position versus time curve (Figure 3.1.6). We will now show that the same result is true if Definition 5.7.5 is used to compute the average velocity.

For this purpose, suppose that $s(t)$ and $v(t)$ are the position and velocity functions of such a particle, and let us use Formula (21) to calculate the average velocity of the particle over a time interval $[t_0, t_1]$. This yields

$$v_{\text{ave}} = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} v(t)\,dt = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} s'(t)\,dt = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

Thus, *the average velocity over a time interval is the displacement divided by the elapsed time*. Geometrically, this is the slope of the secant line shown in Figure 5.7.12. Thus, the discussion of average velocity in Section 3.1 is consistent with Definition 5.7.5.



$$v_{\text{ave}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{\text{displacement}}{\text{elapsed time}}$$

Figure 5.7.12

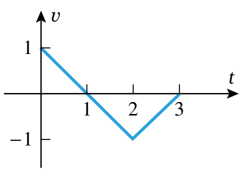
EXERCISE SET 5.7 Graphing Calculator C CAS

1. (a) If $h'(t)$ is the rate of change of a child's height measured in inches per year, what does the integral $\int_0^{10} h'(t)\,dt$ represent, and what are its units?
(b) If $r'(t)$ is the rate of change of the radius of a spherical balloon measured in centimeters per second, what does the integral $\int_1^2 r'(t)\,dt$ represent, and what are its units?
(c) If $H(t)$ is the rate of change of the speed of sound with respect to temperature measured in ft/s per °F, what does the integral $\int_{32}^{100} H(t)\,dt$ represent, and what are its units?
(d) If $v(t)$ is the velocity of a particle in rectilinear motion, measured in cm/h, what does the integral $\int_{t_1}^{t_2} v(t)\,dt$ represent, and what are its units?

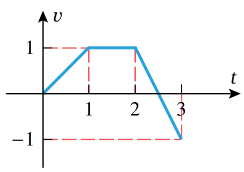
2. (a) Suppose that sludge is emptied into a river at the rate of $V(t)$ gallons per minute, starting at time $t = 0$. Write an integral that represents the total volume of sludge that is emptied into the river during the first hour.
(b) Suppose that the tangent line to a curve $y = f(x)$ has slope $m(x)$ at the point x. What does the integral $\int_{x_1}^{x_2} m(x)\,dx$ represent?

3. In each part, the velocity versus time curve is given for a particle moving along a line. Use the curve to find the displacement and the distance traveled by the particle over the time interval $0 \le t \le 3$.

(a)



(b)



4. Sketch a velocity versus time curve for a particle that travels a distance of 5 units along a coordinate line during the time interval $0 \le t \le 10$ and has a displacement of 0 units.

5. The accompanying figure shows the acceleration versus time curve for a particle moving along a coordinate line. If the initial velocity of the particle is 20 m/s, estimate
(a) the velocity at time $t = 4$ s
(b) the velocity at time $t = 6$ s.

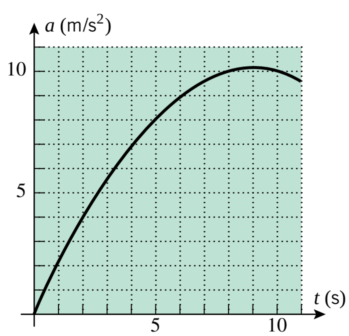


Figure Ex-5

6. Determine whether the particle in Exercise 5 is speeding up or slowing down at times $t = 4$ s and $t = 6$ s.

In Exercises 7–10, a particle moves along an s-axis. Use the given information to find the position function of the particle.

7. (a) $v(t) = t^3 - 2t^2 + 1$; $s(0) = 1$
(b) $a(t) = 4\cos 2t$; $v(0) = -1$; $s(0) = -3$

8. (a) $v(t) = 1 + \sin t$; $s(0) = -3$
(b) $a(t) = t^2 - 3t + 1$; $v(0) = 0$; $s(0) = 0$

9. (a) $v(t) = 2t - 3$; $s(1) = 5$
(b) $a(t) = \cos t$; $v(\pi/2) = 2$; $s(\pi/2) = 0$

10. (a) $v(t) = t^{2/3}$; $s(8) = 0$
(b) $a(t) = \sqrt{t}$; $v(4) = 1$; $s(4) = -5$

In Exercises 11–14, a particle moves with a velocity of $v(t)$ m/s along an s-axis. Find the displacement and the distance traveled by the particle during the given time interval.

11. (a) $v(t) = \sin t$; $0 \le t \le \pi/2$
(b) $v(t) = \cos t$; $\pi/2 \le t \le 2\pi$

12. (a) $v(t) = 2t - 4$; $0 \le t \le 6$
(b) $v(t) = |t - 3|$; $0 \le t \le 5$

13. (a) $v(t) = t^3 - 3t^2 + 2t$; $0 \le t \le 3$
(b) $v(t) = \sqrt{t} - 2$; $0 \le t \le 3$

14. (a) $v(t) = \frac{1}{2} - (1/t^2)$; $1 \le t \le 3$
(b) $v(t) = 3/\sqrt{t}$; $4 \le t \le 9$

In Exercises 15–18, a particle moves with acceleration $a(t)$ m/s^2 along an s-axis and has velocity v_0 m/s at time $t = 0$. Find the displacement and the distance traveled by the particle during the given time interval.

15. $a(t) = -2$; $v_0 = 3$; $1 \le t \le 4$

16. $a(t) = t - 2$; $v_0 = 0$; $1 \le t \le 5$

17. $a(t) = 1/\sqrt{5t + 1}$; $v_0 = 2$; $0 \le t \le 3$

18. $a(t) = \sin t$; $v_0 = 1$; $\pi/4 \le t \le \pi/2$

19. In each part use the given information to find the position, velocity, speed, and acceleration at time $t = 1$.
(a) $v = \sin \frac{1}{2}\pi t$; $s = 0$ when $t = 0$
(b) $a = -3t$; $s = 1$ and $v = 0$ when $t = 0$

20. The accompanying figure shows the velocity versus time curve over the time interval $1 \le t \le 5$ for a particle moving along a horizontal coordinate line.
(a) What can you say about the sign of the acceleration over the time interval?
(b) When is the particle speeding up? Slowing down?
(c) What can you say about the location of the particle at time $t = 5$ relative to its location at time $t = 1$? Explain your reasoning.

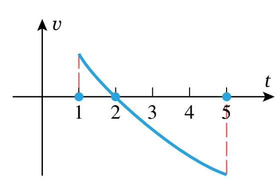


Figure Ex-20

In Exercises 21–24, sketch the curve and find the total area between the curve and the given interval on the x-axis.

21. $y = x^2 - 1$; $[0, 3]$

22. $y = \sin x$; $[0, 3\pi/2]$

23. $y = \sqrt{x+1} - 1$; $[-1, 1]$ **24.** $y = \dfrac{x^2-1}{x^2}$; $\left[\frac{1}{2}, 2\right]$

25. Suppose that the velocity function of a particle moving along an s-axis is $v(t) = 20t^2 - 100t + 50$ ft/s and that the particle is at the origin at time $t = 0$. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for the first 6 s of motion.

26. Suppose that the acceleration function of a particle moving along an s-axis is $a(t) = 4t - 30$ m/s and that the position and velocity at time $t = 0$ are $s_0 = -5$ m and $v_0 = 3$ m/s. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for the first 25 s of motion.

C **27.** Let the velocity function for a particle that is at the origin initially and moves along an s-axis be $v(t) = 0.5 - t \sin t$.
(a) Generate the velocity versus time curve, and use it to make a conjecture about the sign of the displacement over the time interval $0 \le t \le 5$.
(b) Use a CAS to find the displacement.

C **28.** Let the velocity function for a particle that is at the origin initially and moves along an s-axis be $v(t) = 0.5 - t \cos \pi t$.
(a) Generate the velocity versus time curve, and use it to make a conjecture about the sign of the displacement over the time interval $0 \le t \le 1$.
(b) Use a CAS to find the displacement.

29. Suppose that at time $t = 0$ a particle is at the origin of an x-axis and has a velocity of $v_0 = 25$ cm/s. For the first 4 s thereafter it has no acceleration, and then it is acted on by a retarding force that produces a constant negative acceleration of $a = -10$ cm/s^2.
(a) Sketch the acceleration versus time curve over the interval $0 \le t \le 12$.
(b) Sketch the velocity versus time curve over the time interval $0 \le t \le 12$.
(c) Find the x-coordinate of the particle at times $t = 8$ s and $t = 12$ s.
(d) What is the maximum x-coordinate of the particle over the time interval $0 \le t \le 12$?

30. Formulas (8) and (9) for uniformly accelerated motion can be rearranged in various useful ways. For simplicity, let $s = s(t)$ and $v = v(t)$, and derive the following variations of those formulas.
(a) $a = \dfrac{v^2 - v_0^2}{2(s - s_0)}$ (b) $t = \dfrac{2(s - s_0)}{v_0 + v}$
(c) $s = s_0 + vt - \frac{1}{2}at^2$ [Note how this differs from (8).]

Exercises 31–38 involve uniformly accelerated motion. In these exercises assume that the object is moving in the positive direction of a coordinate line, and apply Formulas (8) and (9) or those from Exercise 30, as appropriate. In some of these problems you will need the fact that 88 ft/s = 60 mi/h.

31. (a) An automobile traveling on a straight road decelerates uniformly from 55 mi/h to 25 mi/h in 30 s. Find its acceleration in ft/s^2.
(b) A bicycle rider traveling on a straight path accelerates uniformly from rest to 30 km/h in 1 min. Find his acceleration in km/s^2.

32. A car traveling 60 mi/h along a straight road decelerates at a constant rate of 10 ft/s^2.
(a) How long will it take until the speed is 45 mi/h?
(b) How far will the car travel before coming to a stop?

33. Spotting a police car, you hit the brakes on your new Porsche to reduce your speed from 90 mi/h to 60 mi/h at a constant rate over a distance of 200 ft.
(a) Find the acceleration in ft/s^2.
(b) How long does it take for you to reduce your speed to 55 mi/h?
(c) At the acceleration obtained in part (a), how long would it take for you to bring your Porsche to a complete stop from 90 mi/h?

34. A particle moving along a straight line is accelerating at a constant rate of 3 m/s^2. Find the initial velocity if the particle moves 40 m in the first 4 s.

35. A motorcycle, starting from rest, speeds up with a constant acceleration of 2.6 m/s^2. After it has traveled 120 m, it slows down with a constant acceleration of -1.5 m/s^2 until it attains a speed of 12 m/s. What is the distance traveled by the motorcycle at that point?

36. A sprinter in a 100-m race explodes out of the starting block with an acceleration of 4.0 m/s^2, which she sustains for 2.0 s. Her acceleration then drops to zero for the rest of race.
(a) What is her time for the race?
(b) Make a graph of her distance from the starting block versus time.

37. A car that has stopped at a toll booth leaves the booth with a constant acceleration of 2 ft/s^2. At the time the car leaves the booth it is 5000 ft behind a truck traveling with a constant velocity of 50 ft/s. How long will it take for the car to catch the truck, and how far will the car be from the toll booth at that time?

38. In the final sprint of a rowing race the challenger is rowing at a constant speed of 12 m/s. At the point where the leader is 100 m from the finish line and the challenger is 15 m behind, the leader is rowing at 8 m/s but starts accelerating at a constant 0.5 m/s^2. Who wins?

In Exercises 39–48, assume that a free-fall model applies. Solve these exercises by applying Formulas (12) and (13) or, if appropriate, use those from Exercise 30 with $a = -g$. In these exercises take $g = 32$ ft/s^2 or $g = 9.8$ m/s^2, depending on the units.

39. A projectile is launched vertically upward from ground level with an initial velocity of 112 ft/s.
(a) Find the velocity at $t = 3$ s and $t = 5$ s.
(b) How high will the projectile rise?
(c) Find the speed of the projectile when it hits the ground.

40. A projectile fired downward from a height of 112 ft reaches the ground in 2 s. What is its initial velocity?

41. A projectile is fired vertically upward from ground level with an initial velocity of 16 ft/s.
(a) How long will it take for the projectile to hit the ground?
(b) How long will the projectile be moving upward?

42. A rock is dropped from the top of the Washington Monument, which is 555 ft high.
(a) How long will it take for the rock to hit the ground?
(b) What is the speed of the rock at impact?

43. A helicopter pilot drops a package when the helicopter is 200 ft above the ground and rising at a speed of 20 ft/s.
(a) How long will it take for the package to hit the ground?
(b) What will be its speed at impact?

44. A stone is thrown downward with an initial speed of 96 ft/s from a height of 112 ft.
(a) How long will it take for the stone to hit the ground?
(b) What will be its speed at impact?

45. A projectile is fired vertically upward with an initial velocity of 49 m/s from a tower 150 m high.
(a) How long will it take for the projectile to reach its maximum height?
(b) What is the maximum height?
(c) How long will it take for the projectile to pass its starting point on the way down?
(d) What is the velocity when it passes the starting point on the way down?
(e) How long will it take for the projectile to hit the ground?
(f) What will be its speed at impact?

46. A man drops a stone from a bridge. What is the height of the bridge if
(a) the stone hits the water 4 s later
(b) the sound of the splash reaches the man 4 s later? [Take 1080 ft/s as the speed of sound.]

47. In the final stages of a Moon landing, a lunar module fires its retrorockets and descends to a height of $h = 5$ m above the lunar surface (Figure Ex-47). At that point the retrorockets are cut off, and the module goes into free fall. Given that the Moon's gravity is 1/6 of the Earth's, find the speed of the module when it touches the lunar surface.

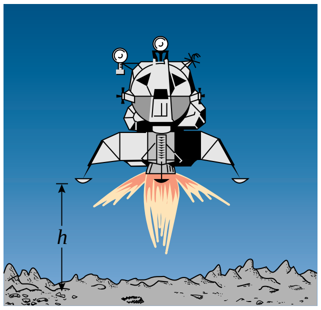


Figure Ex-47

48. Given that the Moon's gravity is 1/6 of the Earth's, how much faster would a projectile have to be launched upward from the surface of the Earth than from the surface of the Moon to reach a height of 1000 ft?

In Exercises 49–52, find the average value of the function over the given interval.

49. $f(x) = 3x$; $[1, 3]$

50. $f(x) = x^2$; $[-1, 2]$

51. $f(x) = \sin x$; $[0, \pi]$

52. $f(x) = \cos x$; $[0, \pi]$

53. (a) Find f_{ave} of $f(x) = x^2$ over $[0, 2]$.
(b) Find a number x^* in $[0, 2]$ such that $f(x^*) = f_{\text{ave}}$.
(c) Sketch the graph of $f(x) = x^2$ over $[0, 2]$ and construct a rectangle over the interval whose area is the same as the area under the graph of f over the interval.

54. (a) Find f_{ave} of $f(x) = 2x$ over $[0, 4]$.
(b) Find a number x^* in $[0, 4]$ such that $f(x^*) = f_{\text{ave}}$.
(c) Sketch the graph of $f(x) = 2x$ over $[0, 4]$ and construct a rectangle over the interval whose area is the same as the area under the graph of f over the interval.

55. (a) Suppose that the velocity function of a particle moving along a coordinate line is $v(t) = 3t^3 + 2$. Find the average velocity of the particle over the time interval $1 \le t \le 4$ by integrating.
(b) Suppose that the position function of a particle moving along a coordinate line is $s(t) = 6t^2 + t$. Find the average velocity of the particle over the time interval $1 \le t \le 4$ algebraically.

56. (a) Suppose that the acceleration function of a particle moving along a coordinate line is $a(t) = t + 1$. Find the average acceleration of the particle over the time interval $0 \le t \le 5$ by integrating.
(b) Suppose that the velocity function of a particle moving along a coordinate line is $v(t) = \cos t$. Find the average acceleration of the particle over the time interval $0 \le t \le \pi/4$ algebraically.

57. Water is run at a constant rate of 1 ft^3/min to fill a cylindrical tank of radius 3 ft and height 5 ft. Assuming that the tank is empty initially, make a conjecture about the average weight of the water in the tank over the time period required to fill it, and then check your conjecture by integrating. [Take the weight density of water to be 62.4 lb/ft^3.]

58. (a) The temperature of a 10-m-long metal bar is 15°C at one end and 30°C at the other end. Assuming that the temperature increases linearly from the cooler end to the hotter end, what is the average temperature of the bar?
(b) Explain why there must be a point on the bar where the temperature is the same as the average, and find it.

59. (a) Suppose that a reservoir supplies water to an industrial park at a constant rate of $r = 4$ gallons per minute (gal/min) between 8:30 A.M. and 9:00 A.M. How much water does the reservoir supply during that time period?

(b) Suppose that one of the industrial plants increases its water consumption between 9:00 A.M. and 10:00 A.M. and that the rate at which the reservoir supplies water increases linearly, as shown in the accompanying figure. How much water does the reservoir supply during that 1-hour time period?

(c) Suppose that from 10:00 A.M. to 12 noon the rate at which the reservoir supplies water is given by the formula $r(t) = 10 + \sqrt{t}$ gal/min, where t is the time (in minutes) since 10:00 A.M. How much water does the reservoir supply during that 2-hour time period?

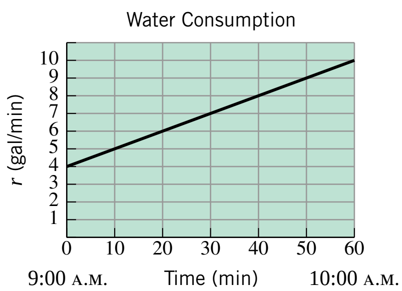


Figure Ex-59

60. A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data she estimates that between 4:30 P.M. and 5:30 P.M. the rate $R(t)$ at which cars enter the highway is given by the formula $R(t) = 100(1 - 0.0001t^2)$ cars per minute, where t is the time (in minutes) since 4:30 P.M.

(a) When does the peak traffic flow into the highway occur?

(b) Estimate the number of cars that enter the highway during the rush hour.

61. (a) Prove: If f is continuous on $[a, b]$, then

$$\int_a^b [f(x) - f_{\text{ave}}]\, dx = 0$$

(b) Does there exist a constant $c \neq f_{\text{ave}}$ such that

$$\int_a^b [f(x) - c]\, dx = 0?$$

5.8 EVALUATING DEFINITE INTEGRALS BY SUBSTITUTION

In this section we will discuss two methods for evaluating definite integrals in which a substitution is required.

TWO METHODS FOR MAKING SUBSTITUTIONS IN DEFINITE INTEGRALS

Recall from Section 5.3 that indefinite integrals of the form

$$\int f(g(x))g'(x)\, dx$$

can sometimes be evaluated by making the u-substitution

$$u = g(x), \quad du = g'(x)\, dx \tag{1}$$

which converts the integral to the form

$$\int f(u)\, du$$

To apply this method to a definite integral of the form

$$\int_a^b f(g(x))g'(x)\, dx$$

we need to account for the effect that the substitution has on the x-limits of integration. There are two ways of doing this.

Method 1 First evaluate the indefinite integral

$$\int f(g(x))g'(x)\, dx$$

by substitution, and then use the relationship

$$\int_a^b f(g(x))g'(x)\, dx = \left[\int f(g(x))g'(x)\, dx\right]_a^b$$

to evaluate the definite integral. This procedure does not require any modification of the x-limits of integration.

Method 2 Make the substitution (1) directly in the definite integral, and then use the relationship $u = g(x)$ to replace the x-limits, $x = a$ and $x = b$, by corresponding u-limits, $u = g(a)$ and $u = g(b)$. This produces a new definite integral

$$\int_{g(a)}^{g(b)} f(u)\,du$$

that is expressed entirely in terms of u.

Example 1 Use the two methods above to evaluate $\int_0^2 x(x^2+1)^3\,dx$.

Solution by Method 1. If we let

$$u = x^2 + 1 \quad \text{so that} \quad du = 2x\,dx \tag{2}$$

then we obtain

$$\int x(x^2+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{u^4}{8} + C = \frac{(x^2+1)^4}{8} + C$$

Thus,

$$\int_0^2 x(x^2+1)^3\,dx = \left[\int x(x^2+1)^3\,dx\right]_{x=0}^{2} = \left.\frac{(x^2+1)^4}{8}\right]_{x=0}^{2}$$

$$= \frac{625}{8} - \frac{1}{8} = 78$$

Solution by Method 2. If we make the substitution $u = x^2 + 1$ in (2), then

if $x = 0$, $u = 1$

if $x = 2$, $u = 5$

Thus,

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int_1^5 u^3\,du = \left.\frac{u^4}{8}\right]_{u=1}^{5} = \frac{625}{8} - \frac{1}{8} = 78$$

which agrees with the result obtained by Method 1. ◀

The following theorem states precise conditions under which Method 2 can be used.

> **5.8.1 THEOREM.** *If g' is continuous on $[a, b]$ and f is continuous on an interval containing the values of $g(x)$ for $a \le x \le b$, then*
>
> $$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof. Since f is continuous on an interval containing the values of $g(x)$ for $a \le x \le b$, it follows that f has an antiderivative F on that interval. If we let $u = g(x)$, then the chain rule implies that

$$\frac{d}{dx}F(g(x)) = \frac{d}{dx}F(u) = \frac{dF}{du}\frac{du}{dx} = f(u)\frac{du}{dx} = f(g(x))g'(x)$$

for each x in $[a, b]$. Thus, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ on $[a, b]$. Therefore, by Part 1 of the Fundamental Theorem of Calculus (Theorem 5.6.1)

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du \quad \blacksquare$$

The choice of methods for evaluating definite integrals by substitution is generally a matter of taste, but in the following examples we will use the second method, since the idea is new.

Example 2 Evaluate

$$\text{(a)} \int_0^{\pi/8} \sin^5 2x \cos 2x\, dx \qquad \text{(b)} \int_2^5 (2x-5)(x-3)^9\, dx$$

Solution (a). Let

$$u = \sin 2x \quad \text{so that} \quad du = 2\cos 2x\, dx \quad \left(\text{or } \tfrac{1}{2}\, du = \cos 2x\, dx\right)$$

With this substitution,

$$\text{if} \quad x = 0, \quad u = \sin(0) = 0$$
$$\text{if} \quad x = \pi/8, \quad u = \sin(\pi/4) = 1/\sqrt{2}$$

so

$$\int_0^{\pi/8} \sin^5 2x \cos 2x\, dx = \frac{1}{2}\int_0^{1/\sqrt{2}} u^5\, du = \frac{1}{2} \cdot \frac{u^6}{6}\bigg]_{u=0}^{1/\sqrt{2}}$$
$$= \frac{1}{2}\left[\frac{1}{6(\sqrt{2})^6} - 0\right] = \frac{1}{96}$$

Solution (b). Let

$$u = x - 3 \quad \text{so that} \quad du = dx$$

This leaves a factor of $2x + 5$ unresolved in the integrand. However,

$$x = u + 3, \quad \text{so} \quad 2x - 5 = 2(u+3) - 5 = 2u + 1$$

With this substitution,

$$\text{if} \quad x = 2, \quad u = 2 - 3 = -1$$
$$\text{if} \quad x = 5, \quad u = 5 - 3 = 2$$

so

$$\int_2^5 (2x-5)(x-3)^9\, dx = \int_{-1}^2 (2u+1)u^9\, du = \int_{-1}^2 (2u^{10} + u^9)\, du$$
$$= \left[\frac{2u^{11}}{11} + \frac{u^{10}}{10}\right]_{u=-1}^2 = \left(\frac{2^{12}}{11} + \frac{2^{10}}{10}\right) - \left(-\frac{2}{11} + \frac{1}{10}\right)$$
$$= \frac{52{,}233}{110} = 474\frac{93}{110}$$

◀

Example 3 Find the average value of the function

$$f(x) = \frac{\cos(\pi/x)}{x^2}$$

over the interval $[1, 3]$.

Solution. From Definition 5.7.5 the average value of f over the interval $[1, 3]$ is

$$f_{\text{ave}} = \frac{1}{3-1}\int_1^3 \frac{\cos(\pi/x)}{x^2}\, dx = \frac{1}{2}\int_1^3 \frac{\cos(\pi/x)}{x^2}\, dx$$

To evaluate this integral, we make the substitution

$$u = \frac{\pi}{x} \quad \text{so that} \quad du = -\frac{\pi}{x^2}\, dx = -\pi \cdot \frac{1}{x^2}\, dx \quad \text{or} \quad -\frac{1}{\pi}\, du = \frac{1}{x^2}\, dx$$

With this substitution,

$$\text{if} \quad x = 1, \quad u = \pi$$
$$\text{if} \quad x = 3, \quad u = \pi/3$$

Thus, the average value of f over the interval $[1, 3]$ is

$$f_{\text{ave}} = \frac{1}{2}\int_1^3 \frac{\cos(\pi/x)}{x^2}\,dx = \frac{1}{2}\cdot\left(-\frac{1}{\pi}\right)\int_{\pi}^{\pi/3} \cos u\,du$$

$$= -\frac{1}{2\pi}\sin u\Big]_{u=\pi}^{\pi/3} = -\frac{1}{2\pi}(\sin(\pi/3) - \sin\pi) = -\frac{\sqrt{3}}{4\pi} \approx -0.1378 \quad \blacktriangleleft$$

REMARK. Observe that the u-substitution in this example produced an integral in which the upper u-limit of integration was smaller than the lower u-limit of integration. In our computations we left the limits of integration in that order, but we could have reversed the order to put the larger limit on top and compensated by reversing the sign of the integral in accordance with Definition 5.5.3(b). The choice of procedures is a matter of taste; both produce the same result (verify).

EXERCISE SET 5.8 [C] CAS

In Exercises 1 and 2, express the integral in terms of the variable u, but do not evaluate it.

1. (a) $\displaystyle\int_0^2 (x+1)^7\,dx;\ u = x+1$

(b) $\displaystyle\int_{-1}^2 x\sqrt{8-x^2}\,dx;\ u = 8-x^2$

(c) $\displaystyle\int_{-1}^1 \sin(\pi\theta)\,d\theta;\ u = \pi\theta$

(d) $\displaystyle\int_0^3 (x+2)(x-3)^{20}\,dx;\ u = x-3$

2. (a) $\displaystyle\int_{-1}^4 (5-2x)^8\,dx;\ u = 5-2x$

(b) $\displaystyle\int_{-\pi/3}^{2\pi/3} \frac{\sin x}{\sqrt{2+\cos x}}\,dx;\ u = 2+\cos x$

(c) $\displaystyle\int_0^{\pi/4} \tan^2 x\sec^2 x\,dx;\ u = \tan x$

(d) $\displaystyle\int_0^1 x^3\sqrt{x^2+3}\,dx;\ u = x^2+3$

In Exercises 3–12, evaluate the definite integral two ways: first by a u-substitution in the definite integral and then by a u-substitution in the corresponding indefinite integral.

3. $\displaystyle\int_0^1 (2x+1)^4\,dx$

4. $\displaystyle\int_1^2 (4x-2)^3\,dx$

5. $\displaystyle\int_{-1}^0 (1-2x)^3\,dx$

6. $\displaystyle\int_1^2 (4-3x)^8\,dx$

7. $\displaystyle\int_0^8 x\sqrt{1+x}\,dx$

8. $\displaystyle\int_{-5}^0 x\sqrt{4-x}\,dx$

9. $\displaystyle\int_0^{\pi/2} 4\sin(x/2)\,dx$

10. $\displaystyle\int_0^{\pi/6} 2\cos 3x\,dx$

11. $\displaystyle\int_{-2}^{-1} \frac{x}{(x^2+2)^3}\,dx$

12. $\displaystyle\int_{1-\pi}^{1+\pi} \sec^2\left(\tfrac{1}{4}x - \tfrac{1}{4}\right)dx$

In Exercises 13–16, evaluate the definite integral by expressing it in terms of u and evaluating the resulting integral using a formula from geometry.

13. $\displaystyle\int_0^{5/3} \sqrt{25-9x^2}\,dx;\ u = 3x$

14. $\displaystyle\int_0^2 x\sqrt{16-x^4}\,dx;\ u = x^2$

15. $\displaystyle\int_{\pi/3}^{\pi/2} \sin\theta\sqrt{1-4\cos^2\theta}\,d\theta;\ u = 2\cos\theta$

16. $\displaystyle\int_{-3}^1 \sqrt{3-2x-x^2}\,dx;\ u = x+1$

17. Find the area under the curve $y = \sin\pi x$ over the interval $[0, 1]$.

18. Find the area under the curve $y = 3\cos 2x$ over the interval $[0, \pi/8]$.

19. Find the area under the curve $y = 1/(x+5)^2$ over the interval $[3, 7]$.

20. Find the area under the curve $y = 1/(3x+1)^2$ over the interval $[0, 1]$.

21. Find the average value of
$$f(x) = \frac{x}{(5x^2+1)^2}$$
over the interval $[0, 2]$.

22. Find the average value of $f(x) = \sec^2\pi x$ over the interval $\left[-\frac{1}{4}, \frac{1}{4}\right]$.

In Exercises 23–36, evaluate the integrals by any method.

23. $\displaystyle\int_0^1 \frac{dx}{\sqrt{3x+1}}$

24. $\displaystyle\int_1^2 \sqrt{5x-1}\,dx$

25. $\displaystyle\int_{-1}^1 \frac{x^2\,dx}{\sqrt{x^3+9}}$

26. $\displaystyle\int_{-1}^0 6t^2(t^3+1)^{19}\,dt$

27. $\displaystyle\int_1^3 \frac{x+2}{\sqrt{x^2+4x+7}}\,dx$

28. $\displaystyle\int_1^2 \frac{dx}{x^2-6x+9}$

29. $\displaystyle\int_{-3\pi/4}^{\pi/4} \sin x\cos x\,dx$

30. $\displaystyle\int_0^{\pi/4} \sqrt{\tan x}\sec^2 x\,dx$

31. $\displaystyle\int_0^{\sqrt{\pi}} 5x\cos(x^2)\,dx$

32. $\displaystyle\int_{\pi^2}^{4\pi^2} \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx$

33. $\displaystyle\int_{\pi/12}^{\pi/9} \sec^2 3\theta\,d\theta$

34. $\displaystyle\int_0^{\pi/2} \sin^2 3\theta\cos 3\theta\,d\theta$

35. $\displaystyle\int_0^1 \frac{y^2\,dy}{\sqrt{4-3y}}$

36. $\displaystyle\int_{-1}^4 \frac{x\,dx}{\sqrt{5+x}}$

C **37.** (a) Use a CAS to find the exact value of the integral
$$\int_0^{\pi/6} \sin^4 x\cos^3 x\,dx$$
(b) Confirm the exact value by hand calculation. [*Hint:* Use the identity $\cos^2 x = 1-\sin^2 x$.]

C **38.** (a) Use a CAS to find the exact value of the integral
$$\int_{-\pi/4}^{\pi/4} \tan^4 x\,dx$$
(b) Confirm the exact value by hand calculation. [*Hint:* Use the identity $1+\tan^2 x = \sec^2 x$.]

39. (a) Find $\displaystyle\int_0^1 f(3x+1)\,dx$ if $\displaystyle\int_1^4 f(x)\,dx = 5$.

(b) Find $\displaystyle\int_0^3 f(3x)\,dx$ if $\displaystyle\int_0^9 f(x)\,dx = 5$.

(c) Find $\displaystyle\int_{-2}^0 xf(x^2)\,dx$ if $\displaystyle\int_0^4 f(x)\,dx = 1$.

40. Given that m and n are positive integers, show that
$$\int_0^1 x^m(1-x)^n\,dx = \int_0^1 x^n(1-x)^m\,dx$$
by making a substitution. Do not attempt to evaluate the integrals.

41. Given that n is a positive integer, show that
$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$
by using a trigonometric identity and making a substitution. Do not attempt to evaluate the integrals.

42. Given that n is a positive integer, evaluate the integral
$$\int_0^1 x(1-x)^n\,dx$$

43. Electricity is supplied to homes in the form of ***alternating current***, which means that the voltage has a sinusoidal waveform described by an equation of the form
$$V = V_p\sin(2\pi ft)$$
(see the accompanying figure). In this equation, V_p is called the ***peak voltage*** or ***amplitude*** of the current, f is called its ***frequency***, and $1/f$ is called its ***period***. The voltages V and V_p are measured in volts (V), the time t is measured in seconds (s), and the frequency is measured in hertz (Hz) or sometimes in cycles per second. (A ***cycle*** is the electrical term for one period of the waveform.) Most alternating-current voltmeters read what is called the ***rms*** or ***root-mean-square*** value of V. By definition, this is the square root of the average value of V^2 over one period.

(a) Show that
$$V_{\text{rms}} = \frac{V_p}{\sqrt{2}}$$
[*Hint:* Compute the average over the cycle from $t=0$ to $t=1/f$, and use the identity $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ to help evaluate the integral.]

(b) In the United States, electrical outlets supply alternating current with an rms voltage of 120 V at a frequency of 60 Hz. What is the peak voltage at such an outlet?

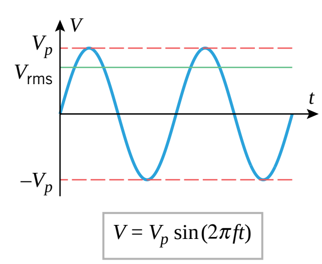


Figure Ex-43

44. Show that if f and g are continuous functions, then
$$\int_0^t f(t-x)g(x)\,dx = \int_0^t f(x)g(t-x)\,dx$$

45. (a) Let $I = \displaystyle\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$. Show that $I = a/2$.

[*Hint:* Let $u = a-x$, and then note the difference between the resulting integrand and 1.]

(b) Use the result of part (a) to find
$$\int_0^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}\,dx$$

(c) Use the result of part (a) to find
$$\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$$

46. Let $I = \displaystyle\int_{-1}^1 \frac{1}{1+x^2}\,dx$. Show that the substitution $x = 1/u$ results in
$$I = -\int_{-1}^1 \frac{1}{1+u^2}\,du = -I$$

so $2I = 0$, which implies that $I = 0$. However, this is impossible since the integrand of the given integral is positive over the interval of integration. Where is the error?

47. Find the limit

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{\sin(k\pi/n)}{n}$$

by evaluating an appropriate definite integral over the interval [0, 1].

C **48.** Check your answer to Exercise 47 by evaluating the limit directly with a CAS.

49. (a) Prove that if f is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

and give a geometric explanation of this result. [*Hint:* One way to prove that a quantity q is zero is to show that $q = -q$.]

(b) Prove that if f is an even function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

and give a geometric explanation of this result. [*Hint:* Split the interval of integration from $-a$ to a into two parts at 0.]

50. Evaluate

(a) $\displaystyle\int_{-1}^{1} x\sqrt{\cos(x^2)}\,dx$ (b) $\displaystyle\int_{0}^{\pi} \sin^8 x\cos^5 x\,dx$.

[*Hint:* Use the substitution $u = x - (\pi/2)$.]

SUPPLEMENTARY EXERCISES

C CAS

1. Write a paragraph that describes the *rectangle method* for defining the area under a curve $y = f(x)$ over an interval $[a, b]$.

2. What is an *integral curve* of a function f? How are two integral curves of a function f related?

3. The *definite integral* of f over the interval $[a, b]$ is defined as the limit

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k\to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Explain what the various symbols on the right side of this equation mean.

4. State the two parts of the Fundamental Theorem of Calculus, and explain what is meant by the phrase "differentiation and integration are inverse processes."

5. Derive the formulas for the position and velocity functions of a particle that moves with uniformly accelerated motion along a coordinate line.

6. (a) Devise a procedure for finding upper and lower estimates of the area of the region in the accompanying figure (in cm^2).

(b) Use your procedure to find upper and lower estimates of the area.

(c) Improve on the estimates you obtained in part (b).

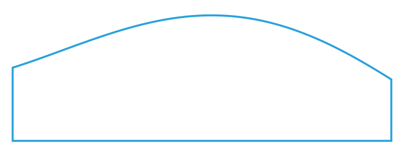


Figure Ex-6

7. Suppose that

$$\int_0^1 f(x)\,dx = \tfrac{1}{2},\quad \int_1^2 f(x)\,dx = \tfrac{1}{4},$$

$$\int_0^3 f(x)\,dx = -1,\quad \int_0^1 g(x)\,dx = 2$$

In each part, use this information to evaluate the given integral, if possible. If there is not enough information to evaluate the integral, then say so.

(a) $\displaystyle\int_0^2 f(x)\,dx$ (b) $\displaystyle\int_1^3 f(x)\,dx$ (c) $\displaystyle\int_2^3 5f(x)\,dx$

(d) $\displaystyle\int_1^0 g(x)\,dx$ (e) $\displaystyle\int_0^1 g(2x)\,dx$ (f) $\displaystyle\int_0^1 [g(x)]^2\,dx$

8. In each part, use the information in Exercise 7 to evaluate the given integral. If there is not enough information to evaluate the integral, then say so.

(a) $\displaystyle\int_0^1 [f(x)+g(x)]\,dx$ (b) $\displaystyle\int_0^1 f(x)g(x)\,dx$

(c) $\displaystyle\int_0^1 \frac{f(x)}{g(x)}\,dx$ (d) $\displaystyle\int_0^1 [4g(x)-3f(x)]\,dx$

9. In each part, evaluate the integral. Where appropriate, you may use a geometric formula.

(a) $\displaystyle\int_{-1}^1 (1+\sqrt{1-x^2})\,dx$

(b) $\displaystyle\int_0^3 (x\sqrt{x^2+1}-\sqrt{9-x^2})\,dx$

(c) $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx$

10. Evaluate the integral $\int_0^1 |2x-1|\,dx$, and sketch the region whose area it represents.

11. One of the numbers π, $\pi/2$, $35\pi/128$, $1-\pi$ is the correct value of the integral

$$\int_0^{\pi} \sin^8 x\,dx$$

Use the accompanying graph of $y = \sin^8 x$ and a logical process of elimination to find the correct value. [Do not attempt to evaluate the integral.]

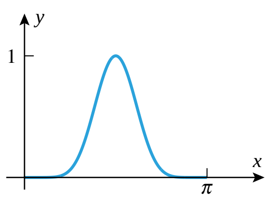


Figure Ex-11

12. In each part, find the limit by interpreting it as a limit of Riemann sums in which the interval $[0, 1]$ is divided into n subintervals of equal length.

(a) $\displaystyle\lim_{n\to+\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n^{3/2}}$

(b) $\displaystyle\lim_{n\to+\infty} \frac{1^4+2^4+3^4+\cdots+n^4}{n^5}$

13. The accompanying figure shows five points on the graph of an unknown function f. Devise a strategy for using the known points to approximate the area A under the graph of $y = f(x)$ over the interval $[1, 5]$. Describe your strategy, and use it to approximate A.

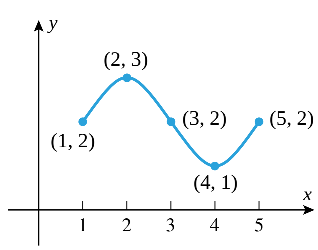


Figure Ex-13

14. The accompanying figure shows the direction field for a differential equation $dy/dx = f(x)$. Which of the following functions is most likely to be $f(x)$?

$\sqrt{x}$, $\sin x$, x^4, x

Explain your reasoning.

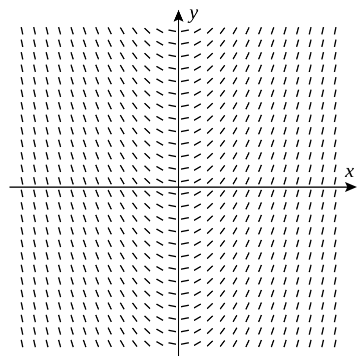


Figure Ex-14

15. In each part, confirm the stated equality.

(a) $1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

(b) $\displaystyle\lim_{n\to+\infty} \sum_{k=1}^{n-1}\left(\frac{9}{n}-\frac{k}{n^2}\right) = \frac{17}{2}$

(c) $\displaystyle\sum_{i=1}^{3}\left(\sum_{j=1}^{2}(i+j)\right) = 21$

16. Express

$$\sum_{k=4}^{18} k(k-3)$$

in sigma notation with

(a) $k = 0$ as the lower limit of summation

(b) $k = 5$ as the lower limit of summation.

17. The accompanying figure shows a square that is n units by n units that has been subdivided into a one-unit square and $n-1$ "L-shaped" regions. Use this figure to show that the sum of the first n consecutive positive odd integers is n^2.

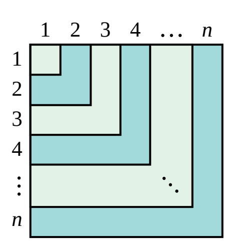


Figure Ex-17

18. Derive the result of Exercise 17 by writing

$$1+3+5+\cdots+2n-1 = \sum_{k=1}^{n}(2k-1)$$

When part of each term of a sum cancels part of the next term, leaving only portions of the first and last terms at the end, the sum is said to ***telescope***. In Exercises 19–22, evaluate the telescoping sum.

19. $\sum_{k=5}^{17}(3^k - 3^{k-1})$

20. $\sum_{k=1}^{50}\left(\frac{1}{k} - \frac{1}{k+1}\right)$

21. $\sum_{k=2}^{20}\left(\frac{1}{k^2} - \frac{1}{(k-1)^2}\right)$

22. $\sum_{k=1}^{100}(2^{k+1} - 2^k)$

23. (a) Show that

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

$$\left[\textit{Hint: } \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).\right]$$

(b) Use the result in part (a) to find

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}$$

24. (a) Show that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

$$\left[\textit{Hint: } \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.\right]$$

(b) Use the result in part (a) to find

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)}$$

25. Let $\bar{x}$ denote the arithmetic average of the n numbers $x_1, x_2, \ldots, x_n$. Use Theorem 5.4.1 to prove that

$$\sum_{i=1}^{n}(x_i - \bar{x}) = 0$$

26. Let

$$S = \sum_{k=0}^{n} ar^k$$

Show that $S - rS = a - ar^{n+1}$ and hence that

$$\sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1-r} \qquad (r \neq 1)$$

(A sum of this form is called a ***geometric sum***.)

27. In each part, rewrite the sum, if necessary, so that the lower limit is 0, and then use the formula derived in Exercise 26 to evaluate the sum. Check your answers using the summation feature of a calculating utility.

(a) $\sum_{k=1}^{20} 3^k$ (b) $\sum_{k=5}^{30} 2^k$ (c) $\sum_{k=0}^{100}(-1)^{k+1}\frac{1}{2^k}$

C **28.** In each part, make a conjecture about the limit by using a CAS to evaluate the sum for $n = 10$, 20, and 50; and then check your conjecture by using the formula in Exercise 26 to express the sum in closed form, and then finding the limit exactly.

(a) $\lim_{n\to+\infty}\sum_{k=0}^{n}\frac{1}{2^k}$ (b) $\lim_{n\to+\infty}\sum_{k=1}^{n}\left(\frac{3}{4}\right)^k$

29. (a) Show that the substitutions $u = \sec x$ and $u = \tan x$ produce different values for the integral

$$\int \sec^2 x \tan x \, dx$$

(b) Explain why both are correct.

30. Use the two substitutions in Exercise 29 to evaluate the definite integral

$$\int_0^{\pi/4} \sec^2 x \tan x \, dx$$

and confirm that they produce the same result.

31. Evaluate the integral

$$\int \sqrt{1 + x^{-2/3}}\, dx$$

by making the substitution $u = 1 + x^{2/3}$.

32. (a) Express the equation

$$\int_a^b [f_1(x) + f_2(x) + \cdots + f_n(x)]\, dx$$
$$= \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx + \cdots + \int_a^b f_n(x)]\,dx$$

in sigma notation.

(b) If $c_1, c_2, \ldots, c_n$ are constants and $f_1, f_2, \ldots, f_n$ are integrable functions on $[a, b]$, do you think it is always true that

$$\int_a^b \left(\sum_{k=1}^{n} c_k f_k(x)\right) dx = \sum_{k=1}^{n}\left[c_k \int_a^b f_k(x)\,dx\right]?$$

Explain your reasoning.

33. Find an integral formula for the antiderivative of $1/(1+x^2)$ on the interval $(-\infty, +\infty)$ whose value at $x = 1$ is (a) 0 and (b) 2.

C **34.** Let $F(x) = \int_0^x \frac{t-3}{t^2+7}\, dt$.

(a) Find the intervals on which F is increasing. Decreasing.
(b) Find the open intervals on which F is concave up. Concave down.
(c) Find the x-values, if any, at which the function F has absolute extrema.
(d) Use a CAS to graph F, and confirm that the results in parts (a), (b), and (c) are consistent with the graph.

35. Prove that the function
$$F(x) = \int_0^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt$$
is constant on the interval $(0, +\infty)$.

36. What is the natural domain of the function
$$F(x) = \int_1^x \frac{1}{t^2-9}\,dt?$$
Explain your reasoning.

37. In each part, determine the values of x for which $F(x)$ is positive, negative, or zero without performing the integration; explain your reasoning.

(a) $F(x) = \int_1^x \frac{t^4}{t^2+3}\,dt$ (b) $F(x) = \int_{-1}^x \sqrt{4-t^2}\,dt$

38. Find a formula (defined piecewise) for the upper boundary of the trapezoid shown in the accompanying figure, and then integrate that function to derive the formula for the area of the trapezoid given on the inside front cover of this text.

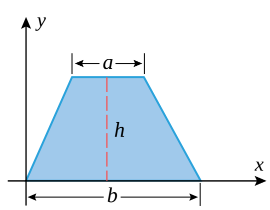


Figure Ex-38

39. The velocity of a particle moving along an s-axis is measured at 5-s intervals for 40 s, and the velocity function is modeled by a smooth curve. The curve and the data points are shown in the accompanying figure.

(a) Does the particle have constant acceleration? Explain your reasoning.

(b) Is there any 15-s time interval during which the acceleration is constant? Explain your reasoning.

(c) Estimate the average velocity of the particle over the 40-s time period.

(d) Estimate the distance traveled by the particle from time $t = 0$ to time $t = 40$.

(e) Is the particle ever slowing down during the 40-s time period? Explain your reasoning.

(f) Is there sufficient information for you to determine the s-coordinate of the particle at time $t = 10$? If so, find it. If not, explain what additional information you need.

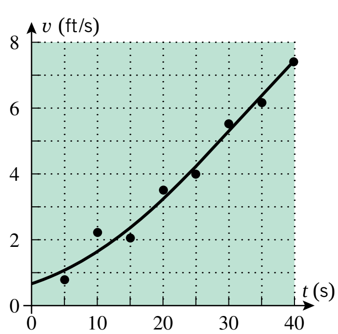


Figure Ex-39

40. Suppose that a tumor grows at the rate of $r(t) = t/7$ grams per week. When, during the second 26 weeks of growth, is the weight of the tumor the same as its average weight during that period?

In Exercises 41–46, evaluate the integrals by hand, and check your answers with a CAS if you have one.

41. $\int \frac{\cos 3x}{\sqrt{5+2\sin 3x}}\,dx$

42. $\int \frac{\sqrt{3+\sqrt{x}}}{\sqrt{x}}\,dx$

43. $\int \frac{x^2}{(ax^3+b)^2}\,dx$

44. $\int x\sec^2(ax^2)\,dx$

45. $\int_{-2}^{-1} \left(u^{-4} + 3u^{-2} - \frac{1}{u^5}\right) du$

46. $\int_0^1 \sin^2(\pi x)\cos(\pi x)\,dx$

C **47.** Use a CAS to approximate the area of the region in the first quadrant that lies below the curve $y = x + x^2 - x^3$ and above the x-axis.

C **48.** In each part, use a CAS to solve the initial-value problem.

(a) $\frac{dy}{dx} = x^2\cos 3x;\ y(\pi/2) = -1$

(b) $\frac{dy}{dx} = \frac{x^3}{(4+x^2)^{3/2}};\ y(0) = -2$

C **49.** In each part, use a CAS, where needed, to solve for k.

(a) $\int_1^k (x^3 - 2x - 1)\,dx = 0, \quad k > 1$

(b) $\int_0^k (x^2 + \sin 2x)\,dx = 3, \quad k \ge 0$

C **50.** Use a CAS to approximate the largest and smallest values of the integral
$$\int_{-1}^x \frac{t}{\sqrt{2+t^3}}\,dt$$
for $1 \le x \le 3$.

C **51.** The function J_0 defined by
$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin t)\,dt$$
is called the ***Bessel function of order zero***.

(a) Use a CAS to graph the equation $y = J_0(x)$ over the interval $0 \le x \le 8$.

(b) Estimate $J_0(1)$.

(c) Estimate the smallest positive zero of $J_0(x)$.

52. Find the area under the graph of $f(x) = 5x - x^2$ over the interval $[0, 5]$ using Definition 5.4.3 with x_k^* as the *left* endpoint of each subinterval.

EXPANDING THE CALCULUS HORIZON

Blammo the Human Cannonball

Blammo the Human Cannonball will be fired from a cannon and hopes to land in a small net at the opposite end of the circus arena. Your job as Blammo's manager is to do the mathematical calculations that will allow Blammo to perform his death-defying act safely. The methods that you will use are from the field of ***ballistics*** (*the study of projectile motion*).

The Problem

Blammo's cannon has a ***muzzle velocity*** of 35 m/s, which means that Blammo will leave the muzzle with that velocity. The muzzle opening will be 5 m above the ground, and Blammo's objective is to land in a net that is also 5 m above the ground and that extends a distance of 10 m between 90 m and 100 m from the cannon opening (Figure 1). Your mathematical problem is to determine the ***elevation angle*** α of the cannon (the angle from the horizontal to the cannon barrel) that will make Blammo land in the net.

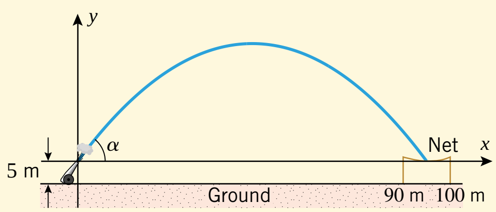


Figure 1

Modeling Assumptions

Blammo's trajectory will be determined by his initial velocity, the elevation angle of the cannon, and the forces that act on him after he leaves the muzzle. We will assume that the only force acting on Blammo after he leaves the muzzle is the downward force of the Earth's gravity. In particular, we will ignore the effect of air resistance. It will be convenient to introduce the xy-coordinate system shown in Figure 1 and to assume that Blammo is at the origin at time $t = 0$. We will also assume that Blammo's motion can be decomposed into two independent components, a horizontal component parallel to the x-axis and a vertical component parallel to the y-axis. We will analyze the horizontal and vertical components of Blammo's motion separately, and then we will combine the information to obtain a complete picture of his trajectory.

Blammo's Equations of Motion

We will denote the position and velocity functions for Blammo's horizontal component of motion by $x(t)$ and $v_x(t)$, and we will denote the position and velocity functions for his vertical component of motion by $y(t)$ and $v_y(t)$.

Since the only force acting on Blammo after he leaves the muzzle is the downward force of the Earth's gravity, there are no horizontal forces to alter his initial horizontal velocity $v_x(0)$. Thus, Blammo will have a constant velocity of $v_x(0)$ in the x-direction; this implies that

$$x(t) = v_x(0)t \tag{1}$$

In the y-direction Blammo is acted on only by the downward force of the Earth's gravity. Thus, his motion in this direction is governed by the free-fall model; hence, from (12) in Section 5.7 his vertical position function is

$$y(t) = y(0) + v_y(0)t - \tfrac{1}{2}gt^2$$

Taking $g = 9.8\ \text{m/s}^2$, and using the fact that $y(0) = 0$, this equation can be written as

$$y(t) = v_y(0)t - 4.9t^2 \tag{2}$$

Exercise 1 At time $t = 0$ Blammo's velocity is 35 m/s, and this velocity is directed at an angle α with the horizontal. It is a fact of physics that the initial velocity components $v_x(0)$ and $v_y(0)$ can be obtained geometrically from the muzzle velocity and the angle of elevation using the triangle shown in Figure 2. We will justify this later in the text, but for now use this fact to show that Equations (1) and (2) can be expressed as

$$x(t) = (35\cos\alpha)t$$
$$y(t) = (35\sin\alpha)t - 4.9t^2$$

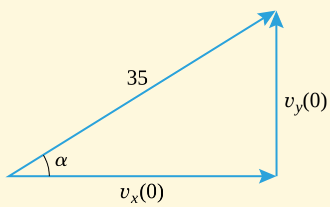


Figure 2

Exercise 2

(a) Use the result in Exercise 1 to find the velocity functions $v_x(t)$ and $v_y(t)$ in terms of the elevation angle α.

(b) Find the time t at which Blammo is at his maximum height above the x-axis, and show that this maximum height (in meters) is

$$y_{\max} = 62.5\sin^2\alpha$$

Exercise 3 The equations obtained in Exercise 1 can be viewed as parametric equations for Blammo's trajectory. Show, by eliminating the parameter t, that if $0 < \alpha < \pi/2$, then Blammo's trajectory is given by the equation

$$y = (\tan\alpha)x - \frac{0.004}{\cos^2\alpha}x^2$$

Explain why Blammo's trajectory is a parabola.

Finding the Elevation Angle

Define Blammo's ***horizontal range*** R to be the horizontal distance he travels until he returns to the height of the muzzle opening ($y = 0$). Your objective is to find elevation angles that will make the horizontal range fall between 90 m and 100 m, thereby ensuring that Blammo lands in the net (Figure 3).

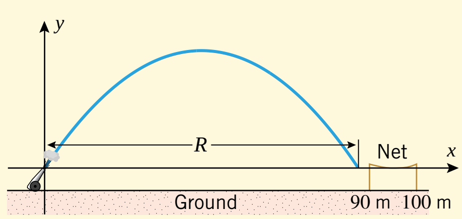


Figure 3

Exercise 4 Use a graphing utility and either the parametric equations obtained in Exercise 1 or the single equation obtained in Exercise 3 to generate Blammo's trajectories, taking elevation angles at increments of 10° from 15° to 85°. In each case, determine visually whether Blammo lands in the net.

Exercise 5 Find the time required for Blammo to return to his starting height ($y = 0$), and use that result to show that Blammo's range R is given by the formula

$$R = 125 \sin 2\alpha$$

Exercise 6

(a) Use the result in Exercise 5 to find two elevation angles that will allow Blammo to hit the midpoint of the net 95 m away.

(b) The tent is 55 m high. Explain why the larger elevation angle cannot be used.

Exercise 7 How much can the smaller elevation angle in Exercise 6 vary and still have Blammo hit the net between 90 m and 100 m?

Blammo's Shark Trick

Blammo is to be fired from 5 m above ground level with a muzzle velocity of 35 m/s over a flaming wall that is 20 m high and past a 5-m-high shark pool (Figure 4). To make the feat impressive, the pool will be made as long as possible. Your job as Blammo's manager is to determine the length of the pool, how far to place the cannon from the wall, and what elevation angle to use to ensure that Blammo clears the pool.

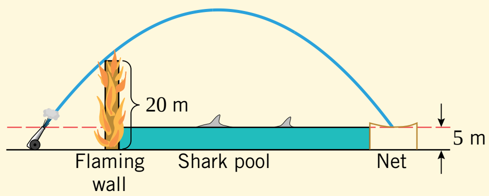


Figure 4

Exercise 8 Prepare a written presentation of the problem and your solution of it that is at an appropriate level for an engineer, physicist, or mathematician to read. Your presentation should contain the following elements: an explanation of all notation, a list and description of all formulas that will be used, a diagram that shows the orientation of any coordinate systems that will be used, a description of any assumptions you make to solve the problem, graphs that you think will enhance the presentation, and a clear step-by-step explanation of your solution.

Module by: *John Rickert*, *Rose-Hulman Institute of Technology*
Howard Anton, *Drexel University*