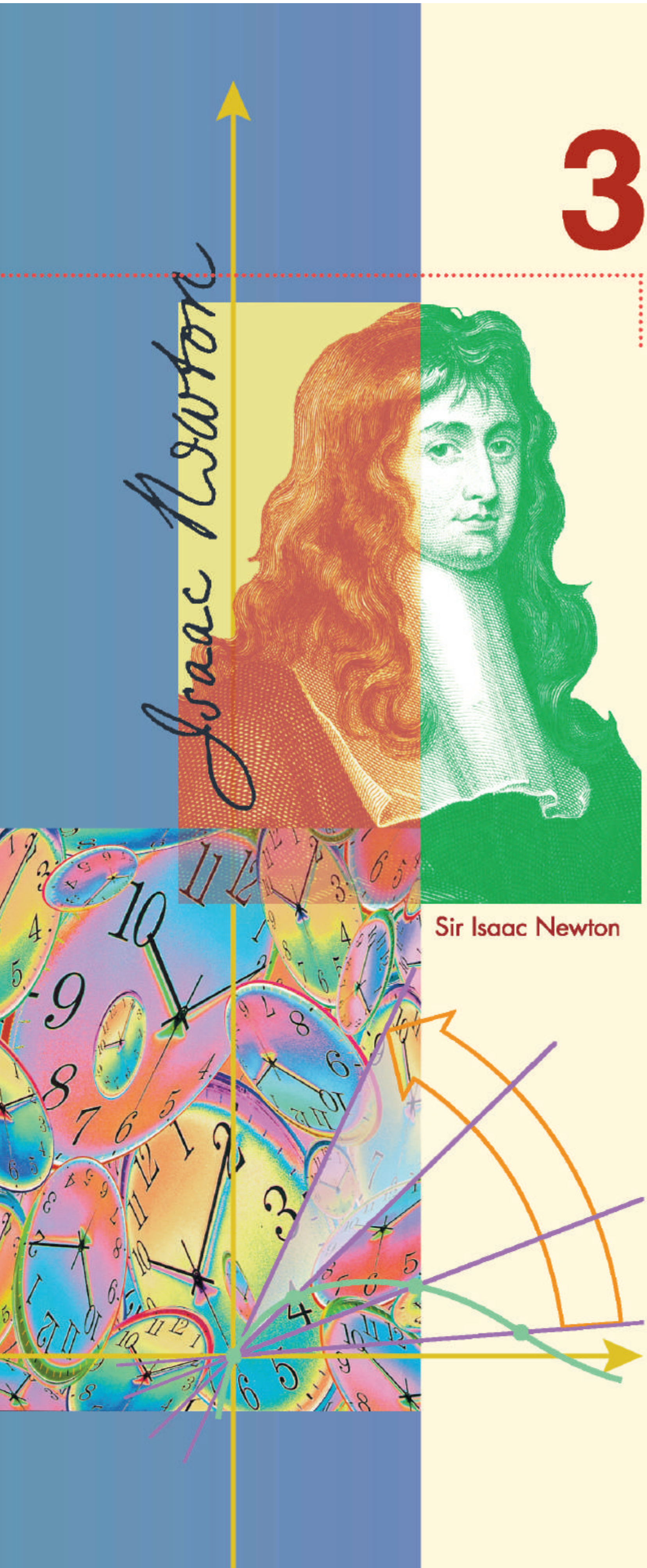


Sir Isaac Newton

3

THE DERIVATIVE

Many real-world phenomena involve changing quantities—the speed of a rocket, the inflation of currency, the number of bacteria in a culture, the shock intensity of an earthquake, the voltage of an electrical signal, and so forth. In this chapter we will develop the concept of a *derivative*, which is the mathematical tool that is used to study rates at which quantities change. In Section 3.1 we will interpret both average and instantaneous velocity geometrically, and we will define the slope of a curve at a point. In Sections 3.2 to 3.6 we will provide a precise definition of the derivative and we will develop mathematical tools for calculating derivatives efficiently. In Section 3.7 we will show how these methods of differentiation can be applied to problems involving rates of change.

One of the important themes of calculus is that many nonlinear functions can be closely approximated by linear functions. In Section 3.8 we will show how derivatives can be used to generate such approximations.

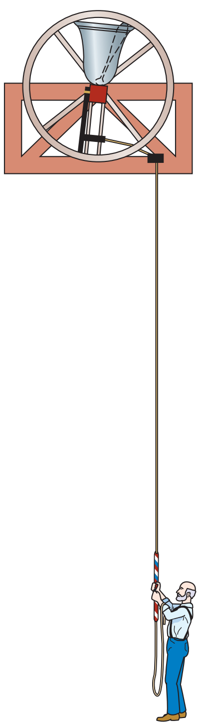
3.1 SLOPES AND RATES OF CHANGE

In this section we will explore the connection between velocity at an instant, the slope of a curve at a point, and rate of change. Our work here is intended to be informal and introductory, and all of the ideas that we develop will be revisited in more detail in later sections.

VELOCITY AND SLOPES

In Section 2.1 we interpreted the *instantaneous velocity* of a particle moving along an s-axis as a limit of average velocities. We begin our introduction to the derivative with another visit to the topic of velocity.

For purposes of illustration, consider a bell ringer practicing for her part in a change-ringing group at an English bell tower. The ringer controls a rope, pulling periodically to ring the bell. We will concentrate on the position of the *sally* (the handgrip on the rope), measured in feet above the floor of the ringing room. Imagine the s-axis as the line of travel of the sally. Figure 3.1.1*a* shows a sequence of "snapshots" of one such scenario, taken at times $t = 0, 1, 2, 3,$ and 4 s.



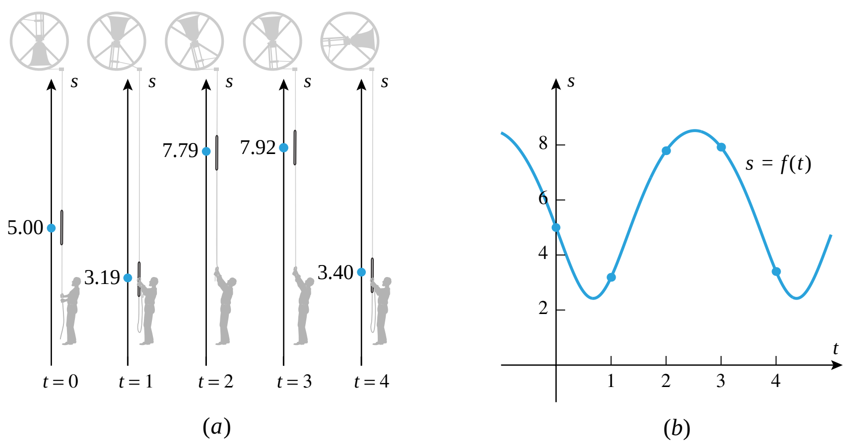


Figure 3.1.1

We may be able to record the height of the sally at various times, as in Table 3.1.1, or even model the motion of the sally by a function, as depicted in the graph in Figure 3.1.1*b*. The velocity of the sally measures the rate of ascent of the sally in its motion during the ringing of the bell. For example, during the first 2 s ($t = 0$ to $t = 2$), the displacement of the sally is $f(2) - f(0) = 7.79 - 5.00 = 2.79$ ft, so the average velocity of the sally during these 2 s is

$$v_{\text{ave}} = \frac{7.79 - 5.00}{2 - 0} \approx 1.39 \text{ ft/s}$$

The average velocity during the next 2 s ($t = 2$ to $t = 4$) is

$$v_{\text{ave}} = \frac{3.40 - 7.79}{4 - 2} \approx -2.19 \text{ ft/s}$$

Note that the displacement of the sally is negative during this latter time interval, since its position at time $t = 4$ is below that at time $t = 2$. Thus, the average velocity is also negative.

Table 3.1.1

t (seconds)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$s = f(t)$ (ft)	5.00	2.66	3.19	5.78	7.79	8.52	7.92	6.02	3.40

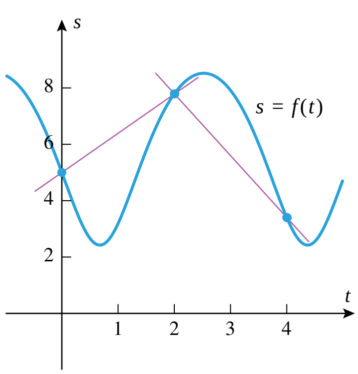


Figure 3.1.2

We can see from the graph of $s = f(t)$ (Figure 3.1.2) that these average velocities are equal to the slopes of the lines through the points (0, 5.00) and (2, 7.79), and through

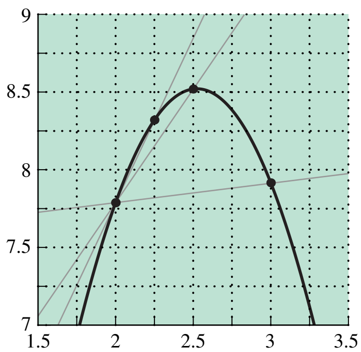


Figure 3.1.3

(2, 7.79) and (4, 3.40). Thus, average velocity can be interpreted as a geometric property of the graph of the position function.

> **3.1.1 GEOMETRIC INTERPRETATION OF AVERAGE VELOCITY.** *If an object moves along an s-axis, and if the position versus time curve is $s = f(t)$, then the average velocity of the object between times t_0 and t_1,*
>
> $$v_{\text{ave}} = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$
>
> *is represented geometrically by the slope of the line joining the points $(t_0, f(t_0))$ and $(t_1, f(t_1))$.*

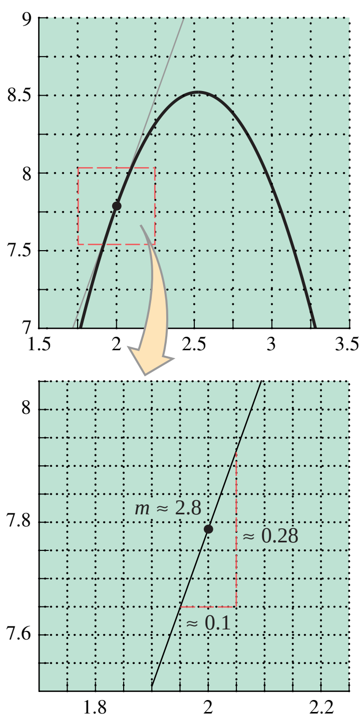


Figure 3.1.4

Now, from the graph of $s = f(t)$ in Figure 3.1.1*b* we can see that the sally is rising more quickly during the time interval $1.5 \le t \le 2$ than during the interval $2 \le t \le 2.5$. This is numerically revealed using the data in Table 3.1.1 to obtain average velocities of 4.02 ft/s and 1.46 ft/s, respectively, for these two time intervals. But what of the velocity, v_{inst}, of the sally at the instant our clock strikes $t = 2$ s? How should v_{inst} be defined? Does it have a geometric interpretation as well? We argued in Section 2.1 that the "instantaneous velocity" at a particular moment in time should be the limiting value of average velocities. This suggests that we define the instantaneous velocity of the sally at time $t = 2$ by

$$v_{\text{inst}} = \lim_{t_1 \to 2} \frac{f(t_1) - f(2)}{t_1 - 2}$$

It follows that we can estimate v_{inst} at $t = 2$ by calculating average velocities over ever smaller intervals anchored at 2. That is, we would expect that the fractions

$$\frac{f(2.2) - f(2)}{2.2 - 2}, \quad \frac{f(2.1) - f(2)}{2.1 - 2}, \quad \frac{f(2.01) - f(2)}{2.01 - 2}$$

would, in turn, each yield a better estimate for v_{inst}. Since Table 3.1.1 is lacking for such refined data, consider the portion of the graph of $s = f(t)$ near $t = 2$ shown in Figure 3.1.3.

The ratios that produce average velocities on an interval $2 \le t \le t_1$ are slopes of lines through the points $(2, f(2))$ and $(t_1, f(t_1))$. Figure 3.1.3 shows such lines for $t_1 = 3, 2.5$, and 2.25. We can infer the limiting value of these slopes as t_1 approaches 2 by magnifying a portion of the graph of f near the point $(2, f(2))$. This is illustrated in Figure 3.1.4, from which it appears that the limiting value is about 2.8. Thus, subject to our crude measuring devices, the instantaneous velocity at time $t = 2$ is given by $v_{\text{inst}} \approx 2.8$ ft/s.

SLOPE OF A CURVE

The preceding discussion of average and instantaneous velocities could be cast as an investigation of slopes related to the position curve. The slope of a general function curve at a point can be translated into useful information in many applications, so a consideration of the notion of the *slope of a curve* is warranted.

Consider the function $y = f(x)$ whose graph is shown in Figure 3.1.5. We focus on the point $P(x_0, f(x_0))$. One has an intuitive notion that the "steepness" of the curve varies at different points. For example, view the graph of $y = f(x)$ in Figure 3.1.5 as the cross

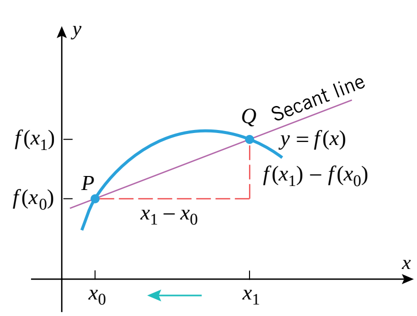


Figure 3.1.5

section of a hill and imagine a hiker walking the hill from left to right. The hiker will find the trek fairly arduous at point P, but the climb gets easier as she approaches the summit. Rather than rely on comparative notions of "less steep" or "more steep," we seek a numeric value to attach to each point on the curve that will describe "how steep" the curve is at that point. For straight lines, steepness is the same at every point, and the measure used to describe steepness is the slope of the line. (Note that slope not only describes "how steep" a line is, but also whether the line rises or falls.) Our goal is to define slope for our curve $y = f(x)$, even though $f(x)$ is not linear.

Since we know how to calculate the slope of a line through two points, let us consider a line joining point P with another point $Q(x_1, f(x_1))$ on the curve. By analogy with secants to circles, a line determined by two points on a curve is called a ***secant line*** to the curve. The slope of the secant line PQ is given by

$$m_{\text{sec}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{1}$$

As the sampling point $Q(x_1, f(x_1))$ is chosen closer to P, that is, as x_1 is selected closer to x_0, the slopes m_{sec} more nearly approximate what we might reasonably call the "slope" of the curve $y = f(x)$ at the point P. Thus, from (1), the slope of the curve $y = f(x)$ at $P(x_0, f(x_0))$ should be defined by

$$m_{\text{curve}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{2}$$

Example 1 Consider the function $f(x) = 6x - x^2$ and the point $P(2, f(2)) = (2, 8)$.

(a) Find the slopes of secant lines to the graph of $y = f(x)$ determined by P and points on the graph at $x = 3$ and $x = 1.5$.

(b) Find the slope of the graph of $y = f(x)$ at the point P.

Solution (a). The secant line to the graph of f through P and $Q(3, f(3)) = (3, 9)$ has slope

$$m_{\text{sec}} = \frac{9 - 8}{3 - 2} = 1$$

The secant line to the graph of f through P and $Q(1.5, f(1.5)) = (1.5, 6.75)$ has slope

$$m_{\text{sec}} = \frac{6.75 - 8}{1.5 - 2} = 2.5$$

Solution (b). The slope of the graph of f at the point P is

$$m_{\text{curve}} = \lim_{x_1 \to 2} \frac{f(x_1) - f(2)}{x_1 - 2} = \lim_{x_1 \to 2} \frac{6x_1 - x_1^2 - 8}{x_1 - 2}$$

$$= \lim_{x_1 \to 2} \frac{(4 - x_1)(x_1 - 2)}{x_1 - 2} = \lim_{x_1 \to 2} (4 - x_1) = 4 - 2 = 2$$ ◀

Recall our discussion of instantaneous velocity as a limit of average velocities, in which average velocities corresponded to slopes of secant lines on the position curve. We now have an interpretation of such a limit of slopes of secant lines as the slope of the position curve at the instant in question. This provides a geometric interpretation of instantaneous velocity as the slope of the graph of the position curve.

3.1.2 GEOMETRIC INTERPRETATION OF INSTANTANEOUS VELOCITY. *If a particle moves along an s-axis, and if the position versus time curve is $s = f(t)$, then the instantaneous velocity of the particle at time t_0,*

$$v_{\text{inst}} = \lim_{t_1 \to t_0} \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

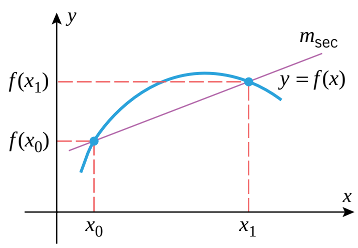
is represented geometrically by the slope of the curve at the point $(t_0, f(t_0))$.

SLOPES AND RATES OF CHANGE

Velocity or slope can be viewed as *rate of change*—the rate of change of position with respect to time, or the rate of change of a function's value with respect to its input. Rates of change occur in many applications. For example:

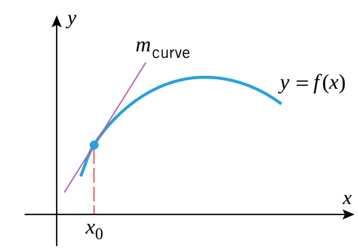
- A microbiologist might be interested in the rate at which the number of bacteria in a colony changes with time.
- An engineer might be interested in the rate at which the length of a metal rod changes with temperature.
- An economist might be interested in the rate at which production cost changes with the quantity of a product that is manufactured.
- A medical researcher might be interested in the rate at which the radius of an artery changes with the concentration of alcohol in the bloodstream.

In general, if x and y are quantities related by an equation $y = f(x)$, we can consider the rate at which y changes with x. As with velocity, we distinguish between an average rate of change, represented by the slope of a secant line to the graph of $y = f(x)$, and an instantaneous rate of change, represented by the slope of the curve at a point.



m_{sec} is the average rate of change of y with respect to x over the interval $[x_0, x_1]$.

(*a*)



m_{curve} is the instantaneous rate of change of y with respect to x when $x = x_0$.

(*b*)

Figure 3.1.6

3.1.3 DEFINITION. If $y = f(x)$, then the ***average rate of change of y with respect to x over the interval*** $[x_0, x_1]$ is

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{3}$$

Geometrically, the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is the slope of the secant line to the graph of $y = f(x)$ through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$:

$$r_{\text{ave}} = m_{\text{sec}}$$

(see Figure 3.1.6*a*).

3.1.4 DEFINITION. If $y = f(x)$, then the ***instantaneous rate of change of y with respect to x when x = x₀*** is

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{4}$$

Geometrically, the instantaneous rate of change of y with respect to x when $x = x_0$ is the slope of the graph of $y = f(x)$ at the point $(x_0, f(x_0))$:

$$r_{\text{inst}} = m_{\text{curve}}$$

(see Figure 3.1.6*b*).

Example 2 Let $y = x^2 + 1$.

(a) Find the average rate of change of y with respect to x over the interval [3, 5].

(b) Find the instantaneous rate of change of y with respect to x when $x = -4$.

(c) Find the instantaneous rate of change of y with respect to x at the general point corresponding to $x = x_0$.

Solution (a). We apply Formula (3) with $f(x) = x^2 + 1$, $x_0 = 3$, and $x_1 = 5$. This yields

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(5) - f(3)}{5 - 3} = \frac{26 - 10}{2} = 8$$

Thus, on the average, y increases 8 units per unit increase in x over the interval [3, 5].

Solution (b). We apply Formula (4) with $f(x) = x^2 + 1$ and $x_0 = -4$. This yields

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to -4} \frac{f(x_1) - f(-4)}{x_1 - (-4)} = \lim_{x_1 \to -4} \frac{(x_1^2 + 1) - 17}{x_1 + 4}$$

$$= \lim_{x_1 \to -4} \frac{x_1^2 - 16}{x_1 + 4} = \lim_{x_1 \to -4} \frac{(x_1 + 4)(x_1 - 4)}{x_1 + 4} = \lim_{x_1 \to -4} (x_1 - 4) = -8$$

Thus, for a small change in x from $x = -4$, the value of y will change approximately eight times as much *in the opposite direction*. That is, because the instantaneous rate of change is negative, the value of y *decreases* as values of x move through $x = -4$ from left to right.

Solution (c). We proceed as in part (b):

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{(x_1^2 + 1) - (x_0^2 + 1)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_1^2 - x_0^2}{x_1 - x_0}$$

$$= \lim_{x_1 \to x_0} \frac{(x_1 + x_0)(x_1 - x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} (x_1 + x_0) = 2x_0$$

Thus, the instantaneous rate of change of y with respect to x at $x = x_0$ is $2x_0$. Observe that the result in part (b) can be obtained from this more general result by setting $x_0 = -4$. ◀

RATES OF CHANGE IN APPLICATIONS

In applied problems, average and instantaneous rates of change must be accompanied by appropriate units. In general, the units for a rate of change of y with respect to x are obtained by "dividing" the units of y by the units of x and then simplifying according to the standard rules of algebra. Here are some examples:

- If y is in degrees Fahrenheit (°F) and x is in inches (in), then a rate of change of y with respect to x has units of degrees Fahrenheit per inch (°F/in).
- If y is in feet per second (ft/s) and x is in seconds (s), then a rate of change of y with respect to x has units of feet per second per second (ft/s/s), which would usually be written as ft/s^2.
- If y is in newton-meters (N·m) and x is in meters (m), then a rate of change of y with respect to x has units of newtons (N), since N·m/m = N.
- If y is in foot-pounds (ft·lb) and x is in hours (h), then a rate of change of y with respect to x has units of foot-pounds per hour (ft·lb/h).

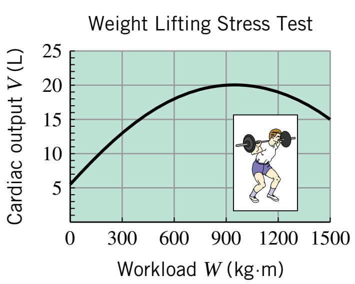


Figure 3.1.7

Example 3 The limiting factor in athletic endurance is cardiac output, that is, the volume of blood that the heart can pump per unit of time during an athletic competition. Figure 3.1.7 shows a stress-test graph of cardiac output V in liters (L) of blood versus workload W in kilogram-meters (kg·m) for 1 minute of weight lifting. This graph illustrates the known medical fact that cardiac output increases with the workload, but after reaching a peak value begins to decrease.

(a) Use the secant line shown in Figure 3.1.8*a* to estimate the average rate of change of cardiac output with respect to workload as the workload increases from 300 to 1200 kg·m.

(b) Use the line segment shown in Figure 3.1.8*b* to estimate the instantaneous rate of change of cardiac output with respect to workload at the point where the workload is 300 kg·m.

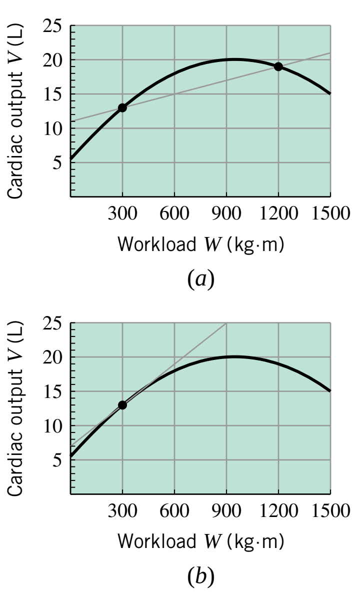


Figure 3.1.8

Solution (a). Using the estimated points (300, 13) and (1200, 19), the slope of the secant line indicated in Figure 3.1.8*a* is

$$m_{\text{sec}} \approx \frac{19 - 13}{1200 - 300} \approx 0.0067 \, \frac{\text{L}}{\text{kg·m}}$$

Since $r_{\text{ave}} = m_{\text{sec}}$, the average rate of change of cardiac output with respect to workload over the interval is approximately 0.0067 L/kg·m. This means that on the average a 1-unit increase in workload produced a 0.0067-L increase in cardiac output over the interval.

Solution (b). We estimate the slope of the cardiac output curve at $W = 300$ by sketching a line that appears to meet the curve at $W = 300$ with slope equal to that of the curve (Figure 3.1.8*b*). Estimating points (0, 7) and (900, 25) on this line, we obtain

$$r_{\text{inst}} \approx \frac{25-7}{900-0} = 0.02\ \frac{\text{L}}{\text{kg}\cdot\text{m}}$$ ◀

EXERCISE SET 3.1

1. The accompanying figure shows the position versus time curve for an elevator that moves upward a distance of 60 m and then discharges its passengers.
(a) Estimate the instantaneous velocity of the elevator at $t = 10$ s.
(b) Sketch a velocity versus time curve for the motion of the elevator for $0 \le t \le 20$.

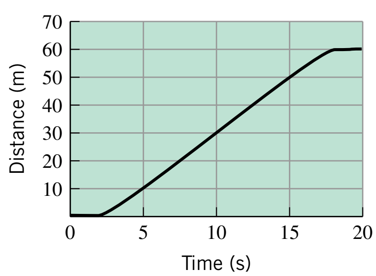


Figure Ex-1

2. The accompanying figure shows the position versus time curve for a certain particle moving along a straight line. Estimate each of the following from the graph:
(a) the average velocity over the interval $0 \le t \le 3$
(b) the values of t at which the instantaneous velocity is zero
(c) the values of t at which the instantaneous velocity is either a maximum or a minimum
(d) the instantaneous velocity when $t = 3$ s.

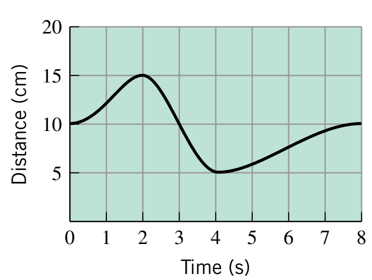


Figure Ex-2

3. The accompanying figure shows the position versus time curve for a certain particle moving on a straight line.
(a) Is the particle moving faster at time t_0 or time t_2? Explain.
(b) The portion of the curve near the origin is horizontal. What does this tell us about the initial velocity of the particle?
(c) Is the particle speeding up or slowing down in the interval $[t_0, t_1]$? Explain.
(d) Is the particle speeding up or slowing down in the interval $[t_1, t_2]$? Explain.

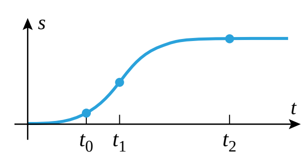


Figure Ex-3

4. An automobile, initially at rest, begins to move along a straight track. The velocity increases steadily until suddenly the driver sees a concrete barrier in the road and applies the brakes sharply at time t_0. The car decelerates rapidly, but it is too late—the car crashes into the barrier at time t_1 and instantaneously comes to rest. Sketch a position versus time curve that might represent the motion of the car.

5. If a particle moves at constant velocity, what can you say about its position versus time curve?

6. The accompanying figure shows the position versus time curves of four different particles moving on a straight line. For each particle, determine whether its instantaneous velocity is increasing or decreasing with time.

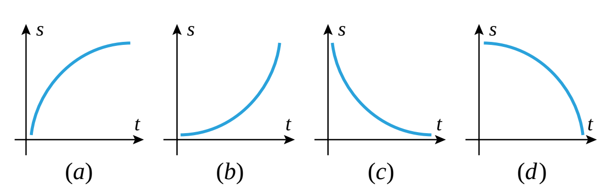


Figure Ex-6

In Exercises 7–10, a function $y = f(x)$ and values of x_0 and x_1 are given.
(a) Find the average rate of change of y with respect to x over the interval $[x_0, x_1]$.
(b) Find the instantaneous rate of change of y with respect to x at the given value of x_0.
(c) Find the instantaneous rate of change of y with respect to x at a general x-value x_0.
(d) Sketch the graph of $y = f(x)$ together with the secant line whose slope is given by the result in part (a), and indicate graphically the slope of the curve given by the result in part (b).

7. $y = \frac{1}{2}x^2;\ x_0 = 3,\ x_1 = 4$

8. $y = x^3;\ x_0 = 1,\ x_1 = 2$

9. $y = 1/x;\ x_0 = 2,\ x_1 = 3$

10. $y = 1/x^2;\ x_0 = 1,\ x_1 = 2$

In Exercises 11–14, a function $y = f(x)$ and an x-value x_0 are given.
(a) Find the slope of the graph of f at a general x-value x_0.
(b) Find the slope of the graph of f at the x-value specified by the given x_0.

11. $f(x) = x^2 + 1;\ x_0 = 2$

12. $f(x) = x^2 + 3x + 2;\ x_0 = 2$

13. $f(x) = \sqrt{x};\ x_0 = 1$

14. $f(x) = 1/\sqrt{x};\ x_0 = 4$

15. Suppose that the outside temperature versus time curve over a 24-hour period is as shown in the accompanying figure.
(a) Estimate the maximum temperature and the time at which it occurs.
(b) The temperature rise is fairly linear from 8 A.M. to 2 P.M. Estimate the rate at which the temperature is increasing during this time period.
(c) Estimate the time at which the temperature is decreasing most rapidly. Estimate the instantaneous rate of change of temperature with respect to time at this instant.

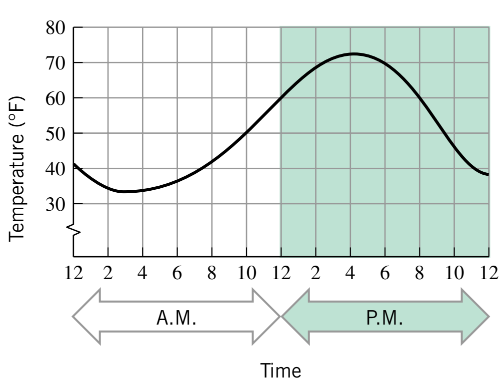


Figure Ex-15

16. The accompanying figure shows the graph of the pressure p in atmospheres (atm) versus the volume V in liters (L) of 1 mole of an ideal gas at a constant temperature of 300 K (kelvins). Use the line segments shown in the figure to estimate the rate of change of pressure with respect to volume at the points where $V = 10$ L and $V = 25$ L.

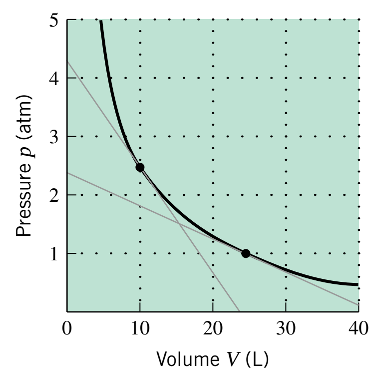


Figure Ex-16

17. The accompanying figure shows the graph of the height h in centimeters versus the age t in years of an individual from birth to age 20.
(a) When is the growth rate greatest?
(b) Estimate the growth rate at age 5.
(c) At approximately what age between 10 and 20 is the growth rate greatest? Estimate the growth rate at this age.
(d) Draw a rough graph of the growth rate versus age.

Figure Ex-17

In Exercises 18–21, use geometric interpretations 3.1.1 and 3.1.2 to find the average and instantaneous velocity.

18. A rock is dropped from a height of 576 ft and falls toward Earth in a straight line. In t seconds the rock drops a distance of $s = 16t^2$ ft.
(a) How many seconds after release does the rock hit the ground?

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(b) What is the average velocity of the rock during the time it is falling?
(c) What is the average velocity of the rock for the first 3 s?
(d) What is the instantaneous velocity of the rock when it hits the ground?

19. During the first 40 s of a rocket flight, the rocket is propelled straight up so that in t seconds it reaches a height of $s = 5t^3$ ft.
(a) How high does the rocket travel in 40 s?
(b) What is the average velocity of the rocket during the first 40 s?
(c) What is the average velocity of the rocket during the first 135 ft of its flight?
(d) What is the instantaneous velocity of the rocket at the end of 40 s?

20. A particle moves on a line away from its initial position so that after t hours it is $s = 3t^2 + t$ miles from its initial position.
(a) Find the average velocity of the particle over the interval $[1, 3]$.
(b) Find the instantaneous velocity at $t = 1$.

21. A particle moves in the positive direction along a straight line so that after t minutes its distance is $s = 6t^4$ feet from the origin.
(a) Find the average velocity of the particle over the interval $[2, 4]$.
(b) Find the instantaneous velocity at $t = 2$.

3.2 THE DERIVATIVE

In this section we will introduce the concept of a "derivative," the primary mathematical tool that is used to calculate rates of change and slopes of curves.

SLOPE OF A CURVE AND TANGENT LINES

In the preceding section we argued that the slope of the graph of $y = f(x)$ at $x = x_0$ should be given by

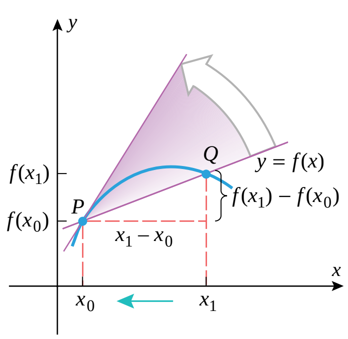
$$m_{\text{curve}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{1}$$

The ratio

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

is called a ***difference quotient***. As we saw in the last section, the difference quotient can also be interpreted as the average rate of change of $f(x)$ over the interval $[x_0, x_1]$, and its limit as $x_1 \to x_0$ is the instantaneous rate of change of $f(x)$ at $x = x_0$.

The geometric problem of finding the slope of a curve, and the somewhat paradoxical notions of instantaneous velocity and instantaneous rate of change, are all resolved by a limit of a difference quotient. The fact that problems in such disparate areas are unified by this expression is celebrated in the definition of the *derivative* of a function at a value in its domain.



$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Figure 3.2.1

3.2.1 DEFINITION. Suppose that x_0 is a number in the domain of a function f. If

$$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

exists, then the value of this limit is called the ***derivative of f at x = x₀*** and is denoted by $f'(x_0)$. That is,

$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{2}$$

(see Figure 3.2.1). If the limit of the difference quotient exists, $f'(x_0)$ is the ***slope of the graph of f at the point P(x₀, f(x₀))*** (or at $x = x_0$). If this limit does not exist, then the slope of the graph of f is ***undefined*** at P (or at $x = x_0$).

Now that we have defined the derivative of a function, we can begin to answer a question that fueled much of the early development of calculus. Mathematicians of the seventeenth century were perplexed by the problem of defining a *tangent line* to a general curve. Of course, in the case of a circle the definition was apparent: a line is tangent to a circle if it meets the circle at a single point. But, it was also clear that this simple definition would not suffice in many cases. For example, the y-axis intersects the parabola $y = x^2$ at a single point but does not appear to be "tangent" to the curve (Figure 3.2.2*a*). On the other hand, the line $y = 1$ does seem to be tangent to the graph of $y = \sin x$, even though it intersects this graph infinitely often (Figure 3.2.2*b*).

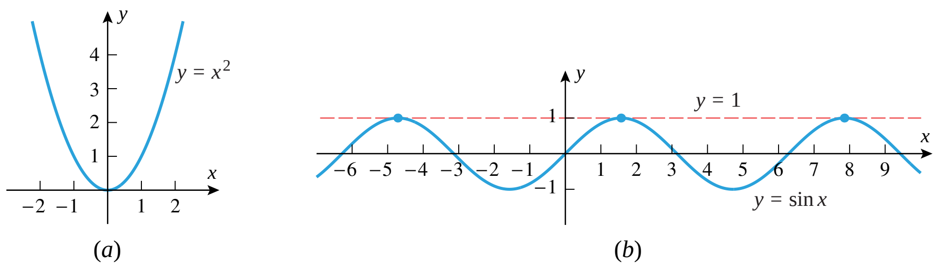


Figure 3.2.2

By the end of the first half of the seventeenth century, mathematicians such as Descartes and Fermat had developed a variety of procedures for *constructing* tangent lines. However, a general definition of a tangent line to a curve was still missing. Roughly speaking, a line should be tangent to the graph of a function $y = f(x)$ at a point $(x_0, f(x_0))$ provided the line has the same *direction* as the graph at the point. Since the direction of a line is determined by its slope, we would expect a line to be tangent to the graph at $(x_0, f(x_0))$ if the slope of the line is equal to the slope of the graph of f at x_0. Thus, we can now use the derivative to define the tangent line to a curve when the curve is the graph of a function $y = f(x)$. (Later we will extend this definition to more general curves.)

3.2.2 DEFINITION. Suppose that x_0 is a number in the domain of a function f. If

$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

exists, then we define the ***tangent line to the graph of f at the point P(x₀, f(x₀))*** to be the line whose equation is

$$y - f(x_0) = f'(x_0)(x - x_0) \tag{3}$$

We also call this the ***tangent line to the graph of f at*** $x = x_0$.

WARNING. Tangent lines to graphs do not have the same properties as tangent lines to circles. For example, a tangent line to a circle intersects the circle only at the point of tangency whereas a tangent line to a general graph may intersect the graph at points other than the point of tangency (Figure 3.2.3).

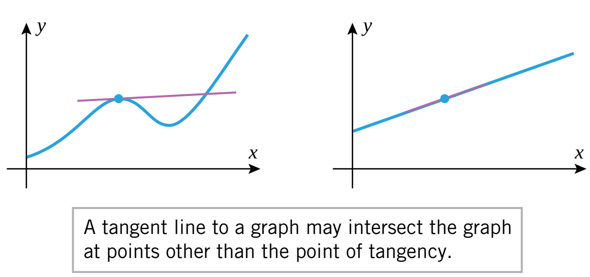


Figure 3.2.3

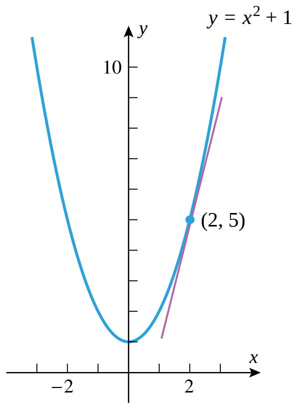


Figure 3.2.4

Example 1 Find the slope of the graph of $y = x^2 + 1$ at the point (2, 5), and use it to find the equation of the tangent line to $y = x^2 + 1$ at $x = 2$ (Figure 3.2.4).

Solution. From (2), the slope of the graph of $y = x^2 + 1$ at the point (2, 5) is given by

$$f'(2) = \lim_{x_1 \to 2} \frac{f(x_1) - f(2)}{x_1 - 2} = \lim_{x_1 \to 2} \frac{(x_1^2 + 1) - 5}{x_1 - 2} = \lim_{x_1 \to 2} \frac{x_1^2 - 4}{x_1 - 2}$$

$$= \lim_{x_1 \to 2} \frac{(x_1 - 2)(x_1 + 2)}{x_1 - 2} = \lim_{x_1 \to 2} (x_1 + 2) = 4$$

The tangent line is the line through the point (2, 5) with slope 4,

$$y - 5 = 4(x - 2)$$

which we may also write in slope-intercept form as $y = 4x - 3$. ◀

SLOPE OF A CURVE BY ZOOMING

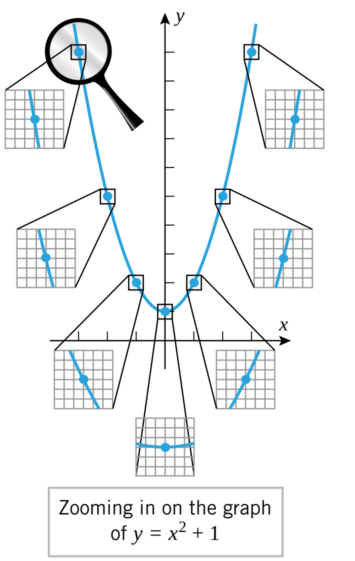


Figure 3.2.5

The slope of a curve at a point can be estimated by zooming on a graphing utility. The idea is to zoom in on the point until the surrounding portion of the curve appears to be a straight line (Figure 3.2.5). The utility's trace operation can then be used to estimate the slope. Figure 3.2.6 illustrates this procedure for the tangent line in Example 1. The first part of the figure shows the graph of $y = x^2 + 1$ in the window

$$[-6.3, 6.3] \times [0, 6.2]$$

and the second part shows the graph after we have zoomed in on the point (2, 5) by a factor of 10. The trace operation produces the points

$$(2.05, 5.2025) \quad \text{and} \quad (1.95, 4.8025)$$

on the curve, so the slope of the tangent line can be approximated as

$$f'(2) \approx \frac{5.2025 - 4.8025}{2.05 - 1.95} = \frac{0.4}{0.1} = 4.0$$

which happens to agree exactly with the result in Example 1. It is important to understand, however, that the exact agreement in this case is accidental; in general, this method will not produce exact results because of roundoff errors in the computations, and also because the magnified portion of the curve may have a slight curvature, even though it appears straight on the screen.

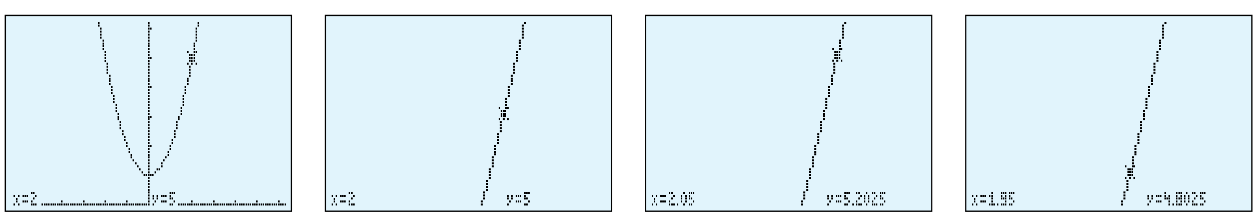


Figure 3.2.6

THE DERIVATIVE

In general, the slope of a curve $y = f(x)$ will depend on the point $(x, f(x))$ at which the slope is computed. That is, the slope is itself a function of x. To illustrate this, let us use (2) to compute $f'(x_0)$ at a general x-value x_0 for the curve $y = x^2 + 1$. The computations are similar to those in Example 1.

$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{(x_1^2 + 1) - (x_0^2 + 1)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_1^2 - x_0^2}{x_1 - x_0}$$

$$= \lim_{x_1 \to x_0} \frac{(x_1 - x_0)(x_1 + x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} (x_1 + x_0) = 2x_0 \qquad (4)$$

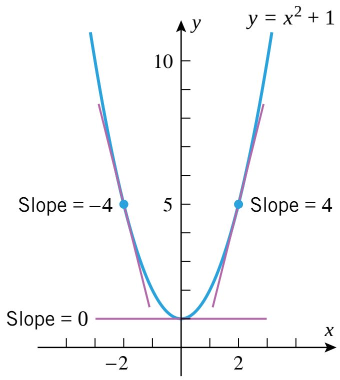


Figure 3.2.7

Now we can use the general formula $f'(x_0) = 2x_0$ to compute the slope of the tangent line at any point along the curve $y = x^2 + 1$ simply by substituting the appropriate value for $x = x_0$. For example, if $x_0 = 2$, $2x_0 = 4$, so $f'(2) = 4$, agreeing with the result in Example 1. Similarly, if $x_0 = 0$, then $2x_0 = 0$, so $f'(0) = 0$; and if $x_0 = -2$, then $2x_0 = -4$, so $f'(-2) = -4$ (Figure 3.2.7).

To generalize this idea, replacing x_0 by x in (2), the slope of the graph of $y = f(x)$ at a general point $(x, f(x))$ is given by

$$f'(x) = \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} \tag{5}$$

The fact that this describes a "slope-producing function" is so important that there is a common terminology associated with it. [To simplify notation, we use w in the place of x_1 in (5).]

> **3.2.3 DEFINITION.** The function f' defined by the formula
>
> $$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x} \tag{6}$$
>
> is called the ***derivative of f with respect to x***. The domain of f' consists of all x in the domain of f for which the limit exists.

REMARK. Despite the presence of the symbol w in the definition, Formula (6) defines the function f' as a function of the single variable x. To calculate the value of $f'(x)$ at a particular input value x, we fix the value of x and let $w \to x$ in (6). The answer to this limit no longer involves the symbol w; w "disappears" at the step in which the limit is evaluated.

REMARK. This is our first encounter with what was alluded to in Section 1.1 as a function that is the result of a "continuing process of incremental refinement." That is, the derivative function f' is *derived* from the function f via a limit. The use of a limiting process to define a new object is a fundamental tool in calculus and will be employed again in later chapters.

Recalling from the last section that the slope of the graph of $y = f(x)$ can be interpreted as the instantaneous rate of change of y with respect to x, it follows that the derivative of a function f can be interpreted in several ways:

> ***Interpretations of the Derivative***. The derivative f' of a function f can be interpreted as a function whose value at x is the slope of the graph of $y = f(x)$ at x, or, alternatively, it can be interpreted as a function whose value at x is the instantaneous rate of change of y with respect to x at x. In particular, when $y = f(t)$ describes the position at time t of an object moving along a straight line, then $f'(t)$ describes the (instantaneous) velocity of the object at time t.

Example 2

(a) Find the derivative with respect to x of $f(x) = x^3 - x$.

(b) Graph f and f' together, and discuss the relationship between the two graphs.

Solution (a). Later in this chapter we will develop efficient methods for finding derivatives, but for now we will find the derivative directly from Formula (6) in the definition of f'. The

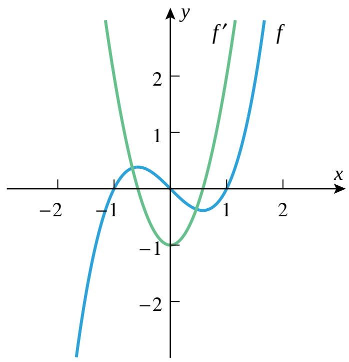
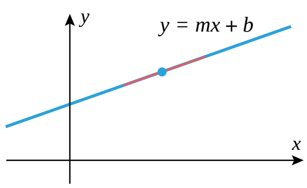


Figure 3.2.8

computations are as follows:

$$f'(x) = \lim_{w\to x}\frac{f(w)-f(x)}{w-x} = \lim_{w\to x}\frac{(w^3-w)-(x^3-x)}{w-x} = \lim_{w\to x}\frac{(w^3-x^3)-(w-x)}{w-x}$$
$$= \lim_{w\to x}\frac{(w-x)[(w^2+wx+x^2)-1]}{w-x} = \lim_{w\to x}(w^2+wx+x^2-1)$$
$$= x^2+x^2+x^2-1 = 3x^2-1$$

Solution (b). Since $f'(x)$ can be interpreted as the slope of the graph of $y = f(x)$ at x, the derivative $f'(x)$ is positive where the graph of f has positive slope, it is negative where the graph of f has negative slope, and it is zero where the graph of f is horizontal. We leave it for the reader to verify that this is consistent with the graphs of $f(x) = x^3 - x$ and $f'(x) = 3x^2 - 1$ shown in Figure 3.2.8. ◀



At each value of x the tangent line has slope m.

Figure 3.2.9

Example 3 At each value of x, the tangent line to a line $y = mx+b$ coincides with the line itself (Figure 3.2.9), and hence all tangent lines have slope m. This suggests geometrically that if $f(x) = mx + b$, then $f'(x) = m$ for all x. This is confirmed by the following computations:

$$f'(x) = \lim_{w\to x}\frac{f(w)-f(x)}{w-x} = \lim_{w\to x}\frac{(mw+b)-(mx+b)}{w-x} = \lim_{w\to x}\frac{mw-mx}{w-x}$$
$$= \lim_{w\to x}\frac{m(w-x)}{w-x} = \lim_{w\to x} m = m$$ ◀

Example 4

(a) Find the derivative with respect to x of $f(x) = \sqrt{x}$.

(b) Find the slope of the curve $y = \sqrt{x}$ at $x = 9$.

(c) Find the limits of $f'(x)$ as $x \to 0^+$ and as $x \to +\infty$, and explain what those limits say about the graph of f.

Solution (a). From Definition 3.2.3,

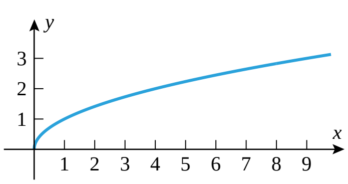
$$f'(x) = \lim_{w\to x}\frac{f(w)-f(x)}{w-x} = \lim_{w\to x}\frac{\sqrt{w}-\sqrt{x}}{w-x} = \lim_{w\to x}\frac{\sqrt{w}-\sqrt{x}}{w-x}\cdot\frac{\sqrt{w}+\sqrt{x}}{\sqrt{w}+\sqrt{x}}$$
$$= \lim_{w\to x}\frac{w-x}{(w-x)(\sqrt{w}+\sqrt{x})} = \lim_{w\to x}\frac{1}{\sqrt{w}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Solution (b). The slope of the curve $y = \sqrt{x}$ at $x = 9$ is $f'(9)$. From part (a), this slope is $f'(9) = 1/(2\sqrt{9}) = \frac{1}{6}$.

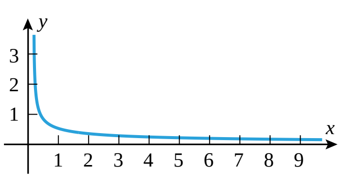
Solution (c). The graphs of $f(x) = \sqrt{x}$ and $f'(x) = 1/(2\sqrt{x})$ are shown in Figure 3.2.10. Observe that $f'(x) > 0$ if $x > 0$, which means that all tangent lines to the graph of $y = \sqrt{x}$ have positive slope at all points in this interval. Since

$$\lim_{x\to 0^+}\frac{1}{2\sqrt{x}} = +\infty \quad\text{and}\quad \lim_{x\to +\infty}\frac{1}{2\sqrt{x}} = 0$$

the graph becomes more and more vertical as $x \to 0^+$ and more and more horizontal as $x \to +\infty$. ◀



$y = f(x) = \sqrt{x}$



$y = f'(x) = \frac{1}{2\sqrt{x}}$

Figure 3.2.10

FOR THE READER. Use a graphing utility to estimate the slope of the curve $y = \sqrt{x}$ at $x = 9$ by zooming, and compare your result to the exact value obtained in the last example. If you have a CAS, read the documentation to determine how it can be used to find derivatives, and then use it to confirm the derivative obtained in Example 4(a).

Example 5 Consider the situation in Example 1 of Section 2.1 where a ball is thrown vertically upward so that the height (in feet) of the ball above the ground t seconds after its release is modeled by the function

$$s(t) = -16t^2 + 29t + 6, \quad 0 \le t \le 2$$

(a) Use the derivative of $s(t)$ at $t = 0.5$ to determine the instantaneous velocity of the ball at time $t = 0.5$ s.

(b) Find the velocity function $v(t) = s'(t)$ for $0 < t < 2$. What is the velocity of the ball just before impacting the ground at time $t = 2$ s?

Solution (a). When $t = 0.5$ s, the height of the ball is $s(0.5) = 16.5$ ft. The ball's instantaneous velocity at time $t = 0.5$ is given by the derivative of s at $t = 0.5$, that is, $s'(0.5)$. From Definition 3.2.1,

$$\begin{aligned} s'(0.5) &= \lim_{w \to 0.5} \frac{s(w) - s(0.5)}{w - 0.5} = \lim_{w \to 0.5} \frac{(-16w^2 + 29w + 6) - 16.5}{w - 0.5} \\ &= \lim_{w \to 0.5} \frac{-16w^2 + 29w - 10.5}{w - 0.5} \cdot \frac{2}{2} = \lim_{w \to 0.5} \frac{-32w^2 + 58w - 21}{2w - 1} \\ &= \lim_{w \to 0.5} \frac{(2w - 1)(-16w + 21)}{2w - 1} = \lim_{w \to 0.5} (-16w + 21) = -8 + 21 = 13 \end{aligned}$$

Thus, the velocity of the ball at time $t = 0.5$ s is $s'(0.5) = 13$ ft/s, which agrees with our estimate from numerical evidence in Example 1 of Section 2.1.

Solution (b). From Definition 3.2.3,

$$\begin{aligned} v(t) = s'(t) &= \lim_{w \to t} \frac{s(w) - s(t)}{w - t} = \lim_{w \to t} \frac{(-16w^2 + 29w + 6) - (-16t^2 + 29t + 6)}{w - t} \\ &= \lim_{w \to t} \frac{-16(w^2 - t^2) + 29(w - t) + (6 - 6)}{w - t} \\ &= \lim_{w \to t} \frac{-16(w - t)(w + t) + 29(w - t)}{w - t} = \lim_{w \to t} \frac{(w - t)[-16(w + t) + 29]}{w - t} \\ &= \lim_{w \to t} [-16(w + t) + 29] = -16(t + t) + 29 = -32t + 29 \end{aligned}$$

Thus, for $0 < t < 2$, the velocity of the ball is given by $v(t) = s'(t) = -32t + 29$. As $t \to 2^-$, $s'(t) = -32t + 29 \to -64 + 29 = -35$ ft/s. That is, the ball is falling at a speed approaching 35 ft/s when its impact with the ground is imminent. ◀

DIFFERENTIABILITY

Observe that a function f must be defined at $x = x_0$ in order for the difference quotient

$$\frac{f(w) - f(x_0)}{w - x_0}$$

to make sense, since this quotient references a value for $f(x_0)$. Since a value for $f(x_0)$ is required before the limit of this quotient can be considered, values in the domain of the derivative function f' must also be in the domain of f.

For a number x_0 in the domain of a function f, we say that f is ***differentiable at x_0***, or that ***the derivative of f exists at x_0***, if

$$\lim_{w \to x_0} \frac{f(w) - f(x_0)}{w - x_0}$$

exists. Thus, the domain of f' consists of all values of x at which f is differentiable. If x_0 is not in the domain of f or if the limit does not exist, then we say that f is ***not differentiable at x_0***, or that ***the derivative of f does not exist at x_0***. If f is differentiable at every value of x in an open interval (a, b), then we say that f is ***differentiable on (a, b)***. This definition also applies to infinite open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the case where f is differentiable on $(-\infty, +\infty)$ we will say that f is ***differentiable everywhere***.

Geometrically, if f is differentiable at a value x_0 for x, then the graph of f has a tangent line at x_0. If f is defined at x_0 but is not differentiable at x_0, then either the graph of f has no well-defined tangent line at x_0 or it has a vertical tangent line at x_0. Informally, the most commonly encountered circumstances of nondifferentiability occur where the graph of f has

- a corner,
- a vertical tangent line, or
- a discontinuity.

Figure 3.2.11 illustrates each of these situations.

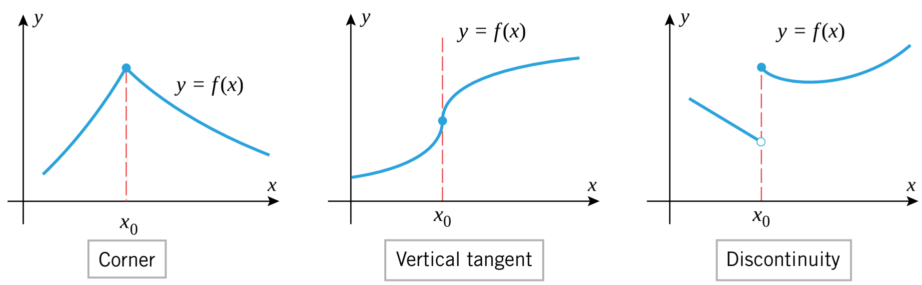


Figure 3.2.11

It makes sense intuitively that a function is not differentiable where its graph has a corner, since there is no reasonable way to define the graph's slope at a corner. For example, Figure 3.2.12*a* shows a typical corner point $P(x_0, f(x_0))$ on the graph of a function f. At this point, the slopes of secant lines joining P and nearby points Q have different limiting values, depending on whether Q is to the left or to the right of P. Hence, the slopes of the secant lines do not have a two-sided limit.

A *vertical tangent line* occurs at a place on a continuous curve where the slopes of secant lines approach $+\infty$ or approach $-\infty$ (Figure 3.2.12*b*). Since an infinite limit is a special way of saying that a limit does not exist, a function f is not differentiable at a point of vertical tangency.

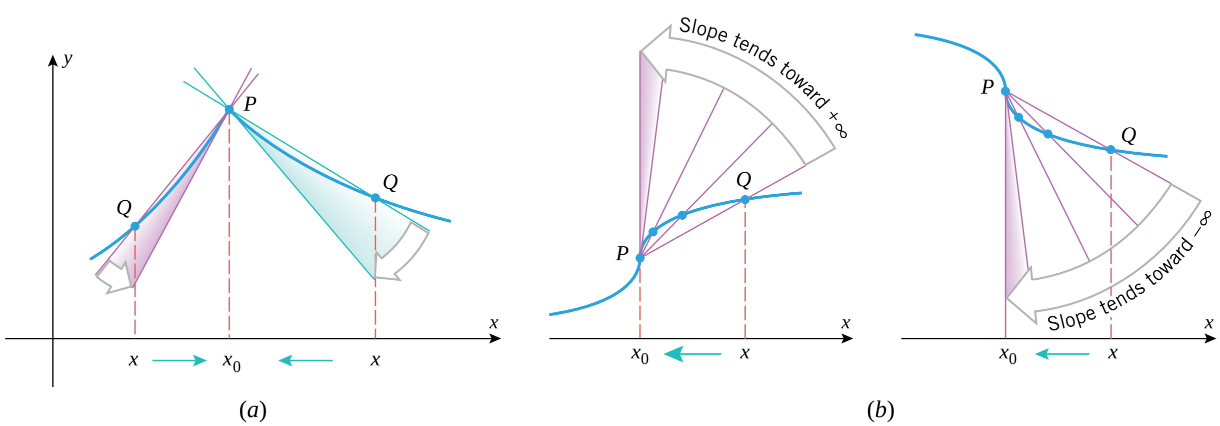


Figure 3.2.12

We will explore the relationship between differentiability and continuity later in this section. It should be noted that there are other, less common, circumstances under which a function may fail to be differentiable. See Exercise 45 for one such example.

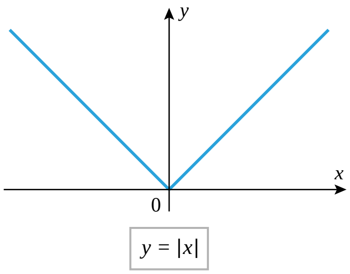


Figure 3.2.13

Example 6 The graph of $y = |x|$ in Figure 3.2.13 has a corner at $x = 0$, which implies that $f(x) = |x|$ is not differentiable at $x = 0$.

(a) Prove that $f(x) = |x|$ is not differentiable at $x = 0$ by showing that the limit in Definition 3.2.3 does not exist at $x = 0$.

(b) Find a formula for $f'(x)$.

Solution (a). From Formula (6) with $x = 0$, the value of $f'(0)$, if it were to exist, would be given by

$$f'(0) = \lim_{w \to 0} \frac{f(w) - f(0)}{w - 0} = \lim_{w \to 0} \frac{|w| - |0|}{w} = \lim_{w \to 0} \frac{|w|}{w}$$

But

$$\frac{|w|}{w} = \begin{cases} 1, & w > 0 \\ -1, & w < 0 \end{cases}$$

so that

$$\lim_{w \to 0^-} \frac{|w|}{w} = -1 \quad \text{and} \quad \lim_{w \to 0^+} \frac{|w|}{w} = 1$$

Thus,

$$f'(0) = \lim_{w \to 0} \frac{|w|}{w}$$

does not exist because the one-sided limits are not equal.

Solution (b). A formula for the derivative of $f(x) = |x|$ can be obtained by writing $|x|$ in piecewise form and treating the cases $x > 0$ and $x < 0$ separately. If $x > 0$, then $f(x) = x$ and $f'(x) = 1$; if $x < 0$, then $f(x) = -x$ and $f'(x) = -1$. Thus,

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

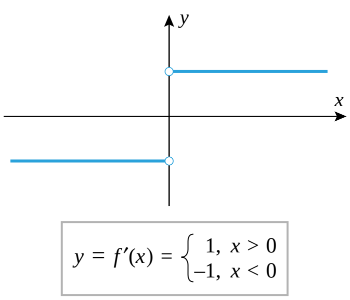


Figure 3.2.14

The graph of f' is shown in Figure 3.2.14. Observe that f' is not continuous at $x = 0$, so this example shows that a function that is continuous everywhere may have a derivative that fails to be continuous everywhere. ◀

DIFFERENTIABILITY AND CONTINUITY

It makes sense intuitively that a function f cannot be differentiable where it has a "jump" discontinuity, since the value of the function changes precipitously at the "jump." The following theorem shows that a function f must be continuous at a value x_0 in order for it to be differentiable there (or stated another way, a function f cannot be differentiable where it is not continuous).

3.2.4 THEOREM. *If f is differentiable at $x = x_0$, then f must also be continuous at x_0.*

Proof. We are given that f is differentiable at x_0, so it follows from (6) that $f'(x_0)$ exists and is given by

$$f'(x_0) = \lim_{w \to x_0} \frac{f(w) - f(x_0)}{w - x_0} \tag{7}$$

To show that f is continuous at x_0, we must show that

$$\lim_{w \to x_0} f(w) = f(x_0)$$

or equivalently,

$$\lim_{w \to x_0} [f(w) - f(x_0)] = 0$$

However, this can be proved using (7) as follows:

$$\begin{aligned}\lim_{w\to x_0}[f(w)-f(x_0)] &= \lim_{w\to x_0}\left[\frac{f(w)-f(x_0)}{w-x_0}\cdot(w-x_0)\right]\\ &= \lim_{w\to x_0}\left[\frac{f(w)-f(x_0)}{w-x_0}\right]\cdot\lim_{w\to x_0}(w-x_0)\\ &= f'(x_0)\cdot 0 = 0\end{aligned}$$

REMARK. The converse to Theorem 3.2.4 is false. That is, *a function may be continuous at an input value, but not differentiable there.* For example, the function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$ (see Example 6). In fact, any function whose graph has a corner and is continuous at the location of the corner will be continuous but not differentiable at the corner.

The relationship between continuity and differentiability was of great historical significance in the development of calculus. In the early nineteenth century mathematicians believed that if a continous function had many points of nondifferentiability, these points, like the tips of a sawblade, would have to be separated from each other and joined by smooth curve segments (Figure 3.2.15). This misconception was shattered by a series of discoveries beginning in 1834. In that year a Bohemian priest, philosopher, and mathematician named Bernhard Bolzano* discovered a procedure for constructing a continuous function that is not differentiable at any point. Later, in 1860, the great German mathematician, Karl Weierstrass produced the first formula for such a function. The graphs of such functions are impossible to draw; it is as if the corners are so numerous that any segment of the curve, when suitably enlarged, reveals more corners. The discovery of these pathological functions was important in that it made mathematicians distrustful of their geometric intuition and more reliant on precise mathematical proof. However, these functions remained only mathematical curiosities until the early 1980s, when applications of them began to emerge. During recent decades, such functions have started to play a fundamental role in the study of geometric objects called ***fractals***. Fractals have revealed an order to natural phenomena that were previously dismissed as random and chaotic.

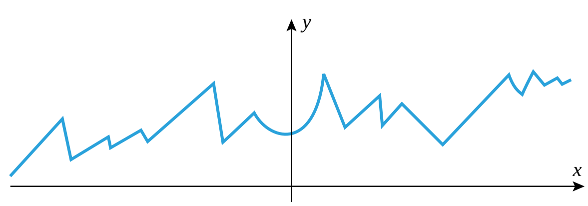


Figure 3.2.15

DERIVATIVE NOTATION

The process of finding a derivative is called ***differentiation***. You can think of differentiation as an operation on functions that associates a function f' with a function f. When the

*BERNHARD BOLZANO (1781–1848). Bolzano, the son of an art dealer, was born in Prague, Bohemia (Czech Republic). He was educated at the University of Prague, and eventually won enough mathematical fame to be recommended for a mathematics chair there. However, Bolzano became an ordained Roman Catholic priest, and in 1805 he was appointed to a chair of Philosophy at the University of Prague. Bolzano was a man of great human compassion; he spoke out for educational reform, he voiced the right of individual conscience over government demands, and he lectured on the absurdity of war and militarism. His views so disenchanted Emperor Franz I of Austria that the emperor pressed the Archbishop of Prague to have Bolzano recant his statements. Bolzano refused and was then forced to retire in 1824 on a small pension. Bolzano's main contribution to mathematics was philosophical. His work helped convince mathematicians that sound mathematics must ultimately rest on rigorous proof rather than intuition. In addition to his work in mathematics, Bolzano investigated problems concerning space, force, and wave propagation.

independent variable is x, the differentiation operation is often denoted by

$$\frac{d}{dx}[f(x)]$$

which is read "***the derivative of f (x) with respect to x***." Thus,

$$\frac{d}{dx}[f(x)] = f'(x) \tag{8}$$

For example, with this notation the derivatives obtained in Examples 2, 3, and 4 can be expressed as

$$\frac{d}{dx}[x^3 - x] = 3x^2 - 1, \quad \frac{d}{dx}[mx + b] = m, \quad \frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} \tag{9}$$

To denote the value of the derivative at a specific value $x = x_0$ with the notation in (8), we would write

$$\frac{d}{dx}[f(x)]\bigg|_{x=x_0} = f'(x_0) \tag{10}$$

For example, from (9)

$$\frac{d}{dx}[x^3 - x]\bigg|_{x=1} = 3(1^2) - 1 = 2, \quad \frac{d}{dx}[mx + b]\bigg|_{x=5} = m, \quad \frac{d}{dx}[\sqrt{x}]\bigg|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Notations (8) and (10) are convenient when no dependent variable is involved. However, if there is a dependent variable, say $y = f(x)$, then (8) and (10) can be written as

$$\frac{d}{dx}[y] = f'(x) \quad \text{and} \quad \frac{d}{dx}[y]\bigg|_{x=x_0} = f'(x_0)$$

It is common to omit the brackets on the left side and write these expressions as

$$\frac{dy}{dx} = f'(x) \quad \text{and} \quad \frac{dy}{dx}\bigg|_{x=x_0} = f'(x_0)$$

where dy/dx is read as "the derivative of y with respect to x." For example, if $y = \sqrt{x}$, then

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad \frac{dy}{dx}\bigg|_{x=x_0} = \frac{1}{2\sqrt{x_0}}, \quad \frac{dy}{dx}\bigg|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

REMARK. Later, the symbols dy and dx will be defined separately. However, for the time being, dy/dx should not be regarded as a ratio; rather, it should be considered as a single symbol denoting the derivative.

When letters other than x and y are used for the independent and dependent variables, then the various notations for the derivative must be adjusted accordingly. For example, if $y = f(u)$, then the derivative with respect to u would be written as

$$\frac{d}{du}[f(u)] = f'(u) \quad \text{and} \quad \frac{dy}{du} = f'(u)$$

In particular, if $y = \sqrt{u}$, then

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}, \quad \frac{dy}{du}\bigg|_{u=u_0} = \frac{1}{2\sqrt{u_0}}, \quad \frac{dy}{du}\bigg|_{u=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

OTHER NOTATIONS

Some writers denote the derivative as $D_x[f(x)] = f'(x)$, but we will not use this notation in this text. In problems where the name of the independent variable is clear from the context, there are some other possible notations for the derivative. For example, if $y = f(x)$, but it is clear from the problem that the independent variable is x, then the derivative with respect to x might be denoted by y' or f'.

Often, you will see Definition 3.2.3 expressed using h or Δx for the difference $w - x$. With $h = w - x$, then $w = x + h$ and $w \to x$ is equivalent to $h \to 0$. Thus, Formula (6) has the form

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{11}$$

Or, using Δx instead of h for $w - x$, Formula (6) has the form

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{12}$$

If $y = f(x)$, then it is also common to let

$$\Delta y = f(w) - f(x) = f(x + \Delta x) - f(x)$$

in which case

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{13}$$

The geometric interpretations of Δx and Δy are shown in Figure 3.2.16.

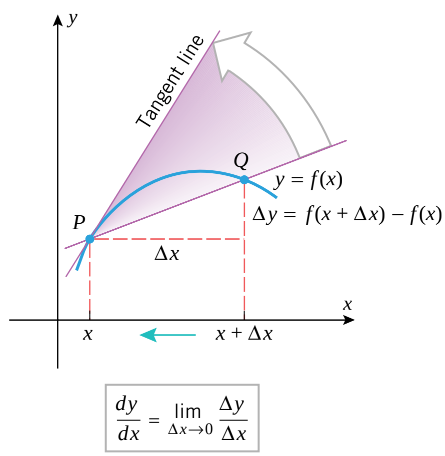


Figure 3.2.16

DERIVATIVES AT THE ENDPOINTS OF AN INTERVAL

If a function f is defined on a closed interval $[a, b]$ and is not defined outside that interval, then the derivative $f'(x)$ is not defined at the endpoints because

$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

is a two-sided limit and only a one-sided limit makes sense at an endpoint. To deal with this situation, we define ***derivatives from the left and right***. These are denoted by f'_- and f'_+, respectively, and are defined by

$$f'_-(x) = \lim_{w \to x^-} \frac{f(w) - f(x)}{w - x} \quad \text{and} \quad f'_+(x) = \lim_{w \to x^+} \frac{f(w) - f(x)}{w - x}$$

At points where $f'_+(x)$ exists we say that the function f is ***differentiable from the right***, and at points where $f'_-(x)$ exists we say that the function f is ***differentiable from the left***. Geometrically, $f'_+(x)$ is the limit of the slopes of the secant lines approaching x from the right, and $f'_-(x)$ is the limit of the slopes of the secant lines approaching x from the left (Figure 3.2.17).

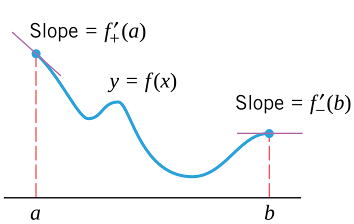


Figure 3.2.17

It can be proved that a function f is continuous from the left at those points where it is differentiable from the left, and f is continuous from the right at those points where it is differentiable from the right.

We say a function f is ***differentiable on an interval*** of the form $[a, b]$, $[a, +\infty)$, $(-\infty, b]$, $[a, b)$, or $(a, b]$ if f is differentiable at all numbers inside the interval, and it is differentiable at the endpoint(s) from the left or right, as appropriate.

EXERCISE SET 3.2 Graphing Calculator

1. Use the graph of $y = f(x)$ in the accompanying figure to estimate the value of $f'(1)$, $f'(3)$, $f'(5)$, and $f'(6)$.

2. For the function graphed in the accompanying figure, arrange the numbers 0, $f'(-3)$, $f'(0)$, $f'(2)$, and $f'(4)$ in increasing order.

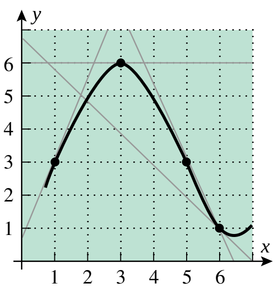


Figure Ex-1

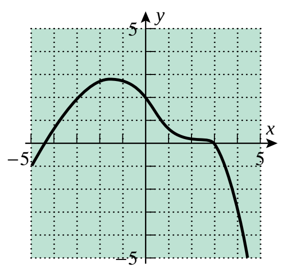


Figure Ex-2

3. (a) If you are given an equation for the tangent line at the point $(a, f(a))$ on a curve $y = f(x)$, how would you go about finding $f'(a)$?
(b) Given that the tangent line to the graph of $y = f(x)$ at the point $(2, 5)$ has the equation $y = 3x - 1$, find $f'(2)$.
(c) For the function $y = f(x)$ in part (b), what is the instantaneous rate of change of y with respect to x at $x = 2$?

4. Given that the tangent line to $y = f(x)$ at the point $(-1, 3)$ passes through the point $(0, 4)$, find $f'(-1)$.

5. Sketch the graph of a function f for which $f(0) = 1$, $f'(0) = 0$, $f'(x) > 0$ if $x < 0$, and $f'(x) < 0$ if $x > 0$.

6. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 0$, and $f'(x) > 0$ if $x < 0$ or $x > 0$.

7. Given that $f(3) = -1$ and $f'(3) = 5$, find an equation for the tangent line to the graph of $y = f(x)$ at $x = 3$.

8. Given that $f(-2) = 3$ and $f'(-2) = -4$, find an equation for the tangent line to the graph of $y = f(x)$ at $x = -2$.

In Exercises 9–14, use Definition 3.2.3 to find $f'(x)$, and then find the equation of the tangent line to $y = f(x)$ at $x = a$.

9. $f(x) = 3x^2$; $a = 3$
10. $f(x) = x^4$; $a = -2$
11. $f(x) = x^3$; $a = 0$
12. $f(x) = 2x^3 + 1$; $a = -1$
13. $f(x) = \sqrt{x+1}$; $a = 8$
14. $f(x) = \sqrt{2x+1}$; $a = 4$

In Exercises 15–20, use Formula (13) to find dy/dx.

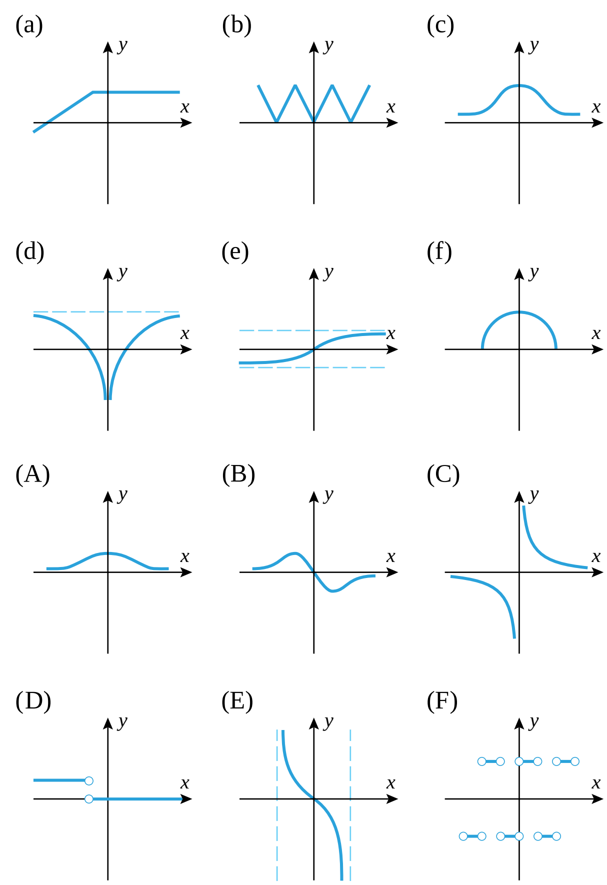
15. $y = \dfrac{1}{x}$
16. $y = \dfrac{1}{x+1}$
17. $y = ax^2 + b$ (a, b constants)
18. $y = x^2 - x$
19. $y = \dfrac{1}{\sqrt{x}}$
20. $y = \dfrac{1}{x^2}$

In Exercises 21 and 22, use Definition 3.2.3 (with appropriate change in notation) to obtain the derivative requested.

21. Find $f'(t)$ if $f(t) = 4t^2 + t$.

22. Find dV/dr if $V = \frac{4}{3}\pi r^3$.

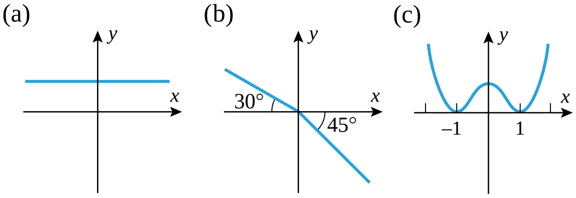
23. Match the graphs of the functions shown in (a)–(f) with the graphs of their derivatives in (A)–(F).



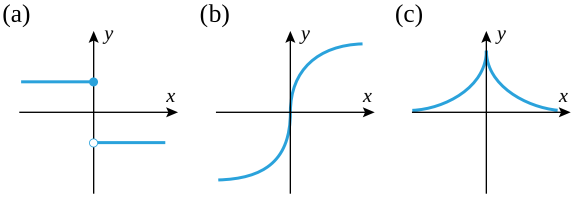
24. Find a function f such that $f'(x) = 1$ for all x, and give an informal argument to justify your answer.

In Exercises 25 and 26, sketch the graph of the derivative of the function whose graph is shown.

25. (a) (b) (c)



26. (a) (b) (c)



In Exercises 27 and 28, the limit represents $f'(a)$ for some function f and some number a. Find $f(x)$ and a in each case.

27. (a) $\lim_{x_1 \to 3} \frac{x_1^2 - 9}{x_1 - 3}$ (b) $\lim_{\Delta x \to 0} \frac{\sqrt{1 + \Delta x} - 1}{\Delta x}$

28. (a) $\lim_{x \to 1} \frac{x^7 - 1}{x - 1}$ (b) $\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$

29. Find $dy/dx|_{x=1}$, given that $y = 4x^2 + 1$.

30. Find $dy/dx|_{x=-2}$, given that $y = (5/x) + 1$.

31. Find an equation for the line that is tangent to the curve $y = x^3 - 2x + 1$ at the point $(0, 1)$, and use a graphing utility to graph the curve and its tangent line on the same screen.

32. Use a graphing utility to graph the following on the same screen: the curve $y = x^2/4$, the tangent line to this curve at $x = 1$, and the secant line joining the points $(0, 0)$ and $(2, 1)$ on this curve.

33. Let $f(x) = 2^x$. Estimate $f'(1)$ by
(a) using a graphing utility to zoom in at an appropriate point until the graph looks like a straight line, and then estimating the slope
(b) using a calculating utility to estimate the limit in Definition 3.2.3 by making a table of values for a succession of values of w approaching 1.

34. Let $f(x) = \sin x$. Estimate $f'(\pi/4)$ by
(a) using a graphing utility to zoom in at an appropriate point until the graph looks like a straight line, and then estimating the slope
(b) using a calculating utility to estimate the limit in Definition 3.2.3 by making a table of values for a succession of values of w approaching $\pi/4$.

35. Suppose that the cost of drilling x feet for an oil well is $C = f(x)$ dollars.
(a) What are the units of $f'(x)$?
(b) In practical terms, what does $f'(x)$ mean in this case?
(c) What can you say about the sign of $f'(x)$?
(d) Estimate the cost of drilling an additional foot, starting at a depth of 300 ft, given that $f'(300) = 1000$.

36. A paint manufacturing company estimates that it can sell $g = f(p)$ gallons of paint at a price of p dollars.
(a) What are the units of dg/dp?
(b) In practical terms, what does dg/dp mean in this case?
(c) What can you say about the sign of dg/dp?
(d) Given that $dg/dp|_{p=10} = -100$, what can you say about the effect of increasing the price from \$10 per gallon to \$11 per gallon?

37. It is a fact that when a flexible rope is wrapped around a rough cylinder, a small force of magnitude F_0 at one end can resist a large force of magnitude F at the other end. The size of F depends on the angle θ through which the rope is wrapped around the cylinder (see the accompanying figure). That figure shows the graph of F (in pounds) versus θ (in radians), where F is the magnitude of the force that can be resisted by a force with magnitude $F_0 = 10$ lb for a certain rope and cylinder.
(a) Estimate the values of F and $dF/d\theta$ when the angle $\theta = 10$ radians.
(b) It can be shown that the force F satisfies the equation $dF/d\theta = \mu F$, where the constant μ is called the ***coefficient of friction***. Use the results in part (a) to estimate the value of μ.

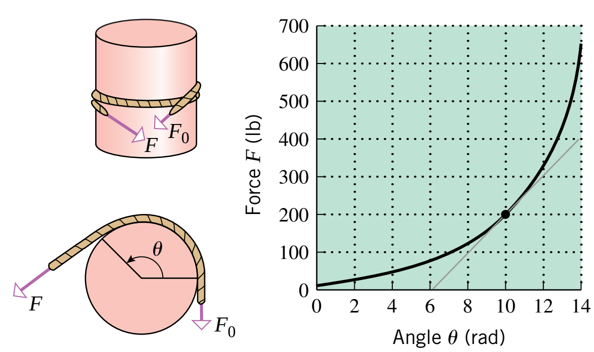


Figure Ex-37

38. According to the U. S. Bureau of the Census, the estimated and projected midyear world population, N, in billions for the years 1950, 1975, 2000, 2025, and 2050 was 2.555, 4.088, 6.080, 7.841, and 9.104, respectively. Although the increase in population is not a continuous function of the time t, we can apply the ideas in this section if we are willing to approximate the graph of N versus t by a continuous curve, as shown in the accompanying figure.
(a) Use the tangent line at $t = 2000$ shown in the figure to approximate the value of dN/dt there. Interpret your result as a rate of change.
(b) The instantaneous ***growth rate*** is defined as

$$\frac{dN/dt}{N}$$

Use your answer to part (a) to approximate the instantaneous growth rate at the start of the year 2000. Express the result as a percentage and include the proper units.

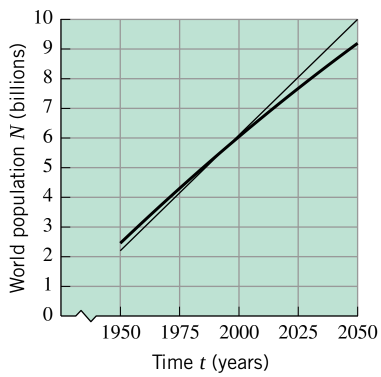


Figure Ex-38

39. According to ***Newton's Law of Cooling***, the rate of change of an object's temperature is proportional to the difference between the temperature of the object and that of the surrounding medium. The accompanying figure shows the graph of the temperature T (in degrees Fahrenheit) versus time t (in minutes) for a cup of coffee, initially with a temperature of 200°F, that is allowed to cool in a room with a constant temperature of 75°F.

(a) Estimate T and dT/dt when $t = 10$ min.

(b) Newton's Law of Cooling can be expressed as

$$\frac{dT}{dt} = k(T - T_0)$$

where k is the constant of proportionality and T_0 is the temperature (assumed constant) of the surrounding medium. Use the results in part (a) to estimate the value of k.

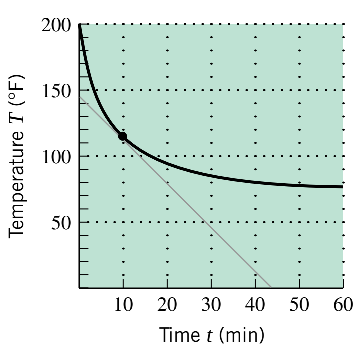


Figure Ex-39

40. Write a paragraph that explains what it means for a function to be differentiable. Include some examples of functions that are not differentiable, and explain the relationship between differentiability and continuity.

41. Show that $f(x) = \sqrt[3]{x}$ is continuous at $x = 0$ but not differentiable at $x = 0$. Sketch the graph of f.

42. Show that $f(x) = \sqrt[3]{(x-2)^2}$ is continuous at $x = 2$ but not differentiable at $x = 2$. Sketch the graph of f.

43. Show that

$$f(x) = \begin{cases} x^2 + 1, & x \le 1 \\ 2x, & x > 1 \end{cases}$$

is continuous and differentiable at $x = 1$. Sketch the graph of f.

44. Show that

$$f(x) = \begin{cases} x^2 + 2, & x \le 1 \\ x + 2, & x > 1 \end{cases}$$

is continuous but not differentiable at $x = 1$. Sketch the graph of f.

45. Show that

$$f(x) = \begin{cases} x \sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is continuous but not differentiable at $x = 0$. Sketch the graph of f near $x = 0$. (See Figure 2.6.7*b* and the remark following Example 3 in Section 2.6.)

46. Show that

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is continuous and differentiable at $x = 0$. Sketch the graph of f near $x = 0$.

47. Suppose that a function f is differentiable at $x = 1$ and

$$\lim_{h \to 0} \frac{f(1+h)}{h} = 5$$

Find $f(1)$ and $f'(1)$.

48. Suppose that f is a differentiable function with the property that

$$f(x+y) = f(x) + f(y) + 5xy \quad \text{and} \quad \lim_{h \to 0} \frac{f(h)}{h} = 3$$

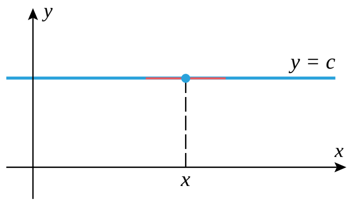
Find $f(0)$ and $f'(x)$.

49. Suppose that f has the property $f(x+y) = f(x)f(y)$ for all values of x and y and that $f(0) = f'(0) = 1$. Show that f is differentiable and $f'(x) = f(x)$. [*Hint:* Start by expressing $f'(x)$ as a limit.]

3.3 TECHNIQUES OF DIFFERENTIATION

In the last section we defined the derivative of a function f as a limit, and we used that limit to calculate a few simple derivatives. In this section we will develop some important theorems that will enable us to calculate derivatives more efficiently.

DERIVATIVE OF A CONSTANT



The tangent line to the graph of $f(x) = c$ has slope 0 for all x.

Figure 3.3.1

The graph of a constant function $f(x) = c$ is the horizontal line $y = c$, and hence the tangent line to this graph has slope 0 at every value of x (Figure 3.3.1). Thus, we should expect the derivative of a constant function to be 0 for all x.

3.3.1 THEOREM. *The derivative of a constant function is* 0; *that is, if c is any real number, then*

$$\frac{d}{dx}[c] = 0$$

Proof. Let $f(x) = c$. Then from the definition of a derivative,

$$\frac{d}{dx}[c] = f'(x) = \lim_{w\to x}\frac{f(w)-f(x)}{w-x} = \lim_{w\to x}\frac{c-c}{w-x} = \lim_{w\to x} 0 = 0$$

Example 1 If $f(x) = 5$ for all x, then $f'(x) = 0$ for all x; that is,

$$\frac{d}{dx}[5] = 0$$

For our next derivative rule, we will need the algebraic identity

$$w^n - x^n = (w-x)(w^{n-1} + w^{n-2}x + w^{n-3}x^2 + \cdots + wx^{n-2} + x^{n-1})$$

which is valid for any positive integer n. This identity may be verified by expanding the right-hand side of the equation and noting the cancellation of terms. For example, with $n = 4$ we have

$$\begin{aligned}(w-x)(w^3 + w^2x + wx^2 + x^3) &= w^4 + (w^3x - xw^3) + (w^2x^2 - xw^2x) \\ &\quad + (wx^3 - xwx^2) - x^4 \\ &= w^4 + 0 + 0 + 0 - x^4 \\ &= w^4 - x^4\end{aligned}$$

3.3.2 THEOREM (*The Power Rule*). *If n is a positive integer, then*

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Proof. Let $f(x) = x^n$. Then from the definition of the derivative we obtain

$$\begin{aligned}\frac{d}{dx}[x^n] = f'(x) &= \lim_{w\to x}\frac{f(w)-f(x)}{w-x} = \lim_{w\to x}\frac{w^n - x^n}{w-x} \\ &= \lim_{w\to x}\frac{(w-x)(w^{n-1} + w^{n-2}x + w^{n-3}x^2 + \cdots + wx^{n-2} + x^{n-1})}{w-x} \\ &= \lim_{w\to x} w^{n-1} + w^{n-2}x + w^{n-3}x^2 + \cdots + wx^{n-2} + x^{n-1} \\ &= x^{n-1} + x^{n-1} + \cdots + x^{n-1} \qquad \text{n terms in all} \\ &= nx^{n-1}\end{aligned}$$

In words, *the derivative of x raised to a positive integer power is the product of the integer exponent and x raised to the next lower integer power.*

Example 2

$$\frac{d}{dx}[x^5] = 5x^4, \quad \frac{d}{dx}[x] = 1\cdot x^0 = 1, \quad \frac{d}{dx}[x^{12}] = 12x^{11}$$

DERIVATIVE OF A CONSTANT TIMES A FUNCTION

3.3.3 THEOREM. *If f is differentiable at x and c is any real number, then cf is also differentiable at x and*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Proof.

$$\frac{d}{dx}[cf(x)] = \lim_{w\to x}\frac{cf(w)-cf(x)}{w-x} = \lim_{w\to x} c\left[\frac{f(w)-f(x)}{w-x}\right]$$

$$= c\lim_{w\to x}\frac{f(w)-f(x)}{w-x} = c\frac{d}{dx}[f(x)]$$

A constant factor can be moved through a limit sign.

In function notation, Theorem 3.3.3 states

$$(cf)' = cf'$$

In words, *a constant factor can be moved through a derivative sign.*

Example 3

$$\frac{d}{dx}[4x^8] = 4\frac{d}{dx}[x^8] = 4[8x^7] = 32x^7$$

$$\frac{d}{dx}[-x^{12}] = (-1)\frac{d}{dx}[x^{12}] = -12x^{11}$$

$$\frac{d}{dx}\left[\frac{x}{\pi}\right] = \frac{1}{\pi}\frac{d}{dx}[x] = \frac{1}{\pi}$$

DERIVATIVES OF SUMS AND DIFFERENCES

3.3.4 THEOREM. *If f and g are differentiable at x, then so are $f+g$ and $f-g$ and*

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Proof.

$$\frac{d}{dx}[f(x)+g(x)] = \lim_{w\to x}\frac{[f(w)+g(w)]-[f(x)+g(x)]}{w-x}$$

$$= \lim_{w\to x}\frac{[f(w)-f(x)]+[g(w)-g(x)]}{w-x}$$

$$= \lim_{w\to x}\frac{f(w)-f(x)}{w-x} + \lim_{w\to x}\frac{g(w)-g(x)}{w-x}$$

The limit of a sum is the sum of the limits.

$$= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

The proof for $f-g$ is similar.

In function notation, Theorem 3.3.4 states

$$(f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

In words, *the derivative of a sum equals the sum of the derivatives*, and *the derivative of a difference equals the difference of the derivatives.*

Example 4

$$\frac{d}{dx}[x^4 + x^2] = \frac{d}{dx}[x^4] + \frac{d}{dx}[x^2] = 4x^3 + 2x$$

$$\frac{d}{dx}[6x^{11} - 9] = \frac{d}{dx}[6x^{11}] - \frac{d}{dx}[9] = 66x^{10} - 0 = 66x^{10}$$ ◀

Although Theorem 3.3.4 was stated for sums and differences of two terms, it can be extended to any mixture of finitely many sums and differences of differentiable functions. For example,

$$\begin{aligned}\frac{d}{dx}[3x^8 - 2x^5 + 6x + 1] &= \frac{d}{dx}[3x^8] - \frac{d}{dx}[2x^5] + \frac{d}{dx}[6x] + \frac{d}{dx}[1]\\ &= 24x^7 - 10x^4 + 6\end{aligned}$$

DERIVATIVE OF A PRODUCT

3.3.5 THEOREM (*The Product Rule*). *If f and g are differentiable at x, then so is the product $f \cdot g$, and*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Proof. The earlier proofs in this section were straightforward applications of the definition of the derivative. However, this proof requires a trick—adding and subtracting the quantity $f(w)g(x)$ to the numerator in the derivative definition as follows:

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= \lim_{w\to x}\frac{f(w)\cdot g(w) - f(x)\cdot g(x)}{w-x}\\ &= \lim_{w\to x}\frac{f(w)g(w) - f(w)g(x) + f(w)g(x) - f(x)g(x)}{w-x}\\ &= \lim_{w\to x}\left[f(w)\cdot\frac{g(w)-g(x)}{w-x} + g(x)\cdot\frac{f(w)-f(x)}{w-x}\right]\\ &= \lim_{w\to x} f(w)\cdot\lim_{w\to x}\frac{g(w)-g(x)}{w-x} + \lim_{w\to x} g(x)\cdot\lim_{w\to x}\frac{f(w)-f(x)}{w-x}\\ &= [\lim_{w\to x} f(w)]\frac{d}{dx}[g(x)] + [\lim_{w\to x} g(x)]\frac{d}{dx}[f(x)]\\ &= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]\end{aligned}$$

[*Note:* In the last step $f(w)\to f(x)$ as $w\to x$ because f is continuous at x by Theorem 3.2.4, and $g(x)\to g(x)$ as $w\to x$ because $g(x)$ does not involve w and hence remains constant.] ▮

The product rule can be written in function notation as

$$(f\cdot g)' = f\cdot g' + g\cdot f'$$

In words, *the derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.*

WARNING. Note that in general $(f \cdot g)' \neq f' \cdot g'$; that is, the derivative of a product is *not* generally the product of the derivatives!

Example 5 Find dy/dx if $y = (4x^2 - 1)(7x^3 + x)$.

Solution. There are two methods that can be used to find dy/dx. We can either use the product rule or we can multiply out the factors in y and then differentiate. We will give both methods.

Method I. (*Using the Product Rule*)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[(4x^2 - 1)(7x^3 + x)] \\ &= (4x^2 - 1)\frac{d}{dx}[7x^3 + x] + (7x^3 + x)\frac{d}{dx}[4x^2 - 1] \\ &= (4x^2 - 1)(21x^2 + 1) + (7x^3 + x)(8x) = 140x^4 - 9x^2 - 1\end{aligned}$$

Method II. (*Multiplying First*)

$$y = (4x^2 - 1)(7x^3 + x) = 28x^5 - 3x^3 - x$$

Thus,

$$\frac{dy}{dx} = \frac{d}{dx}[28x^5 - 3x^3 - x] = 140x^4 - 9x^2 - 1$$

which agrees with the result obtained using the product rule. ◀

DERIVATIVE OF A QUOTIENT

3.3.6 THEOREM (*The Quotient Rule*). *If f and g are differentiable at x and $g(x) \neq 0$, then f/g is differentiable at x and*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Proof.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{w \to x} \frac{\frac{f(w)}{g(w)} - \frac{f(x)}{g(x)}}{w - x} = \lim_{w \to x} \frac{f(w) \cdot g(x) - f(x) \cdot g(w)}{(w - x) \cdot g(x) \cdot g(w)}$$

Adding and subtracting $f(x) \cdot g(x)$ in the numerator yields

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{w \to x} \frac{f(w) \cdot g(x) - f(x) \cdot g(x) - f(x) \cdot g(w) + f(x) \cdot g(x)}{(w - x) \cdot g(x) \cdot g(w)} \\ &= \lim_{w \to x} \frac{\left[g(x) \cdot \frac{f(w) - f(x)}{w - x}\right] - \left[f(x) \cdot \frac{g(w) - g(x)}{w - x}\right]}{g(x) \cdot g(w)} \\ &= \frac{\lim\limits_{w \to x} g(x) \cdot \lim\limits_{w \to x} \frac{f(w) - f(x)}{w - x} - \lim\limits_{w \to x} f(x) \cdot \lim\limits_{w \to x} \frac{g(w) - g(x)}{w - x}}{\lim\limits_{w \to x} g(x) \cdot \lim\limits_{w \to x} g(w)}\end{aligned}$$

$$= \frac{[\lim_{w\to x} g(x)] \cdot \frac{d}{dx}[f(x)] - [\lim_{w\to x} f(x)] \cdot \frac{d}{dx}[g(x)]}{\lim_{w\to x} g(x) \cdot \lim_{w\to x} g(w)}$$

$$= \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

[See the note at the end of the proof of Theorem 3.3.5 for an explanation of the last step.]

The quotient rule can be written in function notation as

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$$

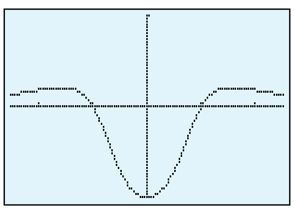
In words, *the derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared*.

WARNING. Note that in general $(f/g)' \neq f'/g'$; that is, the derivative of a quotient is *not* generally the quotient of the derivatives.

Example 6 Let $f(x) = \dfrac{x^2 - 1}{x^4 + 1}$.

(a) Graph $y = f(x)$, and use your graph to make rough estimates of the locations of all horizontal tangent lines.

(b) By differentiating, find the exact locations of the horizontal tangent lines.



[−2.5, 2.5] × [−1, 1]
xScl = 1, yScl = 1

$y = \dfrac{x^2 - 1}{x^4 + 1}$

Figure 3.3.2

Solution (a). In Figure 3.3.2 we have shown the graph of the equation $y = f(x)$ in the window $[-2.5, 2.5] \times [-1, 1]$. This graph suggests that horizontal tangent lines occur at $x = 0$, $x \approx 1.5$, and $x \approx -1.5$.

Solution (b). To find the exact locations of the horizontal tangent lines, we must find the points where $dy/dx = 0$ (why?). We start by finding dy/dx:

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2 - 1}{x^4 + 1}\right] = \frac{(x^4 + 1)\frac{d}{dx}[x^2 - 1] - (x^2 - 1)\frac{d}{dx}[x^4 + 1]}{(x^4 + 1)^2}$$

$$= \frac{(x^4 + 1)(2x) - (x^2 - 1)(4x^3)}{(x^4 + 1)^2}$$

The differentiation is complete. The rest is simplification.

$$= \frac{-2x^5 + 4x^3 + 2x}{(x^4 + 1)^2} = -\frac{2x(x^4 - 2x^2 - 1)}{(x^4 + 1)^2}$$

Now we will set $dy/dx = 0$ and solve for x. We obtain

$$-\frac{2x(x^4 - 2x^2 - 1)}{(x^4 + 1)^2} = 0$$

The solutions of this equation are the values of x for which the numerator is 0:

$$2x(x^4 - 2x^2 - 1) = 0$$

The first factor yields the solution $x = 0$. Other solutions can be found by solving the equation

$$x^4 - 2x^2 - 1 = 0$$

This can be treated as a quadratic equation in x^2 and solved by the quadratic formula. This yields

$$x^2 = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

The minus sign yields imaginary values of x, which we ignore since they are not relevant to the problem. The plus sign yields the solutions

$$x = \pm\sqrt{1 + \sqrt{2}}$$

In summary, horizontal tangent lines occur at

$$x = 0, \quad x = \sqrt{1 + \sqrt{2}} \approx 1.55, \quad \text{and} \quad x = -\sqrt{1 + \sqrt{2}} \approx -1.55$$

which is consistent with the rough estimates that we obtained graphically in part (a). ◀

THE POWER RULE FOR INTEGER EXPONENTS

In Theorem 3.3.2 we established the formula

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

for *positive* integer values of n. Eventually, we will show that this formula applies if n is any real number. As our first step in this direction we will show that it applies for *all integer* values of n.

3.3.7 THEOREM. *If n is any integer, then*

$$\frac{d}{dx}[x^n] = nx^{n-1} \tag{1}$$

Proof. The result has already been established in the case where $n > 0$. If $n < 0$, then let $m = -n$ so that

$$f(x) = x^{-m} = \frac{1}{x^m}$$

From Theorem 3.3.6,

$$f'(x) = \frac{d}{dx}\left[\frac{1}{x^m}\right] = \frac{x^m \frac{d}{dx}[1] - 1\frac{d}{dx}[x^m]}{(x^m)^2} = -\frac{\frac{d}{dx}[x^m]}{(x^m)^2}$$

Since $n < 0$, it follows that $m > 0$, so x^m can be differentiated using Theorem 3.3.2. Thus,

$$f'(x) = -\frac{mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = -mx^{-m-1} = nx^{n-1}$$

which proves (1). In the case $n = 0$ Formula (1) reduces to

$$\frac{d}{dx}[1] = 0 \cdot x^{-1} = 0$$

which is correct by Theorem 3.3.1. ■

Example 7

$$\frac{d}{dx}[x^{-9}] = -9x^{-9-1} = -9x^{-10}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$ ◀

In Example 4 of Section 3.2 we showed that

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} \tag{2}$$

which shows that Formula (1) also works with $n = \frac{1}{2}$, since

$$\frac{d}{dx}[x^{1/2}] = \frac{1}{2x^{1/2}} = \frac{1}{2}x^{-1/2}$$

HIGHER DERIVATIVES

If the derivative f' of a function f is itself differentiable, then the derivative of f' is denoted by f'' and is called the ***second derivative*** of f. As long as we have differentiability, we can continue the process of differentiating derivatives to obtain third, fourth, fifth, and even higher derivatives of f. The successive derivatives of f are denoted by

$$f', \quad f'' = (f')', \quad f''' = (f'')', \quad f^{(4)} = (f''')', \quad f^{(5)} = (f^{(4)})', \ldots$$

These are called the first derivative, the second derivative, the third derivative, and so forth. Beyond the third derivative, it is too clumsy to continue using primes, so we switch from primes to integers in parentheses to denote the ***order*** of the derivative. In this notation it is easy to denote a derivative of arbitrary order by writing

$f^{(n)}$ The nth derivative of f

The significance of the derivatives of order 2 and higher will be discussed later.

Example 8 If $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$, then

$$\begin{aligned}
f'(x) &= 12x^3 - 6x^2 + 2x - 4\\
f''(x) &= 36x^2 - 12x + 2\\
f'''(x) &= 72x - 12\\
f^{(4)}(x) &= 72\\
f^{(5)}(x) &= 0\\
&\vdots\\
f^{(n)}(x) &= 0 \quad (n \geq 5)
\end{aligned}$$

◀

Successive derivatives can also be denoted as follows:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[f(x)]\\
f''(x) &= \frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = \frac{d^2}{dx^2}[f(x)]\\
f'''(x) &= \frac{d}{dx}\left[\frac{d^2}{dx^2}[f(x)]\right] = \frac{d^3}{dx^3}[f(x)]\\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

In general, we write

$$f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$$

which is read "the nth derivative of f with respect to x."

When a dependent variable is involved, say $y = f(x)$, then successive derivatives can be denoted by writing

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}, \ldots$$

or more briefly,

$$y', \quad y'', \quad y''', \quad y^{(4)}, \ldots, y^{(n)}, \ldots$$

EXERCISE SET 3.3 Graphing Calculator

In Exercises 1–12, find dy/dx.

1. $y = 4x^7$
2. $y = -3x^{12}$
3. $y = 3x^8 + 2x + 1$
4. $y = \frac{1}{2}(x^4 + 7)$
5. $y = \pi^3$
6. $y = \sqrt{2}x + (1/\sqrt{2})$
7. $y = -\frac{1}{3}(x^7 + 2x - 9)$
8. $y = \dfrac{x^2+1}{5}$
9. $y = ax^3 + bx^2 + cx + d$ (a, b, c, d constant)
10. $y = \dfrac{1}{a}\left(x^2 + \dfrac{1}{b}x + c\right)$ (a, b, c constant)
11. $y = -3x^{-8} + 2\sqrt{x}$
12. $y = 7x^{-6} - 5\sqrt{x}$

In Exercises 13–20, find $f'(x)$.

13. $f(x) = x^{-3} + \dfrac{1}{x^7}$
14. $f(x) = \sqrt{x} + \dfrac{1}{x}$
15. $f(x) = (3x^2 + 6)\left(2x - \frac{1}{4}\right)$
16. $f(x) = (2 - x - 3x^3)(7 + x^5)$
17. $f(x) = (x^3 + 7x^2 - 8)(2x^{-3} + x^{-4})$
18. $f(x) = \left(\dfrac{1}{x} + \dfrac{1}{x^2}\right)(3x^3 + 27)$
19. $f(x) = (3x^2 + 1)^2$
20. $f(x) = (x^5 + 2x)^2$

In Exercises 21 and 22, find $y'(1)$.

21. $y = \dfrac{1}{5x - 3}$
22. $y = \dfrac{3}{\sqrt{x} + 2}$

In Exercises 23 and 24, find dx/dt.

23. $x = \dfrac{3t}{2t + 1}$
24. $x = \dfrac{t^2 + 1}{3t}$

In Exercises 25–28, find $dy/dx|_{x=1}$.

25. $y = \dfrac{2x - 1}{x + 3}$
26. $y = \dfrac{4x + 1}{x^2 - 5}$
27. $y = \left(\dfrac{3x + 2}{x}\right)(x^{-5} + 1)$
28. $y = (2x^7 - x^2)\left(\dfrac{x - 1}{x + 1}\right)$

In Exercises 29 and 30, approximate $f'(1)$ by considering difference quotients

$$\frac{f(x_1) - f(1)}{x_1 - 1}$$

for values of x_1 near 1, and then find the exact value of $f'(1)$ by differentiating.

29. $f(x) = x^3 - 3x + 1$
30. $f(x) = x\sqrt{x}$

In Exercises 31 and 32, use a graphing utility to estimate the value of $f'(1)$ by zooming in on the graph of f, and then compare your estimate to the exact value obtained by differentiating.

31. $f(x) = \dfrac{x}{x^2 + 1}$
32. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

In Exercises 33–36, find the indicated derivative.

33. $\dfrac{d}{dt}[16t^2]$
34. $\dfrac{dC}{dr}$, where $C = 2\pi r$
35. $V'(r)$, where $V = \pi r^3$
36. $\dfrac{d}{d\alpha}[2\alpha^{-1} + \alpha]$

37. A spherical balloon is being inflated.
    (a) Find a general formula for the instantaneous rate of change of the volume V with respect to the radius r, given that $V = \frac{4}{3}\pi r^3$.
    (b) Find the rate of change of V with respect to r at the instant when the radius is $r = 5$.
38. Find $\dfrac{d}{d\lambda}\left[\dfrac{\lambda\lambda_0 + \lambda^6}{2 - \lambda_0}\right]$ (λ_0 is constant).
39. Find $g'(4)$ given that $f(4) = 3$ and $f'(4) = -5$.
    (a) $g(x) = \sqrt{x}f(x)$ (b) $g(x) = \dfrac{f(x)}{x}$
40. Find $g'(3)$ given that $f(3) = -2$ and $f'(3) = 4$.
    (a) $g(x) = 3x^2 - 5f(x)$ (b) $g(x) = \dfrac{2x + 1}{f(x)}$
41. Find $F'(2)$ given that $f(2) = -1$, $f'(2) = 4$, $g(2) = 1$, and $g'(2) = -5$.
    (a) $F(x) = 5f(x) + 2g(x)$ (b) $F(x) = f(x) - 3g(x)$
    (c) $F(x) = f(x)g(x)$ (d) $F(x) = f(x)/g(x)$
42. Find $F'(\pi)$ given that $f(\pi) = 10$, $f'(\pi) = -1$, $g(\pi) = -3$, and $g'(\pi) = 2$.
    (a) $F(x) = 6f(x) - 5g(x)$ (b) $F(x) = x(f(x) + g(x))$
    (c) $F(x) = 2f(x)g(x)$ (d) $F(x) = \dfrac{f(x)}{4 + g(x)}$
43. Find an equation of the tangent line to the graph of $y = f(x)$ at $x = -3$ if $f(-3) = 2$ and $f'(-3) = 5$.
44. Find an equation for the line that is tangent to the curve $y = (1 - x)/(1 + x)$ at $x = 2$.

In Exercises 45 and 46, find d^2y/dx^2.

45. (a) $y = 7x^3 - 5x^2 + x$ (b) $y = 12x^2 - 2x + 3$
    (c) $y = \dfrac{x + 1}{x}$ (d) $y = (5x^2 - 3)(7x^3 + x)$
46. (a) $y = 4x^7 - 5x^3 + 2x$ (b) $y = 3x + 2$
    (c) $y = \dfrac{3x - 2}{5x}$ (d) $y = (x^3 - 5)(2x + 3)$

In Exercises 47 and 48, find y'''.

47. (a) $y = x^{-5} + x^5$ (b) $y = 1/x$
(c) $y = ax^3 + bx + c$ (a, b, c constant)

48. (a) $y = 5x^2 - 4x + 7$ (b) $y = 3x^{-2} + 4x^{-1} + x$
(c) $y = ax^4 + bx^2 + c$ (a, b, c constant)

49. Find
(a) $f'''(2)$, where $f(x) = 3x^2 - 2$
(b) $\left.\dfrac{d^2y}{dx^2}\right|_{x=1}$, where $y = 6x^5 - 4x^2$
(c) $\left.\dfrac{d^4}{dx^4}[x^{-3}]\right|_{x=1}$

50. Find
(a) $y'''(0)$, where $y = 4x^4 + 2x^3 + 3$
(b) $\left.\dfrac{d^4y}{dx^4}\right|_{x=1}$, where $y = \dfrac{6}{x^4}$.

51. Show that $y = x^3 + 3x + 1$ satisfies $y''' + xy'' - 2y' = 0$.

52. Show that if $x \neq 0$, then $y = 1/x$ satisfies the equation $x^3y'' + x^2y' - xy = 0$.

53. Find a general formula for $F''(x)$ if $F(x) = xf(x)$ and f and f' are differentiable at x.

54. Suppose that the function f is differentiable everywhere and $F(x) = xf(x)$.
(a) Express $F'''(x)$ in terms of x and derivatives of f.
(b) For $n \geq 2$, conjecture a formula for $F^{m}(x)$.

> In Exercises 55 and 56, use a graphing utility to make rough estimates of the locations of all horizontal tangent lines, and then find their exact locations by differentiating.

55. $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$ **56.** $y = \dfrac{x}{x^2 + 9}$

57. Find a function $y = ax^2 + bx + c$ whose graph has an x-intercept of 1, a y-intercept of -2, and a tangent line with a slope of -1 at the y-intercept.

58. Find k if the curve $y = x^2 + k$ is tangent to the line $y = 2x$.

59. Find the x-coordinate of the point on the graph of $y = x^2$ where the tangent line is parallel to the secant line that cuts the curve at $x = -1$ and $x = 2$.

60. Find the x-coordinate of the point on the graph of $y = \sqrt{x}$ where the tangent line is parallel to the secant line that cuts the curve at $x = 1$ and $x = 4$.

61. Find the coordinates of all points on the graph of $y = 1 - x^2$ at which the tangent line passes through the point $(2, 0)$.

62. Show that any two tangent lines to the parabola $y = ax^2$, $a \neq 0$, intersect at a point that is on the vertical line halfway between the points of tangency.

63. Suppose that L is the tangent line at $x = x_0$ to the graph of the cubic equation $y = ax^3 + bx$. Find the x-coordinate of the point where L intersects the graph a second time.

64. Show that the segment of the tangent line to the graph of $y = 1/x$ that is cut off by the coordinate axes is bisected by the point of tangency.

65. Show that the triangle that is formed by any tangent line to the graph of $y = 1/x$, $x > 0$, and the coordinate axes has an area of 2 square units.

66. Find conditions on a, b, c, and d so that the graph of the polynomial $f(x) = ax^3 + bx^2 + cx + d$ has
(a) exactly two horizontal tangents
(b) exactly one horizontal tangent
(c) no horizontal tangents.

67. Newton's Law of Universal Gravitation states that the magnitude F of the force exerted by a point with mass M on a point with mass m is

$$F = \frac{GmM}{r^2}$$

where G is a constant and r is the distance between the bodies. Assuming that the points are moving, find a formula for the instantaneous rate of change of F with respect to r.

68. In the temperature range between 0°C and 700°C the resistance R [in ohms (Ω)] of a certain platinum resistance thermometer is given by

$$R = 10 + 0.04124T - 1.779 \times 10^{-5}T^2$$

where T is the temperature in degrees Celsius. Where in the interval from 0°C to 700°C is the resistance of the thermometer most sensitive and least sensitive to temperature changes? [*Hint:* Consider the size of dR/dT in the interval $0 \leq T \leq 700$.]

> In Exercises 69 and 70, use a graphing utility to make rough estimates of the intervals on which $f'(x) > 0$, and then find those intervals exactly by differentiating.

69. $f(x) = x - \dfrac{1}{x}$ **70.** $f(x) = \dfrac{5x}{x^2 + 4}$

71. Apply the product rule (3.3.5) twice to show that if f, g, and h are differentiable functions, then $f \cdot g \cdot h$ is differentiable, and

$$(f \cdot g \cdot h)' = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$$

72. Based on the result in Exercise 71, make a conjecture about a formula for differentiating a product of n functions.

73. Use the formula in Exercise 71 to find
(a) $\dfrac{d}{dx}\left[(2x + 1)\left(1 + \dfrac{1}{x}\right)(x^{-3} + 7)\right]$
(b) $\dfrac{d}{dx}\left[(x^7 + 2x - 3)^3\right]$.

74. Use the formula you obtained in Exercise 72 to find
(a) $\dfrac{d}{dx}\left[x^{-5}(x^2 + 2x)(4 - 3x)(2x^9 + 1)\right]$
(b) $\dfrac{d}{dx}\left[(x^2 + 1)^{50}\right]$.

In Exercises 75–78, you are asked to determine whether a piecewise-defined function f is differentiable at a value $x = x_0$, where f is defined by different formulas on different sides of x_0. You may use the following result, which is a consequence of the Mean-Value Theorem (discussed in Section 4.8). ***Theorem***. *Let f be continuous at x_0 and suppose that* $\lim_{x \to x_0} f'(x)$ *exists. Then f is differentiable at x_0, and* $f'(x_0) = \lim_{x \to x_0} f'(x)$.

75. Show that

$$f(x) = \begin{cases} x^2 + x + 1, & x \le 1 \\ 3x, & x > 1 \end{cases}$$

is continuous at $x = 1$. Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there. Sketch the graph of f.

76. Let

$$f(x) = \begin{cases} x^2 - 16x, & x < 9 \\ 12\sqrt{x}, & x \ge 9 \end{cases}$$

Is f continuous at $x = 9$? Determine whether f is differentiable at $x = 9$. If so, find the value of the derivative there.

77. Let

$$f(x) = \begin{cases} x^2, & x \le 1 \\ \sqrt{x}, & x > 1 \end{cases}$$

Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there.

78. Let

$$f(x) = \begin{cases} x^3 + \frac{1}{16}, & x < \frac{1}{2} \\ \frac{3}{4}x^2, & x \ge \frac{1}{2} \end{cases}$$

Determine whether f is differentiable at $x = \frac{1}{2}$. If so, find the value of the derivative there.

79. Find all points where f fails to be differentiable. Justify your answer.
(a) $f(x) = |3x - 2|$ (b) $f(x) = |x^2 - 4|$

80. In each part compute f', f'', f''' and then state the formula for $f^{(n)}$.
(a) $f(x) = 1/x$ (b) $f(x) = 1/x^2$
[*Hint:* The expression $(-1)^n$ has a value of 1 if n is even and -1 if n is odd. Use this expression in your answer.]

81. (a) Prove:

$$\frac{d^2}{dx^2}[cf(x)] = c\frac{d^2}{dx^2}[f(x)]$$

$$\frac{d^2}{dx^2}[f(x) + g(x)] = \frac{d^2}{dx^2}[f(x)] + \frac{d^2}{dx^2}[g(x)]$$

(b) Do the results in part (a) generalize to nth derivatives? Justify your answer.

82. Prove:

$$(f \cdot g)'' = f'' \cdot g + 2f' \cdot g' + f \cdot g''$$

83. (a) Find $f^{(n)}(x)$ if $f(x) = x^n$.
(b) Find $f^{(n)}(x)$ if $f(x) = x^k$ and $n > k$, where k is a positive integer.
(c) Find $f^{(n)}(x)$ if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

84. Let $f(x) = x^8 - 2x + 3$; find

$$\lim_{w \to 2} \frac{f'(w) - f'(2)}{w - 2}$$

85. (a) Prove: If $f''(x)$ exists for each x in (a, b), then both f and f' are continuous on (a, b).
(b) What can be said about the continuity of f and its derivatives if $f^{(n)}(x)$ exists for each x in (a, b)?

3.4 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

The main objective of this section is to obtain formulas for the derivatives of trigonometric functions.

DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

For the purpose of finding derivatives of the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$, we will assume that x is measured in radians. We will also need the following limits, which were stated in Theorem 2.6.3 (with x rather than h as the variable):

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{1 - \cos h}{h} = 0$$

We begin with the problem of differentiating $\sin x$. Using the alternative form

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

for the definition of a derivative (Formula (11) of Section 3.2), we have

$$\begin{aligned}\frac{d}{dx}[\sin x] &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\ &= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right]\\ &= \lim_{h\to 0}\left[\cos x\left(\frac{\sin h}{h}\right)-\sin x\left(\frac{1-\cos h}{h}\right)\right]\end{aligned}$$

By the addition formula for sine

Since $\sin x$ and $\cos x$ do not involve h, they remain constant as $h\to 0$; thus,

$$\lim_{h\to 0}(\sin x)=\sin x \quad\text{and}\quad \lim_{h\to 0}(\cos x)=\cos x$$

Consequently,

$$\begin{aligned}\frac{d}{dx}[\sin x] &= \cos x\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right)-\sin x\cdot\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)\\ &= \cos x\cdot(1)-\sin x\cdot(0)=\cos x\end{aligned}$$

Thus, we have shown that

$$\frac{d}{dx}[\sin x]=\cos x \tag{1}$$

The derivative of $\cos x$ can be obtained similarly, resulting in the formula

$$\frac{d}{dx}[\cos x]=-\sin x \tag{2}$$

The derivatives of the remaining trigonometric functions are

$$\frac{d}{dx}[\tan x]=\sec^2 x \qquad \frac{d}{dx}[\sec x]=\sec x\tan x \tag{3–4}$$

$$\frac{d}{dx}[\cot x]=-\csc^2 x \qquad \frac{d}{dx}[\csc x]=-\csc x\cot x \tag{5–6}$$

These can all be obtained from (1) and (2) using the relationships

$$\tan x=\frac{\sin x}{\cos x},\qquad \cot x=\frac{\cos x}{\sin x},\qquad \sec x=\frac{1}{\cos x},\qquad \csc x=\frac{1}{\sin x}$$

For example,

$$\begin{aligned}\frac{d}{dx}[\tan x] &= \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]=\frac{\cos x\cdot\frac{d}{dx}[\sin x]-\sin x\cdot\frac{d}{dx}[\cos x]}{\cos^2 x}\\ &= \frac{\cos x\cdot\cos x-\sin x\cdot(-\sin x)}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x\end{aligned}$$

REMARK. The derivative formulas for the trigonometric functions should be memorized. An easy way of doing this is discussed in Exercise 42. Moreover, we emphasize again that in all of the derivative formulas for the trigonometric functions, x is measured in radians.

Example 1 Find $f'(x)$ if $f(x)=x^2\tan x$.

Solution. Using the product rule and Formula (3), we obtain

$$f'(x) = x^2 \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[x^2] = x^2 \sec^2 x + 2x \tan x$$ ◀

Example 2 Find dy/dx if $y = \dfrac{\sin x}{1 + \cos x}$.

Solution. Using the quotient rule together with Formulas (1) and (2) we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\cos x)\cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[1+\cos x]}{(1+\cos x)^2}\\ &= \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2}\\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}\end{aligned}$$ ◀

Example 3 Find $y''(\pi/4)$ if $y(x) = \sec x$.

Solution.

$$\begin{aligned} y'(x) &= \sec x \tan x\\ y''(x) &= \sec x \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[\sec x]\\ &= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x\\ &= \sec^3 x + \sec x \tan^2 x\end{aligned}$$

Thus,

$$\begin{aligned} y''(\pi/4) &= \sec^3(\pi/4) + \sec(\pi/4)\tan^2(\pi/4)\\ &= (\sqrt{2})^3 + (\sqrt{2})(1)^2 = 3\sqrt{2}\end{aligned}$$ ◀

Example 4 On a sunny day, a 50-ft flagpole casts a shadow that changes with the angle of elevation of the Sun. Let s be the length of the shadow and θ the angle of elevation of the Sun (Figure 3.4.1). Find the rate at which the length of the shadow is changing with respect to θ when $\theta = 45^\circ$. Express your answer in units of feet/degree.

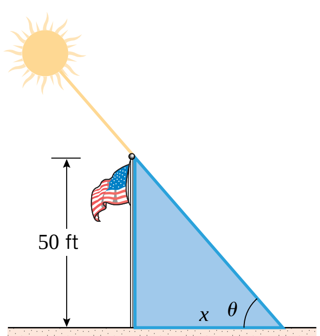


Figure 3.4.1

Solution. The variables s and θ are related by $\tan\theta = 50/s$, or equivalently,

$$s = 50 \cot\theta \tag{7}$$

If θ is measured in radians, then Formula (5) is applicable, which yields

$$\frac{ds}{d\theta} = -50 \csc^2\theta$$

which is the rate of change of shadow length with respect to the elevation angle θ in units of feet/radian. When $\theta = 45^\circ$ (or equivalently, $\theta = \pi/4$ radians), we obtain

$$\left.\frac{ds}{d\theta}\right|_{\theta=\pi/4} = -50\csc^2(\pi/4) = -100 \text{ feet/radian}$$

Converting radians (rad) to degrees (deg) yields

$$-100\frac{\text{ft}}{\text{rad}} \cdot \frac{\pi}{180}\frac{\text{rad}}{\text{deg}} = -\frac{5}{9}\pi \text{ ft/deg} \approx -1.75 \text{ ft/deg}$$

Thus, when $\theta = 45^\circ$, the shadow length is decreasing (because of the minus sign) at an approximate rate of 1.75 ft/deg increase in the angle of elevation. ◀

EXERCISE SET 3.4 Graphing Calculator

In Exercises 1–18, find $f'(x)$.

1. $f(x) = 2\cos x - 3\sin x$

2. $f(x) = \sin x \cos x$

3. $f(x) = \dfrac{\sin x}{x}$

4. $f(x) = x^2 \cos x$

5. $f(x) = x^3 \sin x - 5\cos x$

6. $f(x) = \dfrac{\cos x}{x \sin x}$

7. $f(x) = \sec x - \sqrt{2}\tan x$

8. $f(x) = (x^2 + 1)\sec x$

9. $f(x) = \sec x \tan x$

10. $f(x) = \dfrac{\sec x}{1 + \tan x}$

11. $f(x) = \csc x \cot x$

12. $f(x) = x - 4\csc x + 2\cot x$

13. $f(x) = \dfrac{\cot x}{1 + \csc x}$

14. $f(x) = \dfrac{\csc x}{\tan x}$

15. $f(x) = \sin^2 x + \cos^2 x$

16. $f(x) = \dfrac{1}{\cot x}$

17. $f(x) = \dfrac{\sin x \sec x}{1 + x\tan x}$

18. $f(x) = \dfrac{(x^2 + 1)\cot x}{3 - \cos x \csc x}$

In Exercises 19–24, find d^2y/dx^2.

19. $y = x\cos x$

20. $y = \csc x$

21. $y = x\sin x - 3\cos x$

22. $y = x^2\cos x + 4\sin x$

23. $y = \sin x \cos x$

24. $y = \tan x$

25. Find the equation of the line tangent to the graph of $\tan x$ at
(a) $x = 0$ (b) $x = \pi/4$ (c) $x = -\pi/4$.

26. Find the equation of the line tangent to the graph of $\sin x$ at
(a) $x = 0$ (b) $x = \pi$ (c) $x = \pi/4$.

27. (a) Show that $y = x\sin x$ is a solution to $y'' + y = 2\cos x$.
(b) Show that $y = x\sin x$ is a solution of the equation $y^{(4)} + y'' = -2\cos x$.

28. (a) Show that $y = \cos x$ and $y = \sin x$ are solutions of the equation $y'' + y = 0$.
(b) Show that $y = A\sin x + B\cos x$ is a solution of the equation $y'' + y = 0$ for all constants A and B.

29. Find all values in the interval $[-2\pi, 2\pi]$ at which the graph of f has a horizontal tangent line.
(a) $f(x) = \sin x$ (b) $f(x) = x + \cos x$
(c) $f(x) = \tan x$ (d) $f(x) = \sec x$

30. (a) Use a graphing utility to make rough estimates of the values in the interval $[0, 2\pi]$ at which the graph of $y = \sin x \cos x$ has a horizontal tangent line.
(b) Find the exact locations of the points where the graph has a horizontal tangent line.

31. A 10-ft ladder leans against a wall at an angle θ with the horizontal, as shown in the accompanying figure. The top of the ladder is x feet above the ground. If the bottom of the ladder is pushed toward the wall, find the rate at which x changes with respect to θ when $\theta = 60°$. Express the answer in units of feet/degree.

32. An airplane is flying on a horizontal path at a height of 3800 ft, as shown in the accompanying figure. At what rate is the distance s between the airplane and the fixed point P changing with respect to θ when $\theta = 30°$? Express the answer in units of feet/degree.

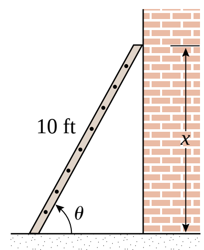


Figure Ex-31

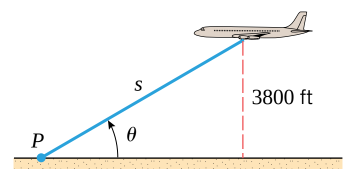


Figure Ex-32

33. A searchlight is trained on the side of a tall building. As the light rotates, the spot it illuminates moves up and down the side of the building. That is, the distance D between ground level and the illuminated spot on the side of the building is a function of the angle θ formed by the light beam and the horizontal (see the accompanying figure). If the searchlight is located 50 m from the building, find the rate at which D is changing with respect to θ when $\theta = 45°$. Express your answer in units of meters/degree.

34. An Earth-observing satellite can see only a portion of the Earth's surface. The satellite has horizon sensors that can detect the angle θ shown in the accompanying figure. Let r be the radius of the Earth (assumed spherical) and h the distance of the satellite from the Earth's surface.
(a) Show that $h = r(\csc\theta - 1)$.
(b) Using $r = 6378$ km, find the rate at which h is changing with respect to θ when $\theta = 30°$. Express the answer in units of kilometers/degree. [Adapted from *Space Mathematics*, NASA, 1985.]

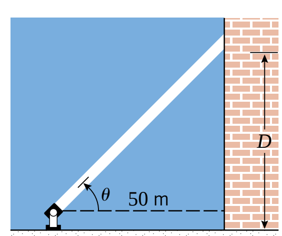


Figure Ex-33

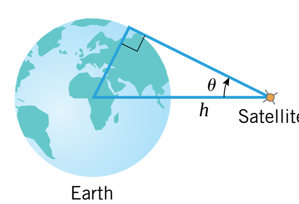


Figure Ex-34

In Exercises 35 and 36, make a conjecture about the derivative by calculating the first few derivatives and observing the resulting pattern.

35. (a) $\dfrac{d^{87}}{dx^{87}}[\sin x]$ (b) $\dfrac{d^{100}}{dx^{100}}[\cos x]$

36. $\dfrac{d^{17}}{dx^{17}}[x \sin x]$

37. In each part, determine where f is differentiable.
(a) $f(x) = \sin x$ (b) $f(x) = \cos x$
(c) $f(x) = \tan x$ (d) $f(x) = \cot x$
(e) $f(x) = \sec x$ (f) $f(x) = \csc x$
(g) $f(x) = \dfrac{1}{1 + \cos x}$ (h) $f(x) = \dfrac{1}{\sin x \cos x}$
(i) $f(x) = \dfrac{\cos x}{2 - \sin x}$

38. (a) Derive Formula (2) using the definition of a derivative.
(b) Use Formulas (1) and (2) to obtain (5).
(c) Use Formula (2) to obtain (4).
(d) Use Formula (1) to obtain (6).

39. Let $f(x) = \cos x$. Find all positive integers n for which $f^{(n)}(x) = \sin x$.

40. (a) Show that $\lim_{h \to 0} \dfrac{\tan h}{h} = 1$.
(b) Use the result in part (a) to help derive the formula for the derivative of $\tan x$ directly from the definition of a derivative.

41. Without using any trigonometric identities, find

$$\lim_{x \to 0} \frac{\tan(x + y) - \tan y}{x}$$

[*Hint:* Relate the given limit to the definition of the derivative of an appropriate function of y.]

42. Let us agree to call the functions $\cos x$, $\cot x$, and $\csc x$ the *cofunctions* of $\sin x$, $\tan x$, and $\sec x$, respectively. Convince yourself that the derivative of any cofunction can be obtained from the derivative of the corresponding function by introducing a minus sign and replacing each function in the derivative by its cofunction. Memorize the derivatives of $\sin x$, $\tan x$, and $\sec x$ and then use the above observation to deduce the derivatives of the cofunctions.

43. The derivative formulas for $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ were obtained under the assumption that x is measured in radians. This exercise shows that different (more complicated) formulas result if x is measured in degrees. Prove that if h and x are degree measures, then
(a) $\lim_{h \to 0} \dfrac{\cos h - 1}{h} = 0$ (b) $\lim_{h \to 0} \dfrac{\sin h}{h} = \dfrac{\pi}{180}$
(c) $\dfrac{d}{dx}[\sin x] = \dfrac{\pi}{180} \cos x$.

3.5 THE CHAIN RULE

In this section we will derive a formula that expresses the derivative of a composition $f \circ g$ in terms of the derivatives of f and g. This formula will enable us to differentiate complicated functions using known derivatives of simpler functions.

DERIVATIVES OF COMPOSITIONS

3.5.1 PROBLEM. *If we know the derivatives of f and g, how can we use this information to find the derivative of the composition $f \circ g$?*

The key to solving this problem is to introduce dependent variables

$$y = (f \circ g)(x) = f(g(x)) \quad \text{and} \quad u = g(x)$$

so that $y = f(u)$. We are interested in using the known derivatives

$$\frac{dy}{du} = f'(u) \quad \text{and} \quad \frac{du}{dx} = g'(x)$$

to find the unknown derivative

$$\frac{dy}{dx} = \frac{d}{dx}[f(g(x))]$$

Stated another way, we are interested in using the known rates of change dy/du and du/dx to find the unknown rate of change dy/dx. But intuition suggests that rates of change multiply. For example, if y changes at 4 times the rate of change of u and u changes at 2 times the rate of change of x, then y changes at $4 \times 2 = 8$ times the rate of change of x. This suggests that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

These ideas are formalized in the following theorem.

3.5.2 THEOREM (*The Chain Rule*). *If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover,*

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Alternatively, if

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

The proof of this result is given in Appendix G.

Example 1 Find $h'(x)$ if $h(x) = 4\cos(x^3)$.

Solution. We first find functions f and g such that $f \circ g = h$. Observe that if $g(x) = x^3$ and $f(u) = 4\cos u$, then

$$(f \circ g)(x) = f(g(x)) = 4\cos(g(x)) = 4\cos(x^3) = h(x)$$

Also,

$$f'(u) = -4\sin u \quad \text{and} \quad g'(x) = 3x^2$$

Using the chain rule,

$$h'(x) = f'(g(x))\,g'(x) = (-4\sin g(x))(3x^2) = -12x^2\sin(x^3)$$

Alternatively, set $y = h(x)$ and let $u = x^3$. Then $y = 4\cos u$. By the form of the chain rule in Formula (1),

$$\begin{aligned} h'(x) = \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}[4\cos u] \cdot \frac{d}{dx}[x^3] \\ &= (-4\sin u) \cdot (3x^2) = (-4\sin(x^3)) \cdot (3x^2) = -12x^2\sin(x^3) \end{aligned}$$ ◀

Formula (1) is easy to remember because the left side is exactly what results if we "cancel" the du's on the right side. This "canceling" device provides a good way to remember the chain rule when variables other than x, y, and u are used.

Example 2 Find dw/dt if $w = \tan x$ and $x = 4t^3 + t$.

Solution. In this case the chain rule takes the form

$$\begin{aligned} \frac{dw}{dt} &= \frac{dw}{dx} \cdot \frac{dx}{dt} = \frac{d}{dx}[\tan x] \cdot \frac{d}{dt}[4t^3 + t] \\ &= (\sec^2 x)(12t^2 + 1) = (12t^2 + 1)\sec^2(4t^3 + t) \end{aligned}$$ ◀

AN ALTERNATIVE APPROACH TO USING THE CHAIN RULE

Although Formula (1) is useful, it is sometimes unwieldy because it involves so many variables. As you become more comfortable with the chain rule, you may want to dispense with actually writing out all these variables. To accomplish this, it is helpful to note that since $(f \circ g)(x) = f(g(x))$, the chain rule may be written in the form

$$\frac{d}{dx}[f(g(x))] = (f \circ g)'(x) = f'(g(x))g'(x)$$

If we call $g(x)$ the "inside function" and f the "outside function," then this equation states that:

The derivative of $f(g(x))$ is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.

That is,

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{\substack{\text{Derivative of}\\ \text{the outside}\\ \text{evaluated at}\\ \text{the inside}}} \cdot \underbrace{g'(x)}_{\substack{\text{Derivative}\\ \text{of the inside}}} \tag{2}$$

For example,

$$\frac{d}{dx}[\cos(x^2+9)] = \underbrace{-\sin(x^2+9)}_{\substack{\text{Derivative of the}\\ \text{outside evaluated}\\ \text{at the inside}}} \cdot \underbrace{2x}_{\substack{\text{Derivative}\\ \text{of the inside}}}$$

$$\frac{d}{dx}[\tan^2 x] = \frac{d}{dx}\left[(\tan x)^2\right] = \underbrace{(2\tan x)}_{\substack{\text{Derivative of}\\ \text{the outside}\\ \text{evaluated at}\\ \text{the inside}}} \cdot \underbrace{(\sec^2 x)}_{\substack{\text{Derivative}\\ \text{of the inside}}} = 2\tan x \sec^2 x$$

Substituting $u = g(x)$ into (2) yields the following alternative form:

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx} \tag{3}$$

For example, to differentiate the function

$$f(x) = \left(x^2 - x + 1\right)^{23} \tag{4}$$

we can let $u = x^2 - x + 1$ and then apply (3) to obtain

$$\frac{d}{dx}\left[(x^2 - x + 1)^{23}\right] = \frac{d}{dx}[u^{23}] = \underbrace{23u^{22}}_{f'(u)}\frac{du}{dx}$$

$$= 23\left(x^2 - x + 1\right)^{22}\frac{d}{dx}[x^2 - x + 1]$$

$$= 23\left(x^2 - x + 1\right)^{22} \cdot (2x - 1)$$

More generally, if u were any other differentiable function of x, the pattern of computations would be virtually the same. For example, if $u = \cos x$, then

$$\frac{d}{dx}[\cos^{23} x] = \frac{d}{dx}[u^{23}] = 23u^{22}\frac{du}{dx} = 23\cos^{22} x \frac{d}{dx}[\cos x]$$

$$= 23\cos^{22} x \cdot (-\sin x) = -23 \sin x \cos^{22} x$$

In both of the preceding computations, the chain rule took the form

$$\frac{d}{dx}[u^{23}] = 23u^{22}\frac{du}{dx} \tag{5}$$

This formula is a generalization of the more basic formula

$$\frac{d}{dx}[x^{23}] = 23x^{22} \tag{6}$$

In fact, in the special case where $u = x$, Formula (5) reduces to (6) since

$$\frac{d}{dx}[u^{23}] = 23u^{22}\frac{du}{dx} = 23x^{22}\frac{d[x]}{dx} = 23x^{22}$$

Table 3.5.1 contains a list of ***generalized derivative formulas*** that are consequences of (3).

Table 3.5.1

GENERALIZED DERIVATIVE FORMULAS	
$\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$ (n an integer)	$\frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx}$
$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}$	$\frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$
$\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}$	$\frac{d}{dx}[\cot u] = -\csc^2 u\frac{du}{dx}$
$\frac{d}{dx}[\sec u] = \sec u \tan u\frac{du}{dx}$	$\frac{d}{dx}[\csc u] = -\csc u \cot u\frac{du}{dx}$

Example 3 Find

(a) $\frac{d}{dx}[\sin(2x)]$ (b) $\frac{d}{dx}[\tan(x^2+1)]$ (c) $\frac{d}{dx}[\sqrt{x^3+\csc x}]$

(d) $\frac{d}{dx}\left[(1+x^5\cot x)^{-8}\right]$ (e) $\frac{d}{dx}\left[\frac{1}{x^3+2x-3}\right]$

Solution (a). Taking $u = 2x$ in the generalized derivative formula for $\sin u$ yields

$$\frac{d}{dx}[\sin(2x)] = \frac{d}{dx}[\sin u] = \cos u\frac{du}{dx} = \cos 2x \cdot \frac{d}{dx}[2x] = \cos 2x \cdot 2 = 2\cos 2x$$

Solution (b). Taking $u = x^2+1$ in the generalized derivative formula for $\tan u$ yields

$$\begin{aligned}\frac{d}{dx}[\tan(x^2+1)] &= \frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}\\ &= \sec^2(x^2+1)\cdot\frac{d}{dx}[x^2+1] = \sec^2(x^2+1)\cdot 2x\\ &= 2x\sec^2(x^2+1)\end{aligned}$$

Solution (c). Taking $u = x^3 + \csc x$ in the generalized derivative formula for $\sqrt{u}$ yields

$$\begin{aligned}\frac{d}{dx}[\sqrt{x^3+\csc x}] = \frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx} &= \frac{1}{2\sqrt{x^3+\csc x}}\cdot\frac{d}{dx}[x^3+\csc x]\\ &= \frac{1}{2\sqrt{x^3+\csc x}}\cdot(3x^2-\csc x\cot x) = \frac{3x^2-\csc x\cot x}{2\sqrt{x^3+\csc x}}\end{aligned}$$

Solution (d). Taking $u = 1+x^5\cot x$ in the generalized derivative formula for u^{-8} yields

$$\begin{aligned}\frac{d}{dx}\left[(1+x^5\cot x)^{-8}\right] &= \frac{d}{dx}[u^{-8}] = -8u^{-9}\frac{du}{dx}\\ &= -8\left(1+x^5\cot x\right)^{-9}\cdot\frac{d}{dx}[1+x^5\cot x]\\ &= -8\left(1+x^5\cot x\right)^{-9}\cdot(x^5(-\csc^2 x)+5x^4\cot x)\\ &= (8x^5\csc^2 x - 40x^4\cot x)\left(1+x^5\cot x\right)^{-9}\end{aligned}$$

Solution (e). Taking $u = x^3+2x-3$ in the generalized derivative formula for u^{-1} yields

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{x^3+2x-3}\right] &= \frac{d}{dx}[(x^3+2x-3)^{-1}] = \frac{d}{dx}[u^{-1}]\\ &= -u^{-2}\frac{du}{dx} = -(x^3+2x-3)^{-2}\frac{d}{dx}[x^3+2x-3]\\ &= -(x^3+2x-3)^{-2}(3x^2+2) = -\frac{3x^2+2}{(x^3+2x-3)^2}\end{aligned}$$

◀

Sometimes you will have to make adjustments in notation or apply the chain rule more than once to calculate a derivative.

Example 4 Find

(a) $\dfrac{d}{dx}[\sin(\sqrt{1+\cos x})]$ (b) $\dfrac{d\mu}{dt}$ if $\mu = \sec\sqrt{\omega t}$ (ω constant)

Solution (a). Taking $u = \sqrt{1+\cos x}$ in the generalized derivative formula for $\sin u$ yields

$$\frac{d}{dx}[\sin(\sqrt{1+\cos x})] = \frac{d}{dx}[\sin u] = \cos u \frac{du}{dx}$$

$$= \cos(\sqrt{1+\cos x}) \cdot \frac{d}{dx}[\sqrt{1+\cos x}]$$

We use the generalized derivative formula for $\sqrt{u}$ with $u = 1+\cos x$.

$$= \cos(\sqrt{1+\cos x}) \cdot \frac{-\sin x}{2\sqrt{1+\cos x}}$$

$$= -\frac{\sin x \cos(\sqrt{1+\cos x})}{2\sqrt{1+\cos x}}$$

Solution (b).

$$\frac{d\mu}{dt} = \frac{d}{dt}[\sec\sqrt{\omega t}] = \sec\sqrt{\omega t}\tan\sqrt{\omega t}\frac{d}{dt}[\sqrt{\omega t}]$$

We used the generalized derivative formula for $\sec u$ with $u = \sqrt{\omega t}$.

$$= \sec\sqrt{\omega t}\tan\sqrt{\omega t}\frac{\omega}{2\sqrt{\omega t}}$$

We used the generalized derivative formula for $\sqrt{u}$ with $u = \omega t$. ◀

DIFFERENTIATING USING COMPUTER ALGEBRA SYSTEMS

Although the chain rule makes it possible to differentiate extremely complicated functions, the computations can be time-consuming to execute by hand. For complicated derivatives engineers and scientists often use computer algebra systems such as *Mathematica*, *Maple*, and *Derive*. For example, although we have all of the mathematical tools to perform the differentiation

$$\frac{d}{dx}\left[\frac{(x^2+1)^{10}\sin^3(\sqrt{x})}{\sqrt{1+\csc x}}\right] \tag{7}$$

by hand, the computations are sufficiently tedious that it would be more efficient to use a computer algebra system.

FOR THE READER. If you have a CAS, use it to perform the differentiation in (7).

EXERCISE SET 3.5 Graphing Calculator C CAS

1. Given that $f'(0) = 2$, $g(0) = 0$, and $g'(0) = 3$, find $(f\circ g)'(0)$.
2. Given that $f'(9) = 5$, $g(2) = 9$, and $g'(2) = -3$, find $(f\circ g)'(2)$.
3. Let $f(x) = x^5$ and $g(x) = 2x - 3$.
   (a) Find $(f\circ g)(x)$ and $(f\circ g)'(x)$.
   (b) Find $(g\circ f)(x)$ and $(g\circ f)'(x)$.
4. Let $f(x) = 5\sqrt{x}$ and $g(x) = 4 + \cos x$.
   (a) Find $(f\circ g)(x)$ and $(f\circ g)'(x)$.
   (b) Find $(g\circ f)(x)$ and $(g\circ f)'(x)$.
5. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
3	5	–2	5	7
5	3	–1	12	4

   (a) $F'(3)$, where $F(x) = f(g(x))$
   (b) $G'(3)$, where $G(x) = g(f(x))$

6. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
–1	2	3	2	–3
2	0	4	1	–5

(a) $F'(-1)$, where $F(x) = f(g(x))$
(b) $G'(-1)$, where $G(x) = g(f(x))$

In Exercises 7–26, find $f'(x)$.

7. $f(x) = (x^3 + 2x)^{37}$
8. $f(x) = (3x^2 + 2x - 1)^6$
9. $f(x) = \left(x^3 - \dfrac{7}{x}\right)^{-2}$
10. $f(x) = \dfrac{1}{(x^5 - x + 1)^9}$
11. $f(x) = \dfrac{4}{(3x^2 - 2x + 1)^3}$
12. $f(x) = \sqrt{x^3 - 2x + 5}$
13. $f(x) = \sqrt{4 + \sqrt{3x}}$
14. $f(x) = \sin^3 x$
15. $f(x) = \sin(x^3)$
16. $f(x) = \cos^2(3\sqrt{x})$
17. $f(x) = 4\cos^5 x$
18. $f(x) = \csc(x^3)$
19. $f(x) = \sin\left(\dfrac{1}{x^2}\right)$
20. $f(x) = \tan^4(x^3)$
21. $f(x) = 2\sec^2(x^7)$
22. $f(x) = \cos^3\left(\dfrac{x}{x+1}\right)$
23. $f(x) = \sqrt{\cos(5x)}$
24. $f(x) = \sqrt{3x - \sin^2(4x)}$
25. $f(x) = [x + \csc(x^3 + 3)]^{-3}$
26. $f(x) = [x^4 - \sec(4x^2 - 2)]^{-4}$

In Exercises 27–40, find dy/dx.

27. $y = x^3 \sin^2(5x)$
28. $y = \sqrt{x}\tan^3(\sqrt{x})$
29. $y = x^5 \sec(1/x)$
30. $y = \dfrac{\sin x}{\sec(3x + 1)}$
31. $y = \cos(\cos x)$
32. $y = \sin(\tan 3x)$
33. $y = \cos^3(\sin 2x)$
34. $y = \dfrac{1 + \csc(x^2)}{1 - \cot(x^2)}$
35. $y = (5x + 8)^{13}(x^3 + 7x)^{12}$
36. $y = (2x - 5)^2(x^2 + 4)^3$
37. $y = \left(\dfrac{x - 5}{2x + 1}\right)^3$
38. $y = \left(\dfrac{1 + x^2}{1 - x^2}\right)^{17}$
39. $y = \dfrac{(2x + 3)^3}{(4x^2 - 1)^8}$
40. $y = [1 + \sin^3(x^5)]^{12}$

In Exercises 41 and 42, use a CAS to find dy/dx.

C **41.** $y = [x\sin 2x + \tan^4(x^7)]^5$

C **42.** $y = \tan^4\left(2 + \dfrac{(7 - x)\sqrt{3x^2 + 5}}{x^3 + \sin x}\right)$

In Exercises 43–50, find an equation for the tangent line to the graph at the specified value of x.

43. $y = x\cos 3x, \ x = \pi$
44. $y = \sin(1 + x^3), \ x = -3$
45. $y = \sec^3\left(\dfrac{\pi}{2} - x\right), \ x = -\dfrac{\pi}{2}$
46. $y = \left(x - \dfrac{1}{x}\right)^3, \ x = 2$
47. $y = \tan(4x^2), \ x = \sqrt{\pi}$
48. $y = 3\cot^4 x, \ x = \dfrac{\pi}{4}$
49. $y = x^2\sqrt{5 - x^2}, \ x = 1$
50. $y = \dfrac{x}{\sqrt{1 - x^2}}, \ x = 0$

In Exercises 51–54, find d^2y/dx^2.

51. $y = x\cos(5x) - \sin^2 x$
52. $y = \sin(3x^2)$
53. $y = \dfrac{1 + x}{1 - x}$
54. $y = x\tan\left(\dfrac{1}{x}\right)$

In Exercises 55–58, find the indicated derivative.

55. $y = \cot^3(\pi - \theta)$; find $\dfrac{dy}{d\theta}$.

56. $\lambda = \left(\dfrac{au + b}{cu + d}\right)^6$; find $\dfrac{d\lambda}{du}$ $(a, b, c, d$ constants).

57. $\dfrac{d}{d\omega}[a\cos^2 \pi\omega + b\sin^2 \pi\omega]$ $(a, b$ constants).

58. $x = \csc^2\left(\dfrac{\pi}{3} - y\right)$; find $\dfrac{dx}{dy}$.

59. (a) Use a graphing utility to obtain the graph of the function $f(x) = x\sqrt{4 - x^2}$.
(b) Use the graph in part (a) to make a rough sketch of the graph of f'.
(c) Find $f'(x)$, and then check your work in part (b) by using the graphing utility to obtain the graph of f'.
(d) Find the equation of the tangent line to the graph of f at $x = 1$, and graph f and the tangent line together.

60. (a) Use a graphing utility to obtain the graph of the function $f(x) = \sin x^2 \cos x$ over the interval $[-\pi/2, \pi/2]$.
(b) Use the graph in part (a) to make a rough sketch of the graph of f' over the interval.
(c) Find $f'(x)$, and then check your work in part (b) by using the graphing utility to obtain the graph of f' over the interval.
(d) Find the equation of the tangent line to the graph of f at $x = 1$, and graph f and the tangent line together over the interval.

61. If an object suspended from a spring is displaced vertically from its equilibrium position by a small amount and released, and if the air resistance and the mass of the spring are ignored, then the resulting oscillation of the object is called ***simple harmonic motion***. Under appropriate conditions the displacement y from equilibrium in terms of time t is given by

$$y = A\cos\omega t$$

where A is the initial displacement at time $t = 0$, and ω is a constant that depends on the mass of the object and the stiffness of the spring (see the accompanying figure). The constant $|A|$ is called the ***amplitude*** of the motion and ω the ***angular frequency***.

(a) Show that

$$\frac{d^2y}{dt^2} = -\omega^2 y$$

(b) The ***period*** T is the time required to make one complete oscillation. Show that $T = 2\pi/\omega$.

(c) The ***frequency*** f of the vibration is the number of oscillations per unit time. Find f in terms of the period T.

(d) Find the amplitude, period, and frequency of an object that is executing simple harmonic motion given by $y = 0.6\cos 15t$, where t is in seconds and y is in centimeters.

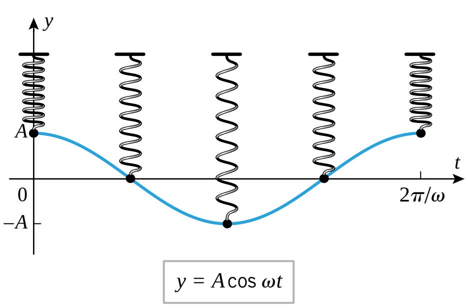


Figure Ex-61

62. Find the value of the constant A so that $y = A\sin 3t$ satisfies the equation

$$\frac{d^2y}{dt^2} + 2y = 4\sin 3t$$

63. The accompanying figure shows the graph of atmospheric pressure p (lb/in^2) versus the altitude h (mi) above sea level.

(a) From the graph and the tangent line at $h = 2$ shown on the graph, estimate the values of p and dp/dh at an altitude of 2 mi.

(b) If the altitude of a space vehicle is increasing at the rate of 0.3 mi/s at the instant when it is 2 mi above sea level, how fast is the pressure changing with time at this instant?

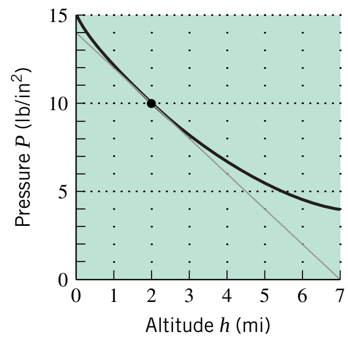


Figure Ex-63

64. The force F (in pounds) acting at an angle θ with the horizontal that is needed to drag a crate weighing W pounds along a horizontal surface at a constant velocity is given by

$$F = \frac{\mu W}{\cos\theta + \mu\sin\theta}$$

where μ is a constant called the ***coefficient of sliding friction*** between the crate and the surface (see the accompanying figure). Suppose that the crate weighs 150 lb and that $\mu = 0.3$.

(a) Find $dF/d\theta$ when $\theta = 30°$. Express the answer in units of pounds/degree.

(b) Find dF/dt when $\theta = 30°$ if θ is decreasing at the rate of 0.5°/s at this instant.

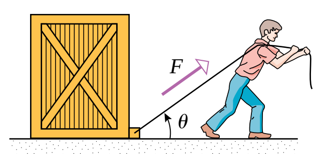


Figure Ex-64

65. Recall that

$$\frac{d}{dx}(|x|) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Use this result and the chain rule to find

$$\frac{d}{dx}(|\sin x|)$$

for nonzero x in the interval $(-\pi, \pi)$.

66. Use the derivative formula for $\sin x$ and the identity

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

to obtain the derivative formula for $\cos x$.

67. Let

$$f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Show that f is continuous at $x = 0$.

(b) Use Definition 3.2.1 to show that $f'(0)$ does not exist.
(c) Find $f'(x)$ for $x \neq 0$.
(d) Determine whether $\lim_{x\to 0} f'(x)$ exists.

68. Let

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Show that f is continuous at $x = 0$.
(b) Use Definition 3.2.1 to find $f'(0)$.
(c) Find $f'(x)$ for $x \neq 0$.
(d) Show that f' is not continuous at $x = 0$.

69. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$
2	1	7
8	5	–3

(a) $g'(2)$, where $g(x) = [f(x)]^3$
(b) $h'(2)$, where $h(x) = f(x^3)$

70. Given that $f'(x) = \sqrt{3x+4}$ and $g(x) = x^2 - 1$, find $F'(x)$ if $F(x) = f(g(x))$.

71. Given that $f'(x) = \dfrac{x}{x^2+1}$ and $g(x) = \sqrt{3x-1}$, find $F'(x)$ if $F(x) = f(g(x))$.

72. Find $f'(x^2)$ if $\dfrac{d}{dx}[f(x^2)] = x^2$.

73. Find $\dfrac{d}{dx}[f(x)]$ if $\dfrac{d}{dx}[f(3x)] = 6x$.

74. Recall that a function f is ***even*** if $f(-x) = f(x)$ and ***odd*** if $f(-x) = -f(x)$, for all x in the domain of f. Assuming that f is differentiable, prove:
(a) f' is odd if f is even
(b) f' is even if f is odd.

75. Draw some pictures to illustrate the results in Exercise 74, and write a paragraph that gives an informal explanation of why the results are true.

76. Let $y = f_1(u)$, $u = f_2(v)$, $v = f_3(w)$, and $w = f_4(x)$. Express dy/dx in terms of dy/du, dw/dx, du/dv, and dv/dw.

77. Find a formula for

$$\frac{d}{dx}[f(g(h(x)))]$$

3.6 IMPLICIT DIFFERENTIATION

In earlier sections we were concerned with differentiating functions that were given by equations of the form $y = f(x)$. In this section we will consider methods for differentiating functions for which it is inconvenient or impossible to express them in this form.

FUNCTIONS DEFINED EXPLICITLY AND IMPLICITLY

An equation of the form $y = f(x)$ is said to define y ***explicitly*** as a function of x because the variable y appears alone on one side of the equation. However, sometimes functions are defined by equations in which y is not alone on one side; for example, the equation

$$yx + y + 1 = x \tag{1}$$

is not of the form $y = f(x)$. However, this equation still defines y as a function of x since it can be rewritten as

$$y = \frac{x-1}{x+1}$$

Thus, we say that (1) defines y ***implicitly*** as a function of x, the function being

$$f(x) = \frac{x-1}{x+1}$$

An equation in x and y can implicitly define more than one function of x; for example, if we solve the equation

$$x^2 + y^2 = 1 \tag{2}$$

for y in terms of x, we obtain $y = \pm\sqrt{1-x^2}$, so we have found two functions that are defined implicitly by (2), namely

$$f_1(x) = \sqrt{1-x^2} \quad \text{and} \quad f_2(x) = -\sqrt{1-x^2} \tag{3}$$

The graphs of these functions are the upper and lower semicircles of the circle $x^2 + y^2 = 1$ (Figure 3.6.1).

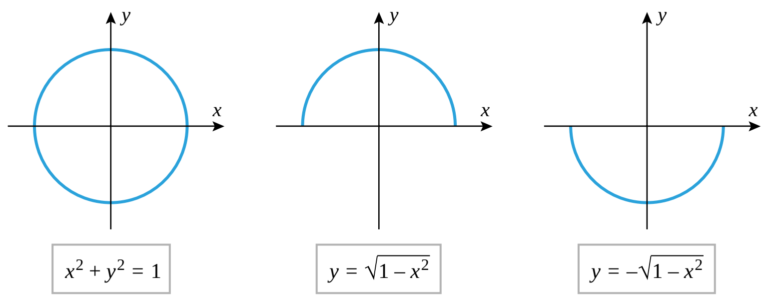


Figure 3.6.1

Observe that the complete circle $x^2 + y^2 = 1$ does not pass the vertical line test, and hence is not itself the graph of a function of x. However, the upper and lower semicircles (which are only portions of the entire circle) do pass the vertical line test, and hence are graphs of functions. In general, if we have an equation in x and y, then any portion of its graph that passes the vertical line test can be viewed as the graph of a function defined by the equation. Thus, we make the following definition.

3.6.1 DEFINITION. We will say that a given equation in x and y defines the function f ***implicitly*** if the graph of $y = f(x)$ coincides with a portion of the graph of the equation.

Thus, for example, the equation $x^2 + y^2 = 1$ defines the functions $f_1(x) = \sqrt{1-x^2}$ and $f_2(x) = -\sqrt{1-x^2}$ implicitly, since the graphs of these functions are contained in the circle $x^2 + y^2 = 1$.

Sometimes it may be difficult or impossible to solve an equation in x and y for y in terms of x. For example, with persistence the equation

$$x^3 + y^3 = 3xy \tag{4}$$

can be solved for y in terms of x, but the algebra is tedious and the resulting formulas are complicated. On the other hand, the equation

$$\sin(xy) = y$$

cannot be solved for y in terms of x by any elementary method. Thus, even though an equation in x and y may define one or more functions of x, it may not be practical or possible to find explicit formulas for those functions.

GRAPHS OF EQUATIONS IN *x* AND *y*

When an equation in x and y cannot be solved for y in terms of x (or x in terms of y), it may be difficult or time-consuming to obtain even a rough sketch of the graph, so the graphing of such equations is usually best left for graphing utilities. In particular, the CAS programs *Mathematica* and *Maple* both have "implicit plotting" capabilities for graphing such equations. For example, Figure 3.6.2 shows the graph of Equation (4), which is called the ***Folium of Descartes***.

FOR THE READER. Figure 3.6.3 shows the graphs of two functions (in solid color) that are defined implicitly by (4). Sketch some more.

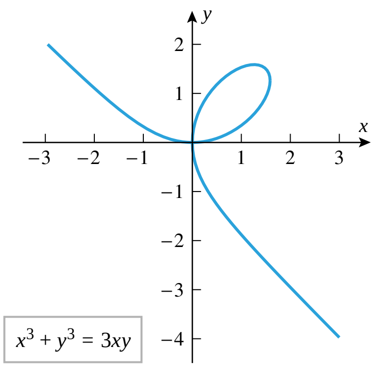


Figure 3.6.2

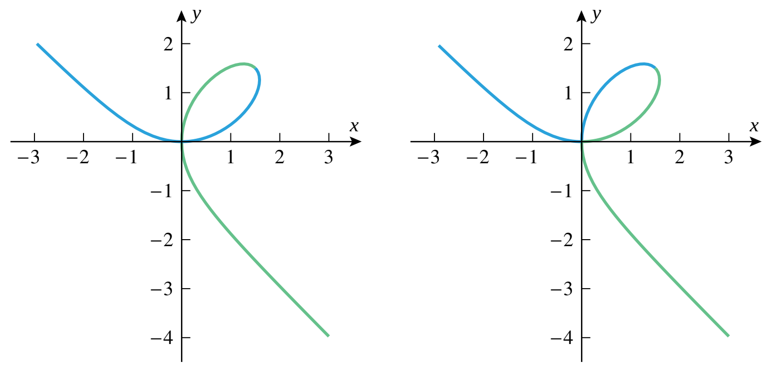


Figure 3.6.3

IMPLICIT DIFFERENTIATION

In general, it is not necessary to solve an equation for y in terms of x in order to differentiate the functions defined implicitly by the equation. To illustrate this, let us consider the simple equation

$$xy = 1 \tag{5}$$

One way to find dy/dx is to rewrite this equation as

$$y = \frac{1}{x} \tag{6}$$

from which it follows that

$$\frac{dy}{dx} = -\frac{1}{x^2} \tag{7}$$

However, there is another way to obtain this derivative. We can differentiate both sides of (5) *before* solving for y in terms of x, treating y as a (temporarily unspecified) differentiable function of x. With this approach we obtain

$$\frac{d}{dx}[xy] = \frac{d}{dx}[1]$$

$$x\frac{d}{dx}[y] + y\frac{d}{dx}[x] = 0$$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

If we now substitute (6) into the last expression, we obtain

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

which agrees with (7). This method of obtaining derivatives is called ***implicit differentiation***.

Example 1 Use implicit differentiation to find dy/dx if $5y^2 + \sin y = x^2$.

$$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$$

$$5\frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] = 2x$$

$$5\left(2y\frac{dy}{dx}\right) + (\cos y)\frac{dy}{dx} = 2x$$

The chain rule was used here because y is a function of x.

$$10y\frac{dy}{dx} + (\cos y)\frac{dy}{dx} = 2x$$

Solving for dy/dx we obtain

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y} \tag{8}$$

Note that this formula involves both x and y. In order to obtain a formula for dy/dx that involves x alone, we would have to solve the original equation for y in terms of x and then substitute in (8). However, it is impossible to do this, so we are forced to leave the formula for dy/dx in terms of x and y. ◀

Example 2 Use implicit differentiation to find d^2y/dx^2 if $4x^2 - 2y^2 = 9$.

Solution. Differentiating both sides of $4x^2 - 2y^2 = 9$ implicitly yields

$$8x - 4y\frac{dy}{dx} = 0$$

from which we obtain

$$\frac{dy}{dx} = \frac{2x}{y} \tag{9}$$

Differentiating both sides of (9) implicitly yields

$$\frac{d^2y}{dx^2} = \frac{(y)(2) - (2x)(dy/dx)}{y^2} \tag{10}$$

Substituting (9) into (10) and simplifying using the original equation, we obtain

$$\frac{d^2y}{dx^2} = \frac{2y - 2x(2x/y)}{y^2} = \frac{2y^2 - 4x^2}{y^3} = -\frac{9}{y^3}$$ ◀

In Examples 1 and 2, the resulting formulas for dy/dx involved both x and y. Although it is usually more desirable to have the formula for dy/dx expressed in terms of x alone, having the formula in terms of x and y is not an impediment to finding slopes and equations of tangent lines provided the x- and y-coordinates of the point of tangency are known. This is illustrated in the following example.

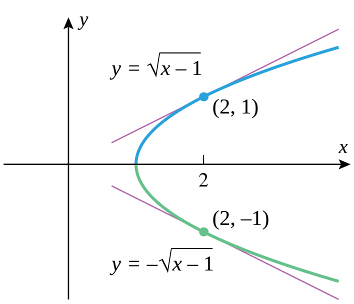


Figure 3.6.4

Example 3 Find the slopes of the curve $y^2 - x + 1 = 0$ at the points $(2, -1)$ and $(2, 1)$.

Solution. We could proceed by solving the equation for y in terms of x, and then evaluating the derivative of $y = \sqrt{x-1}$ at $(2, 1)$ and the derivative of $y = -\sqrt{x-1}$ at $(2, -1)$ (Figure 3.6.4). However, implicit differentiation is more efficient since it gives the slopes of *both* functions. Differentiating implicitly yields

$$\frac{d}{dx}[y^2 - x + 1] = \frac{d}{dx}[0]$$

$$\frac{d}{dx}[y^2] - \frac{d}{dx}[x] + \frac{d}{dx}[1] = \frac{d}{dx}[0]$$

$$2y\frac{dy}{dx} - 1 = 0$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

At $(2, -1)$ we have $y = -1$, and at $(2, 1)$ we have $y = 1$, so the slopes of the curve at those points are

$$\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-1}} = -\frac{1}{2} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=1}} = \frac{1}{2}$$ ◀

Example 4

(a) Use implicit differentiation to find dy/dx for the Folium of Descartes $x^3 + y^3 = 3xy$.

(b) Find an equation for the tangent line to the Folium of Descartes at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

(c) At what point(s) in the first quadrant is the tangent line to the Folium of Descartes horizontal?

Solution (a). Differentiating both sides of the given equation implicitly yields

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$(y^2 - x)\frac{dy}{dx} = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x} \tag{11}$$

Solution (b). At the point $\left(\frac{3}{2}, \frac{3}{2}\right)$, we have $x = \frac{3}{2}$ and $y = \frac{3}{2}$, so from (11) the slope $m_{\tan}$ of the tangent line at this point is

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=3/2\\y=3/2}} = \frac{(3/2) - (3/2)^2}{(3/2)^2 - (3/2)} = -1$$

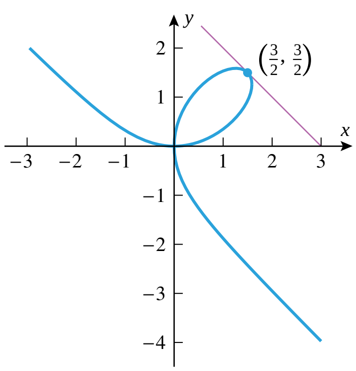


Figure 3.6.5

Thus, the equation of the tangent line at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ is

$$y - \tfrac{3}{2} = -1\left(x - \tfrac{3}{2}\right) \quad \text{or} \quad x + y = 3$$

which is consistent with Figure 3.6.5.

Solution (c). The tangent line is horizontal at the points where $dy/dx = 0$, and from (11) this occurs only where $y - x^2 = 0$ or

$$y = x^2 \tag{12}$$

Substituting this expression for y in the equation $x^3 + y^3 = 3xy$ for the curve yields

$$x^3 + \left(x^2\right)^3 = 3x^3$$

$$x^6 - 2x^3 = 0$$

$$x^3(x^3 - 2) = 0$$

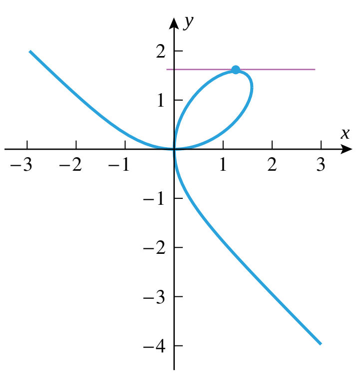


Figure 3.6.6

whose solutions are $x = 0$ and $x = 2^{1/3}$. From (12), the solutions $x = 0$ and $x = 2^{1/3}$ yield the points $(0, 0)$ and $(2^{1/3}, 2^{2/3}) \approx (1.26, 1.59)$, respectively. Of these two, only $(2^{1/3}, 2^{2/3})$ is in the first quadrant. Substituting $x = 2^{1/3}$, $y = 2^{2/3}$ into (11) yields

$$\left.\frac{dy}{dx}\right|_{\substack{x=2^{1/3}\\y=2^{2/3}}} = \frac{0}{2^{4/3} - 2^{2/3}} = 0$$

We conclude that $(2^{1/3}, 2^{2/3})$ is the only point on the Folium of Descartes in the first quadrant at which the tangent line is horizontal (Figure 3.6.6). ◀

REMARK. Note that (11) gives an undefined expression for dy/dx at $(0, 0)$. However, using more advanced techniques it can be shown that the x-axis is tangent to a portion of the Folium of Descartes at the origin.

DIFFERENTIABILITY OF FUNCTIONS DEFINED IMPLICITLY

When differentiating implicitly, it is assumed that y represents a differentiable function of x. If this is not so, then the resulting calculations may be nonsense. For example, if we differentiate the equation

$$x^2 + y^2 + 1 = 0 \tag{13}$$

we obtain

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}$$

However, this derivative is meaningless because (13) does not define a function of x. (The left side of the equation is greater than zero.)

In general, differentiability of implicitly defined functions can be difficult to determine analytically. For example, the first function in Figure 3.6.3 appears to have zero derivative at the origin, whereas the second function in that figure is not differentiable at the origin. However, from Example 4(a) we note that the formula derived for the implicit derivative cannot be evaluated at the origin. This results from the ambiguity created by the curve crossing itself at the origin. We leave a more careful discussion of differentiability for implicitly defined functions for an advanced course in analysis.

DERIVATIVES OF RATIONAL POWERS OF *x*

In Theorem 3.3.7 and the discussion immediately following it, we showed that the formula

$$\frac{d}{dx}[x^n] = nx^{n-1} \tag{14}$$

holds for integer values of n and for $n = \frac{1}{2}$. We will now use implicit differentiation to show that this formula holds for any rational exponent. More precisely, we will show that if r is a rational number, then

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{15}$$

wherever x^r and x^{r-1} are defined. For now, we will assume without proof that x^r is differentiable; the justification for this will be considered later.

Let $y = x^r$. Since r is a rational number, it can be expressed as a ratio of integers $r = m/n$. Thus, $y = x^r = x^{m/n}$ can be written as

$$y^n = x^m \quad \text{so that} \quad \frac{d}{dx}[y^n] = \frac{d}{dx}[x^m]$$

By differentiating implicitly with respect to x and using (14), we obtain

$$ny^{n-1}\frac{dy}{dx} = mx^{m-1} \tag{16}$$

But

$$y^{n-1} = \left[x^{m/n}\right]^{n-1} = x^{m-(m/n)}$$

Thus, (16) can be written as

$$nx^{m-(m/n)}\frac{dy}{dx} = mx^{m-1}$$

so that

$$\frac{dy}{dx} = \frac{m}{n}x^{(m/n)-1} = rx^{r-1}$$

which establishes (15).

Example 5 From (15)

$$\frac{d}{dx}[x^{4/5}] = \frac{4}{5}x^{(4/5)-1} = \frac{4}{5}x^{-1/5}$$

$$\frac{d}{dx}[x^{-7/8}] = -\frac{7}{8}x^{(-7/8)-1} = -\frac{7}{8}x^{-15/8}$$

$$\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$ ◀

If u is a differentiable function of x, and r is a rational number, then the chain rule yields the following generalization of (15):

$$\frac{d}{dx}[u^r] = ru^{r-1} \cdot \frac{du}{dx} \tag{17}$$

Example 6

$$\frac{d}{dx}\left[x^2 - x + 2\right]^{3/4} = \frac{3}{4}\left(x^2 - x + 2\right)^{-1/4} \cdot \frac{d}{dx}[x^2 - x + 2]$$
$$= \frac{3}{4}\left(x^2 - x + 2\right)^{-1/4}(2x - 1)$$

$$\frac{d}{dx}[(\sec \pi x)^{-4/5}] = -\frac{4}{5}(\sec \pi x)^{-9/5} \cdot \frac{d}{dx}[\sec \pi x]$$
$$= -\frac{4}{5}(\sec \pi x)^{-9/5} \cdot \sec \pi x \tan \pi x \cdot \pi$$
$$= -\frac{4\pi}{5}(\sec \pi x)^{-4/5} \tan \pi x$$ ◀

Exercise Set 3.6 C CAS

In Exercises 1–8, find dy/dx.

1. $y = \sqrt[3]{2x - 5}$ **2.** $y = \sqrt[3]{2 + \tan(x^2)}$

3. $y = \left(\frac{x-1}{x+2}\right)^{3/2}$ **4.** $y = \sqrt{\frac{x^2+1}{x^2-5}}$

5. $y = x^3\left(5x^2 + 1\right)^{-2/3}$ **6.** $y = \frac{(3-2x)^{4/3}}{x^2}$

7. $y = [\sin(3/x)]^{5/2}$ **8.** $y = \left[\cos(x^3)\right]^{-1/2}$

In Exercises 9 and 10: (a) Find dy/dx by differentiating implicitly. (b) Solve the equation for y as a function of x, and find dy/dx from that equation. (c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone.

9. $x^3 + xy - 2x = 1$ **10.** $\sqrt{y} - \sin x = 2$

In Exercises 11–20, find dy/dx by implicit differentiation.

11. $x^2 + y^2 = 100$ **12.** $x^3 - y^3 = 6xy$

13. $x^2y + 3xy^3 - x = 3$ **14.** $x^3y^2 - 5x^2y + x = 1$

15. $\frac{1}{y} + \frac{1}{x} = 1$ **16.** $x^2 = \frac{x+y}{x-y}$

17. $\sin(x^2y^2) = x$ **18.** $x^2 = \frac{\cot y}{1 + \csc y}$

19. $\tan^3(xy^2 + y) = x$ **20.** $\frac{xy^3}{1 + \sec y} = 1 + y^4$

In Exercises 21–26, find d^2y/dx^2 by implicit differentiation.

21. $3x^2 - 4y^2 = 7$ **22.** $x^3 + y^3 = 1$

23. $x^3y^3 - 4 = 0$ **24.** $2xy - y^2 = 3$

25. $y + \sin y = x$ **26.** $x \cos y = y$

In Exercises 27 and 28, find the slope of the tangent line to the curve at the given points in two ways: first by solving for y in terms of x and differentiating and then by implicit differentiation.

27. $x^2 + y^2 = 1$; $(1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$

28. $y^2 - x + 1 = 0$; $(10, 3)$, $(10, -3)$

In Exercises 29–32, use implicit differentiation to find the slope of the tangent line to the curve at the specified point, and check that your answer is consistent with the accompanying graph.

29. $x^4 + y^4 = 16$; $(1, \sqrt[4]{15})$ [***Lamé's special quartic***]

30. $y^3 + yx^2 + x^2 - 3y^2 = 0$; $(0, 3)$ [***trisectrix***]

31. $2\left(x^2 + y^2\right)^2 = 25(x^2 - y^2)$; $(3, 1)$ [***lemniscate***]

32. $x^{2/3} + y^{2/3} = 4$; $(-1, 3\sqrt{3})$ [***four-cusped hypocycloid***]

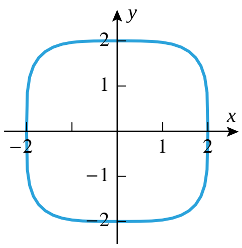


Figure Ex-29

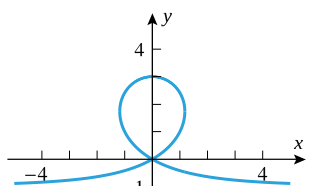


Figure Ex-30

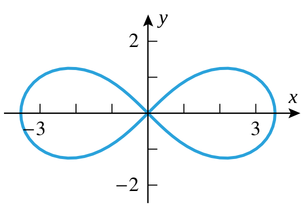


Figure Ex-31

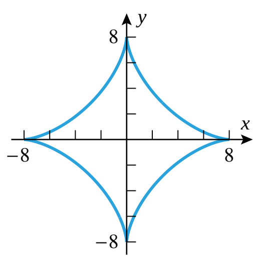


Figure Ex-32

C **33.** If you have a CAS, read the documentation on "implicit plotting," and then generate the four curves in Exercises 29–32.

C **34.** Curves with equations of the form $y^2 = x(x-a)(x-b)$, where $a < b$ are called ***bipartite cubics***.
(a) Use the implicit plotting capability of a CAS to graph the bipartite cubic $y^2 = x(x-1)(x-2)$.
(b) At what points does the curve in part (a) have a horizontal tangent line?
(c) Solve the equation in part (a) for y in terms of x, and use the result to explain why the graph consists of two separate parts (i.e., is *bipartite*).
(d) Graph the equation in part (a) without using the implicit plotting capability of the CAS.

C **35.** (a) Use the implicit plotting capability of a CAS to graph the rotated ellipse $x^2 - xy + y^2 = 4$.
(b) Use the graph to estimate the x-coordinates of all horizontal tangent lines.
(c) Find the exact values for the x-coordinates in part (b).

In Exercises 36–39, use implicit differentiation to find the specified derivative.

36. $\sqrt{u} + \sqrt{v} = 5$; du/dv

37. $a^4 - t^4 = 6a^2t$; da/dt

38. $y = \sin x$; dx/dy.

39. $a^2\omega^2 + b^2\lambda^2 = 1$ (a, b constants); $d\omega/d\lambda$

40. At what point(s) is the tangent line to the curve $y^2 = 2x^3$ perpendicular to the line $4x - 3y + 1 = 0$?

41. Find the values of a and b for the curve $x^2y + ay^2 = b$ if the point $(1, 1)$ is on its graph and the tangent line at $(1, 1)$ has the equation $4x + 3y = 7$.

42. Find the coordinates of the point in the first quadrant at which the tangent line to the curve $x^3 - xy + y^3 = 0$ is parallel to the x-axis.

43. Find equations for two lines through the origin that are tangent to the curve $x^2 - 4x + y^2 + 3 = 0$.

44. Use implicit differentiation to show that the equation of the tangent line to the curve $y^2 = kx$ at (x_0, y_0) is

$$y_0 y = \tfrac{1}{2}k(x + x_0)$$

45. Find dy/dx if

$$2y^3t + t^3y = 1 \quad \text{and} \quad \frac{dt}{dx} = \frac{1}{\cos t}$$

In Exercises 46 and 47, find dy/dt in terms of x, y, and dx/dt, assuming that x and y are differentiable functions of the variable t. [*Hint:* Differentiate both sides of the given equation with respect to t.]

46. $x^3y^2 + y = 3$

47. $xy^2 = \sin 3x$

48. (a) Show that $f(x) = x^{4/3}$ is differentiable at 0, but not twice differentiable at 0.
(b) Show that $f(x) = x^{7/3}$ is twice differentiable at 0, but not three times differentiable at 0.
(c) Find an exponent k such that $f(x) = x^k$ is $(n-1)$ times differentiable at 0, but not n times differentiable at 0.

In Exercises 49 and 50, find all rational values of r such that $y = x^r$ satisfies the given equation.

49. $3x^2y'' + 4xy' - 2y = 0$

50. $16x^2y'' + 24xy' + y = 0$

Two curves are said to be ***orthogonal*** if their tangent lines are perpendicular at each point of intersection, and two families of curves are said to be ***orthogonal trajectories*** of one another if each member of one family is orthogonal to each member of the other family. This terminology is used in Exercises 51 and 52.

51. The accompanying figure shows some typical members of the families of circles $x^2+(y-c)^2=c^2$ (black curves)and $(x-k)^2+y^2=k^2$ (gray curves). Show that these families are orthogonal trajectories of one another. [*Hint:* For the tangent lines to be perpendicular at a point of intersection, the slopes of those tangent lines must be negative reciprocals of one another.]

52. The accompanying figure shows some typical members of the families of hyperbolas $xy=c$ (black curves) and $x^2-y^2=k$ (gray curves), where $c\neq 0$ and $k\neq 0$. Use the hint in Exercise 51 to show that these families are orthogonal trajectories of one another.

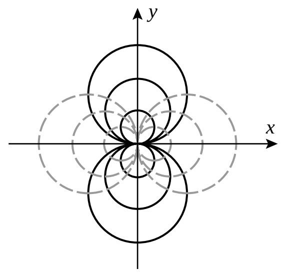


Figure Ex-51

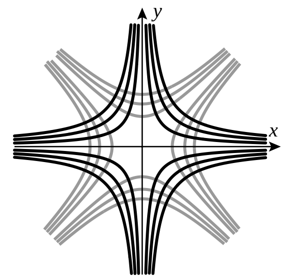


Figure Ex-52

3.7 RELATED RATES

In this section we will study related rates problems. In such problems one tries to find the rate at which some quantity is changing by relating the quantity to other quantities whose rates of change are known.

DIFFERENTIATING EQUATIONS TO RELATE RATES

Figure 3.7.1 shows a liquid draining through a conical filter. As the liquid drains, its volume V, height h, and radius r are functions of the elapsed time t, and at each instant these variables are related by the equation

$$V=\frac{\pi}{3}r^2h$$

If we differentiate both sides of this equation with respect to t, then we obtain

$$\frac{dV}{dt}=\frac{\pi}{3}\left[r^2\frac{dh}{dt}+h\left(2r\frac{dr}{dt}\right)\right]=\frac{\pi}{3}\left(r^2\frac{dh}{dt}+2rh\frac{dr}{dt}\right)$$

Thus, if at a given instant we have values for r, h, and two of the three rates in this equation, then we can solve for the value of the third rate at this instant. In this section we present some specific examples that use this basic idea.

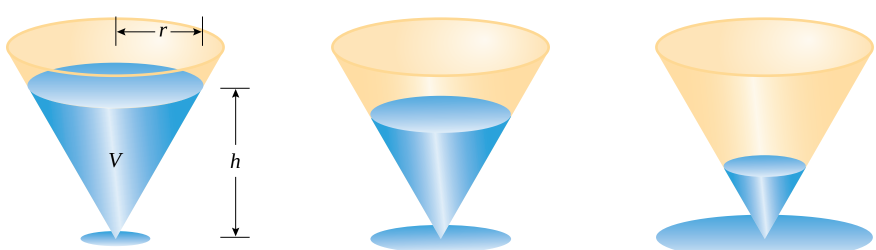


Figure 3.7.1

Example 1 Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

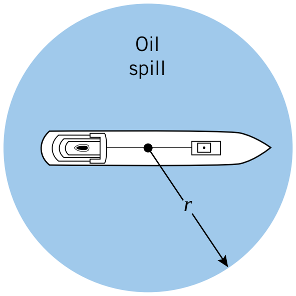


Figure 3.7.2

Solution. Let

t = number of seconds elapsed from the time of the spill
r = radius of the spill in feet after t seconds
A = area of the spill in square feet after t seconds

(Figure 3.7.2). We know the rate at which the radius is increasing, and we want to find the rate at which the area is increasing at the instant when $r = 60$; that is, we want to find

$$\left.\frac{dA}{dt}\right|_{r=60} \quad \text{given that} \quad \frac{dr}{dt} = 2 \text{ ft/s}$$

From the formula for the area of a circle we obtain

$$A = \pi r^2 \tag{1}$$

Because A and r are functions of t, we can differentiate both sides of (1) with respect to t to obtain

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

Thus, when $r = 60$ the area of the spill is increasing at the rate of

$$\left.\frac{dA}{dt}\right|_{r=60} = 2\pi(60)(2) = 240\pi \text{ ft}^2/\text{s}$$

or approximately 754 ft^2/s. ◀

With only minor variations, the method used in Example 1 can be used to solve a variety of related rates problems. The method consists of five steps:

A Strategy for Solving Related Rates Problems

Step 1. Identify the rates of change that are known and the rate of change that is to be found. Interpret each rate as a derivative of a variable with respect to time, and provide a description of each variable involved.

Step 2. Find an equation relating those quantities whose rates are identified in Step 1. In a geometric problem, this is aided by drawing an appropriately labeled figure that illustrates a relationship involving these quantities.

Step 3. Obtain an equation involving the rates in Step 1 by differentiating both sides of the equation in Step 2 with respect to the time variable.

Step 4. Evaluate the equation found in Step 3 using the known values for the quantities and their rates of change at the moment in question.

Step 5. Solve for the value of the remaining rate of change at this moment.

WARNING. Do not substitute prematurely; that is, always perform the differentiation in Step 3 *before* performing the substitution in Step 4.

Example 2 A baseball diamond is a square whose sides are 90 ft long (Figure 3.7.3). Suppose that a player running from second base to third base has a speed of 30 ft/s at the instant when he is 20 ft from third base. At what rate is the player's distance from home plate changing at that instant?

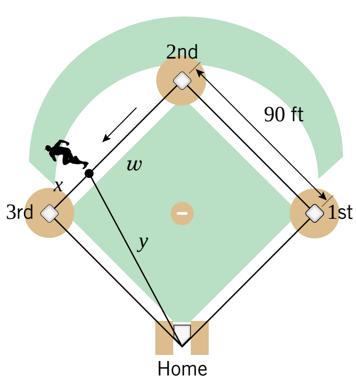


Figure 3.7.3

Solution. The rate we wish to find is the rate of change of the distance from the player to home plate. We are given the speed of the player as he moves along the base path from second to third base, which tells us both the speed with which he is moving away from second base and the speed with which he is approaching third base. Let

t = number of seconds since the player left second base
w = distance in feet from the player to second base
x = distance in feet from the player to third base
y = distance in feet from the player to home plate

Thus, we want to find

$$\left.\frac{dy}{dt}\right|_{x=20} \quad \text{given that} \quad \left.\frac{dw}{dt}\right|_{x=20} = 30 \text{ ft/s} \quad \text{and} \quad \left.\frac{dx}{dt}\right|_{x=20} = -30 \text{ ft/s}$$

[Note that $(dy/dx)_{x=20}$ is negative because x is decreasing with respect to t.]

From the Theorem of Pythagoras,

$$x^2 + 90^2 = y^2 \tag{2}$$

Differentiating both sides of this equation with respect to t yields

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt} \quad \text{or} \quad x\frac{dx}{dt} = y\frac{dy}{dt} \tag{3}$$

To evaluate (3) at the instant when $x = 20$ we need a value for y at this instant. Substituting $x = 20$ into (2) yields

$$400 + 8100 = (y|_{x=20})^2 \quad \text{or} \quad y|_{x=20} = \sqrt{8500} = 10\sqrt{85}$$

Then, evaluating (3) when $x = 20$ yields

$$20 \cdot (-30) = 10\sqrt{85} \cdot \left.\frac{dy}{dt}\right|_{x=20} \quad \text{or} \quad \left.\frac{dy}{dt}\right|_{x=20} = \frac{-600}{10\sqrt{85}} = -\frac{60}{\sqrt{85}} \approx -6.51 \text{ ft/s}$$

The negative sign in the answer tells us that y is decreasing, which makes sense in the physical situation of the problem (Figure 3.7.3). ◀

FOR THE READER. In our solution for Example 2 we chose to relate x and y. An alternative approach would be to relate w and y. Solve the problem using this alternative approach.

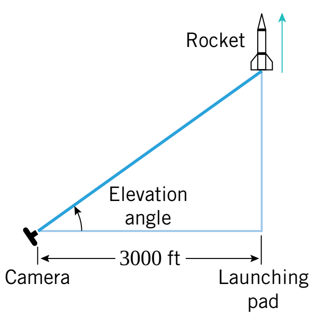


Figure 3.7.4

Example 3 In Figure 3.7.4 we have shown a camera mounted at a point 3000 ft from the base of a rocket launching pad. If the rocket is rising vertically at 880 ft/s when it is 4000 ft above the launching pad, how fast must the camera elevation angle change at that instant to keep the camera aimed at the rocket?

Solution. Let

t = number of seconds elapsed from the time of launch
ϕ = camera elevation angle in radians after t seconds
h = height of the rocket in feet after t seconds

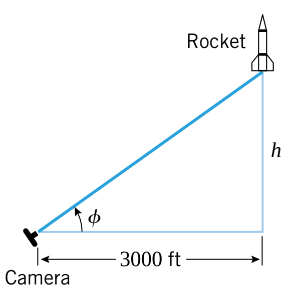


Figure 3.7.5

(Figure 3.7.5). At each instant the rate at which the camera elevation angle must change is $d\phi/dt$, and the rate at which the rocket is rising is dh/dt. We want to find

$$\left.\frac{d\phi}{dt}\right|_{h=4000} \quad \text{given that} \quad \left.\frac{dh}{dt}\right|_{h=4000} = 880 \text{ ft/s}$$

From Figure 3.7.5 we see that

$$\tan\phi = \frac{h}{3000} \tag{4}$$

Because ϕ and h are functions of t, we can differentiate both sides of (4) with respect to t to obtain

$$(\sec^2\phi)\frac{d\phi}{dt} = \frac{1}{3000}\frac{dh}{dt} \tag{5}$$

When $h = 4000$, it follows that

$$(\sec\phi)_{h=4000} = \frac{5000}{3000} = \frac{5}{3}$$

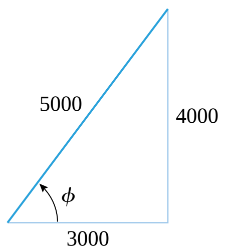


Figure 3.7.6

(Figure 3.7.6), so that from (5)

$$\left(\frac{5}{3}\right)^2 \frac{d\phi}{dt}\bigg|_{h=4000} = \frac{1}{3000} \cdot 880 = \frac{22}{75}$$

$$\frac{d\phi}{dt}\bigg|_{h=4000} = \frac{22}{75} \cdot \frac{9}{25} = \frac{66}{625} \approx 0.11 \text{ rad/s} \approx 6.05 \text{ deg/s}$$ ◀

Example 4 Suppose that liquid is to be cleared of sediment by allowing it to drain through a conical filter that is 16 cm high and has a radius of 4 cm at the top (Figure 3.7.7). Suppose also that the liquid flows out of the cone at a constant rate of 2 cm^3/min.

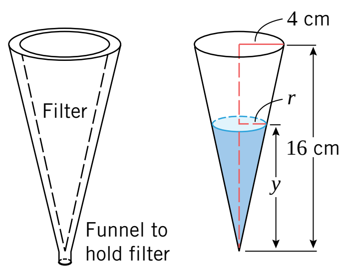


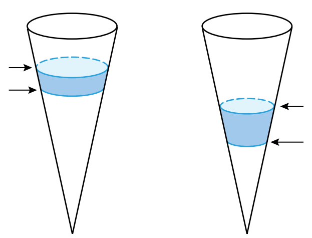
Figure 3.7.7

(a) Do you think that the depth of the liquid will decrease at a constant rate? Give a verbal argument that justifies your conclusion.

(b) Find a formula that expresses the rate at which the depth of the liquid is changing in terms of the depth, and use that formula to determine whether your conclusion in part (a) is correct.

(c) At what rate is the depth of the liquid changing at the instant when the liquid in the cone is 8 cm deep?

Solution (a). For the volume of liquid to decrease by a *fixed amount*, it requires a greater decrease in depth when the cone is close to empty than when it is almost full (Figure 3.7.8). This suggests that for the volume to decrease at a constant rate, the depth must decrease at an increasing rate.



The same volume has drained, but the change in height is greater near the bottom than near the top.

Figure 3.7.8

Solution (b). Let

t = time elapsed from the initial observation (min)
V = volume of liquid in the cone at time t (cm^3)
y = depth of the liquid in the cone at time t (cm)
r = radius of the liquid surface at time t (cm)

(Figure 3.7.7). At each instant the rate at which the volume of liquid is changing is dV/dt, and the rate at which the depth is changing is dy/dt. We want to express dy/dt in terms of y given that dV/dt has a constant value of $dV/dt = -2$. (We must use a minus sign here because V *decreases* as t increases.)

From the formula for the volume of a cone, the volume V, the radius r, and the depth y are related by

$$V = \tfrac{1}{3}\pi r^2 y \tag{6}$$

If we differentiate both sides of (6) with respect to t, the right side will involve the quantity dr/dt. Since we have no direct information about dr/dt, it is desirable to eliminate r from (6) before differentiating. This can be done using similar triangles. From Figure 3.7.7 we see that

$$\frac{r}{y} = \frac{4}{16} \quad \text{or} \quad r = \frac{1}{4}y$$

Substituting this expression in (6) gives

$$V = \frac{\pi}{48}y^3 \tag{7}$$

Differentiating both sides of (7) with respect to t we obtain

$$\frac{dV}{dt} = \frac{\pi}{48}\left(3y^2\frac{dy}{dt}\right)$$

or

$$\frac{dy}{dt} = \frac{16}{\pi y^2}\frac{dV}{dt} = \frac{16}{\pi y^2}(-2) = -\frac{32}{\pi y^2} \tag{8}$$

which expresses dy/dt in terms of y. The minus sign tells us that y is decreasing with time, and

$$\left|\frac{dy}{dt}\right| = \frac{32}{\pi y^2}$$

tells us how fast y is decreasing. From this formula we see that $|dy/dt|$ increases as y decreases, which confirms our conjecture in part (a) that the depth of the liquid decreases more quickly as the liquid drains through the filter.

Solution (c). The rate at which the depth is changing when the depth is 8 cm can be obtained from (8) with $y = 8$:

$$\left.\frac{dy}{dt}\right|_{y=8} = -\frac{32}{\pi(8^2)} = -\frac{1}{2\pi} \approx -0.16 \text{ cm/min}$$ ◀

Exercise Set 3.7

In Exercises 1–4, both x and y denote functions of t that are related by the given equation. Use this equation and the given derivative information to find the specified derivative.

1. Equation: $y = 3x + 5$.
(a) Given that $dx/dt = 2$, find dy/dt when $x = 1$.
(b) Given that $dy/dt = -1$, find dx/dt when $x = 0$.

2. Equation: $x + 4y = 3$.
(a) Given that $dx/dt = 1$, find dy/dt when $x = 2$.
(b) Given that $dy/dt = 4$, find dx/dt when $x = 3$.

3. Equation: $x^2 + y^2 = 1$.
(a) Given that $dx/dt = 1$, find dy/dt when
$$(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$
(b) Given that $dy/dt = -2$, find dx/dt when
$$(x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

4. Equation: $x^2 + y^2 = 2x$.
(a) Given that $dx/dt = -2$, find dy/dt when $(x, y) = (1, 1)$.
(b) Given that $dy/dt = 3$, find dx/dt when
$$(x, y) = \left(\frac{2+\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

5. Let A be the area of a square whose sides have length x, and assume that x varies with the time t.
(a) Draw a picture of the square with the labels A and x placed appropriately.
(b) Write an equation that relates A and x.
(c) Use the equation in part (b) to find an equation that relates dA/dt and dx/dt.
(d) At a certain instant the sides are 3 ft long and increasing at a rate of 2 ft/min. How fast is the area increasing at that instant?

6. Let A be the area of a circle of radius r, and assume that r increases with the time t.
(a) Draw a picture of the circle with the labels A and r placed appropriately.
(b) Write an equation that relates A and r.
(c) Use the equation in part (b) to find an equation that relates dA/dt and dr/dt.
(d) At a certain instant the radius is 5 cm and increasing at the rate of 2 cm/s. How fast is the area increasing at that instant?

7. Let V be the volume of a cylinder having height h and radius r, and assume that h and r vary with time.
(a) How are dV/dt, dh/dt, and dr/dt related?
(b) At a certain instant, the height is 6 in and increasing at 1 in/s, while the radius is 10 in and decreasing at 1 in/s. How fast is the volume changing at that instant? Is the volume increasing or decreasing at that instant?

8. Let l be the length of a diagonal of a rectangle whose sides have lengths x and y, and assume that x and y vary with time.
(a) How are dl/dt, dx/dt, and dy/dt related?
(b) If x increases at a constant rate of $\frac{1}{2}$ ft/s and y decreases at a constant rate of $\frac{1}{4}$ ft/s, how fast is the size of the diagonal changing when $x = 3$ ft and $y = 4$ ft? Is the diagonal increasing or decreasing at that instant?

9. Let θ (in radians) be an acute angle in a right triangle, and let x and y, respectively, be the lengths of the sides adjacent to and opposite θ. Suppose also that x and y vary with time.
(a) How are $d\theta/dt$, dx/dt, and dy/dt related?
(b) At a certain instant, $x = 2$ units and is increasing at 1 unit/s, while $y = 2$ units and is decreasing at $\frac{1}{4}$ unit/s. How fast is θ changing at that instant? Is θ increasing or decreasing at that instant?

10. Suppose that $z = x^3y^2$, where both x and y are changing with time. At a certain instant when $x = 1$ and $y = 2$, x is decreasing at the rate of 2 units/s, and y is increasing at the rate of 3 units/s. How fast is z changing at this instant? Is z increasing or decreasing?

11. The minute hand of a certain clock is 4 in long. Starting from the moment when the hand is pointing straight up, how fast is the area of the sector that is swept out by the hand increasing at any instant during the next revolution of the hand?

12. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 ft/s. How rapidly is the area enclosed by the ripple increasing at the end of 10 s?

13. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 mi^2/h. How fast is the radius of the spill increasing when the area is 9 mi^2?

14. A spherical balloon is inflated so that its volume is increasing at the rate of 3 ft^3/min. How fast is the diameter of the balloon increasing when the radius is 1 ft?

15. A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must air be removed when the radius is 9 cm?

16. A 17-ft ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 5 ft/s, how fast will the top of the ladder be moving down the wall when it is 8 ft above the ground?

17. A 13-ft ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2 ft/s, how fast will the foot be moving away from the wall when the top is 5 ft above the ground?

18. A 10-ft plank is leaning against a wall. If at a certain instant the bottom of the plank is 2 ft from the wall and is being pushed toward the wall at the rate of 6 in/s, how fast is the acute angle that the plank makes with the ground increasing?

19. A softball diamond is a square whose sides are 60 ft long. Suppose that a player running from first to second base has a speed of 25 ft/s at the instant when she is 10 ft from second base. At what rate is the player's distance from home plate changing at that instant?

20. A rocket, rising vertically, is tracked by a radar station that is on the ground 5 mi from the launchpad. How fast is the rocket rising when it is 4 mi high and its distance from the radar station is increasing at a rate of 2000 mi/h?

21. For the camera and rocket shown in Figure 3.7.4, at what rate is the camera-to-rocket distance changing when the rocket is 4000 ft up and rising vertically at 880 ft/s?

22. For the camera and rocket shown in Figure 3.7.4, at what rate is the rocket rising when the elevation angle is $\pi/4$ radians and increasing at a rate of 0.2 radian/s?

23. A satellite is in an elliptical orbit around the Earth. Its distance r (in miles) from the center of the Earth is given by

$$r = \frac{4995}{1 + 0.12\cos\theta}$$

where θ is the angle measured from the point on the orbit nearest the Earth's surface (see the accompanying figure).

(a) Find the altitude of the satellite at ***perigee*** (the point nearest the surface of the Earth) and at ***apogee*** (the point farthest from the surface of the Earth). Use 3960 mi as the radius of the Earth.

(b) At the instant when θ is 120°, the angle θ is increasing at the rate of 2.7°/min. Find the altitude of the satellite and the rate at which the altitude is changing at this instant. Express the rate in units of mi/min.

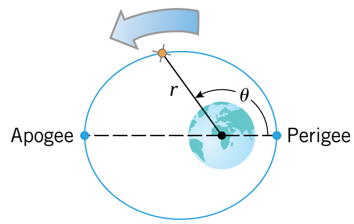


Figure Ex-23

24. An aircraft is flying horizontally at a constant height of 4000 ft above a fixed observation point (see the accompanying figure). At a certain instant the angle of elevation θ is 30° and decreasing, and the speed of the aircraft is 300 mi/h.

(a) How fast is θ decreasing at this instant? Express the result in units of degrees/s.

(b) How fast is the distance between the aircraft and the observation point changing at this instant? Express the result in units of ft/s. Use 1 mi = 5280 ft.

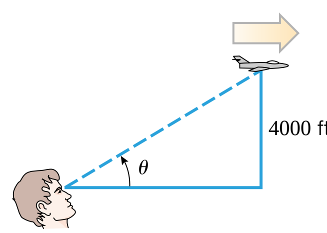


Figure Ex-24

25. A conical water tank with vertex down has a radius of 10 ft at the top and is 24 ft high. If water flows into the tank at a rate of 20 ft^3/min, how fast is the depth of the water increasing when the water is 16 ft deep?

26. Grain pouring from a chute at the rate of 8 ft^3/min forms a conical pile whose altitude is always twice its radius. How fast is the altitude of the pile increasing at the instant when the pile is 6 ft high?

27. Sand pouring from a chute forms a conical pile whose height is always equal to the diameter. If the height increases at a constant rate of 5 ft/min, at what rate is sand pouring from the chute when the pile is 10 ft high?

28. Wheat is poured through a chute at the rate of 10 ft^3/min, and falls in a conical pile whose bottom radius is always half the altitude. How fast will the circumference of the base be increasing when the pile is 8 ft high?

29. An aircraft is climbing at a 30° angle to the horizontal. How fast is the aircraft gaining altitude if its speed is 500 mi/h?

30. A boat is pulled into a dock by means of a rope attached to a pulley on the dock (see the accompanying figure). The rope is attached to the bow of the boat at a point 10 ft below the pulley. If the rope is pulled through the pulley at a rate of 20 ft/min, at what rate will the boat be approaching the dock when 125 ft of rope is out?

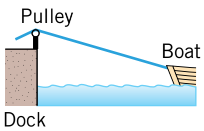


Figure Ex-30

31. For the boat in Exercise 30, how fast must the rope be pulled if we want the boat to approach the dock at a rate of 12 ft/min at the instant when 125 ft of rope is out?

32. A man 6 ft tall is walking at the rate of 3 ft/s toward a streetlight 18 ft high (see the accompanying figure).
(a) At what rate is his shadow length changing?
(b) How fast is the tip of his shadow moving?

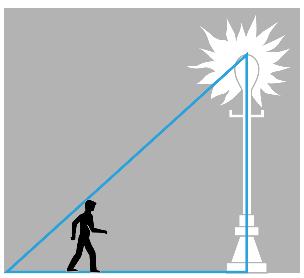


Figure Ex-32

33. A beacon that makes one revolution every 10 s is located on a ship anchored 4 kilometers from a straight shoreline. How fast is the beam moving along the shoreline when it makes an angle of 45° with the shore?

34. An aircraft is flying at a constant altitude with a constant speed of 600 mi/h. An antiaircraft missile is fired on a straight line perpendicular to the flight path of the aircraft so that it will hit the aircraft at a point P (see the accompanying figure). At the instant the aircraft is 2 mi from the impact point P the missile is 4 mi from P and flying at 1200 mi/h. At that instant, how rapidly is the distance between missile and aircraft decreasing?

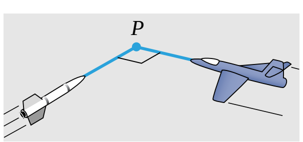


Figure Ex-34

35. Solve Exercise 34 under the assumption that the angle between the flight paths is 120° instead of the assumption that the paths are perpendicular. [*Hint:* Use the law of cosines.]

36. A police helicopter is flying due north at 100 mi/h and at a constant altitude of $\frac{1}{2}$ mi. Below, a car is traveling west on a highway at 75 mi/h. At the moment the helicopter crosses over the highway the car is 2 mi east of the helicopter.
(a) How fast is the distance between the car and helicopter changing at the moment the helicopter crosses the highway?
(b) Is the distance between the car and helicopter increasing or decreasing at that moment?

37. A particle is moving along the curve whose equation is

$$\frac{xy^3}{1+y^2} = \frac{8}{5}$$

Assume that the x-coordinate is increasing at the rate of 6 units/s when the particle is at the point $(1, 2)$.
(a) At what rate is the y-coordinate of the point changing at that instant?
(b) Is the particle rising or falling at that instant?

38. A point P is moving along the curve whose equation is $y = \sqrt{x^3 + 17}$. When P is at $(2, 5)$, y is increasing at the rate of 2 units/s. How fast is x changing?

39. A point P is moving along the line whose equation is $y = 2x$. How fast is the distance between P and the point $(3, 0)$ changing at the instant when P is at $(3, 6)$ if x is decreasing at the rate of 2 units/s at that instant?

40. A point P is moving along the curve whose equation is $y = \sqrt{x}$. Suppose that x is increasing at the rate of 4 units/s when $x = 3$.
(a) How fast is the distance between P and the point $(2, 0)$ changing at this instant?
(b) How fast is the angle of inclination of the line segment from P to $(2, 0)$ changing at this instant?

41. A particle is moving along the curve $y = x/(x^2 + 1)$. Find all values of x at which the rate of change of x with respect to time is three times that of y. [Assume that dx/dt is never zero.]

42. A particle is moving along the curve $16x^2 + 9y^2 = 144$. Find all points (x, y) at which the rates of change of x and y with respect to time are equal. [Assume that dx/dt and dy/dt are never both zero at the same point.]

43. The ***thin lens equation*** in physics is

$$\frac{1}{s} + \frac{1}{S} = \frac{1}{f}$$

where s is the object distance from the lens, S is the image distance from the lens, and f is the focal length of the lens. Suppose that a certain lens has a focal length of 6 cm and that an object is moving toward the lens at the rate of 2 cm/s. How fast is the image distance changing at the instant when the object is 10 cm from the lens? Is the image moving away from the lens or toward the lens?

44. Water is stored in a cone-shaped reservoir (vertex down). Assuming the water evaporates at a rate proportional to the surface area exposed to the air, show that the depth of the water will decrease at a constant rate that does not depend on the dimensions of the reservoir.

45. A meteor enters the Earth's atmosphere and burns up at a rate that, at each instant, is proportional to its surface area. Assuming that the meteor is always spherical, show that the radius decreases at a constant rate.

46. On a certain clock the minute hand is 4 in long and the hour hand is 3 in long. How fast is the distance between the tips of the hands changing at 9 o'clock?

47. Coffee is poured at a uniform rate of 20 cm^3/s into a cup whose inside is shaped like a truncated cone (see the accompanying figure). If the upper and lower radii of the cup are 4 cm and 2 cm and the height of the cup is 6 cm, how fast will the coffee level be rising when the coffee is halfway up? [*Hint:* Extend the cup downward to form a cone.]

Figure Ex-47

3.8 LOCAL LINEAR APPROXIMATION; DIFFERENTIALS

In this section we will show how derivatives can be used to approximate nonlinear functions by simpler linear functions. We will also define the differentials dy and dx and use them to interpret the derivative dy/dx as a ratio of differentials.

LOCAL LINEAR APPROXIMATION

In the solution of certain problems, it can be useful (and sometimes even necessary) to approximate a nonlinear function by a linear function. For example, the equations that describe the motion of a swinging pendulum may be greatly simplified by using the fact that if x is close to 0, then $\sin x \approx x$. The existence of such linear approximations provides us with a geometric interpretation of differentiability. We saw in Section 3.2 that if a function f is differentiable at a number x_0, then the tangent line to the graph of f through the point $P = (x_0, f(x_0))$ will very closely approximate the graph of f for values of x near x_0 (Figure 3.8.1). This linear approximation may be described informally in terms of the behavior of the graph of f under magnification: if f is differentiable at x_0, then stronger and stronger magnifications at P eventually make the curve segment containing P look more and more like a nonvertical line segment, that line being the tangent line to the graph of f at P. For this reason, a function that is differentiable at x_0 is said to be ***locally linear*** at the point $P(x_0, f(x_0))$ (Figure 3.8.2*a*). By contrast, the graph of a function that is not differentiable at x_0 due to a corner at the point $P(x_0, f(x_0))$ cannot be magnified to resemble a straight line segment at that point (Figure 3.8.2*b*).

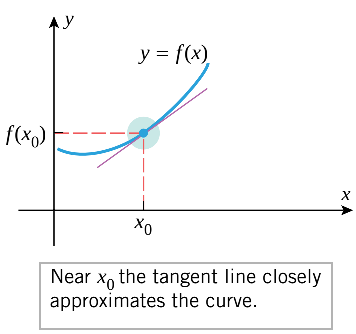


Figure 3.8.1

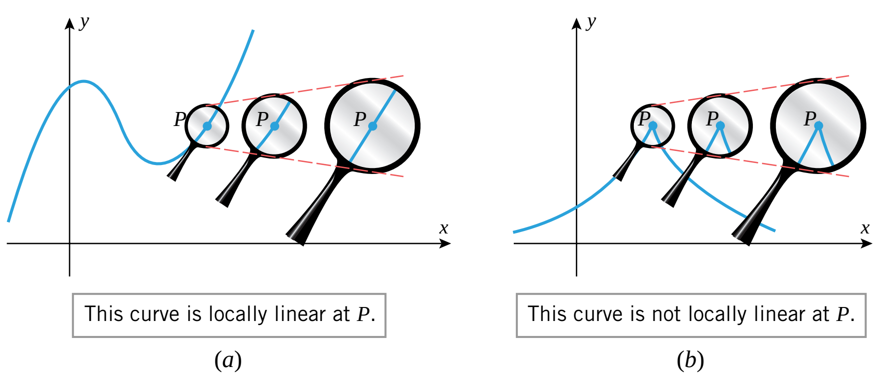


Figure 3.8.2

To capture this intuitive idea analytically, assume that a function f is differentiable at x_0 and recall that the equation of the tangent line to the graph of the function f through $P = (x_0, f(x_0))$ is $y = f(x_0) + f'(x_0)(x - x_0)$. Since this line closely approximates the graph of f for values of x near x_0, it follows that

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \qquad (1)$$

provided x is close to x_0. We call (1) the ***local linear approximation of f at x_0***. Furthermore, it can be shown that (1) is actually the *best* linear approximation of f near x_0 in the sense that any other linear function will fail to give as good an approximation to f for values of x very close to x_0. An alternative version of this formula can be obtained by letting $\Delta x = x - x_0$, in which case (1) can be expressed as

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\,\Delta x \qquad (2)$$

Example 1

(a) Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{1.1}$, and compare your approximation to the result produced directly by a calculating utility.

Solution (a). Since $f'(x) = 1/(2\sqrt{x})$, it follows from (1) that the local linear approximation of $\sqrt{x}$ at $x_0 = 1$ is

$$\sqrt{x} \approx \sqrt{1} + \frac{1}{2\sqrt{1}}(x-1) = 1 + \frac{1}{2}(x-1) = \frac{1}{2}(x+1)$$

In other words, if x is close to 1, then we expect $\sqrt{x}$ to be about $\frac{1}{2}(x+1)$. Figure 3.8.3 shows both the graph of $f(x) = \sqrt{x}$ and the local linear approximation $y = \frac{1}{2}(x+1)$.

Solution (b). Applying the local linear approximation from part (a) yields

$$\sqrt{1.1} \approx \tfrac{1}{2}(1.1 + 1) = 1.05$$

Since the tangent line $y = \frac{1}{2}(x+1)$ in Figure 3.8.3 lies above the graph of $f(x) = \sqrt{x}$, we would expect this approximation to be slightly too large. This expectation is confirmed by the calculator approximation $\sqrt{1.1} \approx 1.04881$. ◀

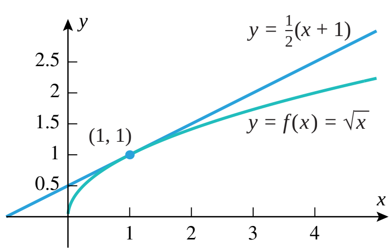


Figure 3.8.3

Example 2

(a) Show that if x is close to 0, then $\sin x \approx x$.

(b) Use the approximation from part (a) to approximate $\sin 2^\circ$, and compare your approximation to the result produced directly by your calculating utility.

Solution (a). Since we are interested in approximating $\sin x$ for values of x close to 0, we compute the local linear approximation of $f(x) = \sin x$ at $x_0 = 0$. With $f(x) = \sin x$, $f'(x) = \cos x$, and $x_0 = 0$, the approximation in (1) becomes

$$\sin x \approx \sin 0 + \cos 0(x - 0) = 0 + 1(x) = x$$

Figure 3.8.4 shows both the graph of $f(x) = \sin x$ and the local linear approximation $y = x$.

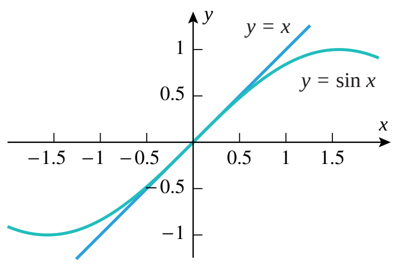


Figure 3.8.4

Solution (b). In the approximation $\sin x \approx x$, the variable x is in radian measure, so we must first convert 2° to radians before we can apply this approximation. Since

$$2^\circ = 2\left(\frac{\pi}{180}\right) = \frac{\pi}{90} \approx 0.0349066 \text{ radian}$$

it follows that $\sin 2^\circ \approx 0.0349066$. Comparing the two graphs in Figure 3.8.4, we would expect this approximation to be slightly too large. The calculator approximation $\sin 2^\circ \approx 0.0348995$ shows that this is indeed the case. ◀

REMARK. Part (b) in both Example 1 and Example 2 is meant to be illustrative only. We are not suggesting that you replace individual calculator computations with the local linear approximation. Local linear approximations are significant because they allow us to model a complicated function by a simple one. This idea will be pursued in greater detail in Chapter 10.

ERROR IN LOCAL LINEAR APPROXIMATIONS

As a general rule, the accuracy of the local linear approximation to $f(x)$ at x_0 will deteriorate as x gets progressively farther from x_0. To illustrate this for the approximation $\sin x \approx x$ in Example 2, let us graph the function

$$E(x) = |\sin x - x|$$

which is the absolute value of the error in the approximation (Figure 3.8.5).

In Figure 3.8.5, the graph shows how the absolute error in the local linear approximation of $\sin x$ increases as x moves progressively farther from 0 in either the positive or negative direction. The graph also tells us that for values of x between the two vertical lines, the absolute error does not exceed 0.01. Thus, for example, we could use the local linear approximation $\sin x \approx x$ for all values of x in the interval $-0.35 < x < 0.35$ (radians) with confidence that the approximation is within 0.01 of the exact value.

DIFFERENTIALS

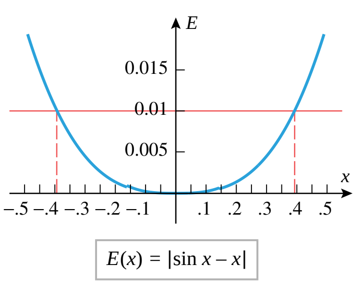


Figure 3.8.5

Newton and Leibniz independently developed different notations for the derivative. This created a notational divide between Britain and the European continent that lasted for more than 50 years. The ***Leibniz notation*** dy/dx eventually prevailed for its superior utility. For example, we have already mentioned that the Leibniz notation makes the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

easy to remember.

Up to now we have been interpreting dy/dx as a single entity representing the derivative of y with respect to x, but we have not attached any meaning to the individual symbols "dy" and "dx." Early in the development of calculus, these symbols represented "infinitely small changes" in the variables y and x and the derivative dy/dx was thought to be a ratio of these infinitely small changes. However, the precise meaning of an "infinitely small change" in a variable turned out to be logically elusive and eventually such arguments were replaced by an analysis that was based on the more modern concept of a limit.

Our next objective is to define the symbols dy and dx so that dy/dx can actually be treated as a ratio. We begin by defining the symbol "dx" to be a *variable* that can assume any real number as its value. The variable dx is called the ***differential of x***. If we are given a function $y = f(x)$ that is differentiable at $x = x_0$, then we define the ***differential of f at x_0*** to be the function of dx given by the formula

$$dy = f'(x_0)\,dx \tag{3}$$

where the symbol "dy" is simply the dependent variable of this function. The variable dy is called the differential of y and we note that it is proportional to dx with constant of proportionality $f'(x_0)$. If $dx \neq 0$, then we can divide both sides of (3) by dx to obtain

$$\frac{dy}{dx} = f'(x_0)$$

Thus, we have achieved our goal of defining dy and dx so that their ratio is a derivative. It is customary to omit the subscript on x and simply write the differential dy as

$$dy = f'(x)\,dx \tag{4}$$

where it is understood that x is regarded as fixed at some value.

Because $f'(x)$ is equal to the slope of the tangent line to the graph of f at the point $(x, f(x))$, the differentials dy and dx can be viewed as a corresponding rise and run of this tangent line (Figure 3.8.6).

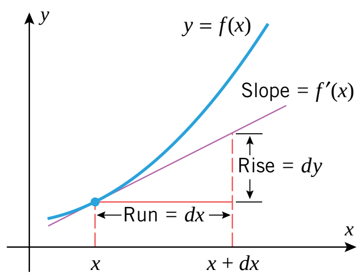


Figure 3.8.6

Example 3 Given the function $y = x^2$, geometrically interpret the relationship between the differentials dx and dy when $x = 3$.

Solution. Since $dy/dx = 2x$, we have $dy = 2x\,dx = 6\,dx$ when $x = 3$. This tells us that if we travel along the tangent line to the curve $y = x^2$ at the point (3, 9), then any change of dx units in the horizonal direction produces a change of $dy = 6\,dx$ units in the vertical direction. ◀

Recall that given a function $y = f(x)$, we defined $\Delta y = f(x + \Delta x) - f(x)$ to denote the signed change in y from its value at some initial number x to its value at a new number $x + \Delta x$. It is important to understand the distinction between the increment Δy and the differential dy. To see the difference, let us assign the independent variables dx and Δx the same value, so $dx = \Delta x$. Then Δy represents the change in y that occurs when we start at x and travel *along the curve* $y = f(x)$ until we have moved Δx $(= dx)$ units in the x-direction, and dy represents the change in y that occurs if we start at x and travel *along the tangent line* until we have moved dx $(= \Delta x)$ units in the x-direction (Figure 3.8.7).

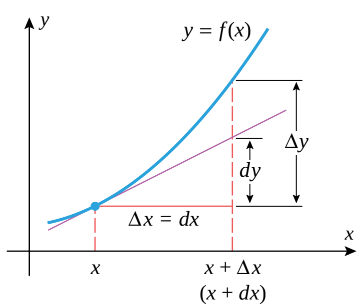


Figure 3.8.7

Example 4 Let $y = \sqrt{x}$. Find dy and Δy at $x = 4$ with $dx = \Delta x = 3$. Then make a sketch of $y = \sqrt{x}$, showing dy and Δy in the picture.

Solution. With $f(x) = \sqrt{x}$ we obtain

$$\Delta y = f(x + \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{7} - \sqrt{4} \approx 0.65$$

If $y = \sqrt{x}$, then

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad dy = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4} = 0.75$$

Figure 3.8.8 shows the curve $y = \sqrt{x}$ together with dy and Δy. ◀

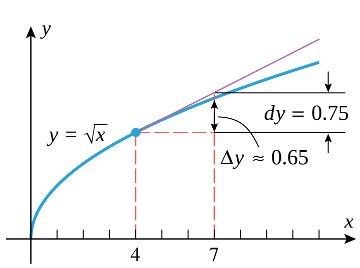


Figure 3.8.8

Although Δy and dy are generally different, the differential dy will nonetheless be a good approximation for Δy provided $dx = \Delta x$ is close to 0. To see this, recall from Section

3.2 that

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

It follows that if Δx is close to 0, then we will have $f'(x) \approx \Delta y/\Delta x$ or, equivalently,

$$\Delta y \approx f'(x)\Delta x = f'(x)\,dx = dy \tag{5}$$

As the reader might guess by comparing Figure 3.8.1 with Figure 3.8.7, the approximation $\Delta y \approx dy$ is simply a restatement of the local linear approximation of a function.

FOR THE READER. Obtain the approximation $\Delta y \approx dy$ directly from the local linear approximation (2) by renaming some parameters and using some algebra.

ERROR PROPAGATION IN APPLICATIONS

In applications, small errors invariably occur in measured quantities. When these quantities are used in computations, those errors are propagated in turn to the computed quantities. For example, suppose that in an application the variables x and y are related by a function $y = f(x)$. If x_a is the actual value of x, but it is measured to be x_0, then we define the difference $dx = x_0 - x_a$ to be the *error* in the measurement of x. Note that if the error is positive, the measured value is larger than the actual value, and if the error is negative, the measured value is smaller than the actual value. Since y is determined from x by the function $y = f(x)$, the true value of y is $f(x_a)$ and the value of y computed from the measured value of x is $f(x_0)$. The *propagated error* in the computed value of y is then defined to be $f(x_0) - f(x_a)$. Note that if the propagated error is positive, the calculated value of y will be too large, and if this error is negative, the calculated value of y will be too small. If f is differentiable at the measured value x_0, and if the error in the measurement of x is close to 0, then the local linear approximation (1) (with x replaced by x_a) becomes

$$f(x_a) \approx f(x_0) + f'(x_0)(x_a - x_0) = f(x_0) - f'(x_0)(x_0 - x_a) = f(x_0) - f'(x_0)\,dx$$

We can now use this approximation in our formula for the propagated error to obtain

$$f(x_0) - f(x_a) \approx f(x_0) - (f(x_0) - f'(x_0)\,dx) = f'(x_0)\,dx$$

In other words, the propagated error may be approximated by

$$f(x_0) - f(x_a) \approx dy \tag{6}$$

where $dy = f'(x_0)\,dx$ is the value of the differential of f at x_0 when $dx = x_0 - x_a$ is the error in the measurement of x.

Unfortunately, this approximation cannot be used directly in applied problems because the measurement error $dx = x_0 - x_a$ will in general be unknown. (Keep in mind that the only value of x that is available to the researcher is the measured value x_0.) However, although the exact value of the error in measuring x will generally be unknown, it is often possible to determine upper and lower bounds for this error. Upper and lower bounds for the propagated error can then be approximated by using the differential $dy = f'(x_0)\,dx$.

Example 5 Suppose that the side of a square is measured with a ruler to be 10 inches with a measurement error of at most $\pm\frac{1}{32}$ of an inch.

(a) Use a differential to estimate the error in the computed area of the square.

(b) Compare the estimate from part (a) with the actual possible error computed using a calculating utility.

Solution (a). The side of a square x and the area of the square y are related by the equation $y = x^2$. Since $dy = 2x\,dx$, if we set $x = 10$, then $dy = 20\,dx$. To say that the measurement error is at most $\pm\frac{1}{32}$ of an inch means that the measurement error $dx = x_0 - x_a$ satisfies the inequalities $-\frac{1}{32} \le dx \le \frac{1}{32}$. Multiplying each term by 20 yields the equivalent inequalities

$$20\left(-\tfrac{1}{32}\right) \le dy \le 20\left(\tfrac{1}{32}\right) \quad \text{or} \quad -\tfrac{5}{8} \le dy \le \tfrac{5}{8}$$

Since we are using the differential dy to approximate the propagated error, we estimate this propagated error to be between $-\frac{5}{8}$ and $\frac{5}{8}$ of a square inch. In other words, we estimate the propagated error to be at most $\pm\frac{5}{8}$ of a square inch.

Solution (b). The area of the square is computed to be 100 square inches, but the actual area could be as much as

$$\left(10 + \tfrac{1}{32}\right)^2 = 100 + \tfrac{5}{8} + \tfrac{1}{1024} \quad \text{or as little as} \quad \left(10 - \tfrac{1}{32}\right)^2 = 100 - \tfrac{5}{8} + \tfrac{1}{1024}$$

The propagated error is therefore between $-\frac{5}{8} + \frac{1}{1024}$ and $\frac{5}{8} + \frac{1}{1024}$. Therefore, the upper and lower bounds for the propagated error that we found in part (a) differ from the actual upper and lower bounds by $\frac{1}{1024}$ of a square inch. ◀

FOR THE READER. Examine a ruler and explain why a measurement error of at most $\frac{1}{32}$ of an inch is reasonable.

The ratio of the error in some measured or calculated quantity to the true value of the quantity is called the ***relative error*** of the measurement or calculation. When expressed as a percentage, the relative error is called the ***percentage error***. For example, suppose that the side of a square is measured to be 10 inches, but the actual length of the side is 9.98 inches. The relative error in this measurement is then $0.02/9.98 \approx 0.002004008$ or about 0.2004008%. However, as a practical matter the relative error cannot be computed exactly, since both the error and the true value of the quantity are usually unknown. To approximate the relative error in the measurement or computation of some quantity q, we use the ratio dq/q, where q is the measured or calculated value of the quantity. If q is a measured quantity, the numerator dq of this ratio denotes a measurement error, and if q is a computed quantity, dq is an estimate of the propagated error given by (6).

Example 6 The radius of a sphere is measured with a percentage error within ±0.04%. Estimate the percentage error in the calculated volume of the sphere.

Solution. The volume V of a sphere is $V = \frac{4}{3}\pi r^3$, so $dV = 4\pi r^2\, dr$. It then follows from the formulas for V and dV that

$$\frac{dV}{V} = \frac{4\pi r^2\, dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r}$$

If dr denotes the error in measurement of the radius of the sphere, then the relative error in this measurement is estimated by the ratio dr/r, where r is the measured value of the radius. Our assumption that the percentage error in this measurement is within ±0.04% then becomes $-0.0004 \le dr/r \le 0.0004$. Multiplying each term by 3 yields the equivalent inequalities

$$-0.0012 = 3(-0.0004) \le dV/V \le 3(0.0004) = 0.0012$$

Since we are using dV/V to approximate the relative error in the calculated volume of the sphere, we estimate this percentage error to be within ±0.12%. ◀

MORE NOTATION; DIFFERENTIAL FORMULAS

The symbol df is another common notation for the differential of a function $y = f(x)$. For example, if $f(x) = \sin x$, then we can write $df = \cos x\, dx$. We can also view the symbol "d" as an *operator* that acts on a function to produce the corresponding differential.

For example, $d[x^2] = 2x\,dx$, $d[\sin x] = \cos x\,dx$, and so on. All of the general rules of differentiation then have corresponding differential versions:

DERIVATIVE FORMULA	DIFFERENTIAL FORMULA
$\dfrac{d}{dx}[c] = 0$	$d[c] = 0$
$\dfrac{d}{dx}[cf] = c\dfrac{df}{dx}$	$d[cf] = c\,df$
$\dfrac{d}{dx}[f+g] = \dfrac{df}{dx} + \dfrac{dg}{dx}$	$d[f+g] = df + dg$
$\dfrac{d}{dx}[fg] = f\dfrac{dg}{dx} + g\dfrac{df}{dx}$	$d[fg] = f\,dg + g\,df$
$\dfrac{d}{dx}\left[\dfrac{f}{g}\right] = \dfrac{g\dfrac{df}{dx} - f\dfrac{dg}{dx}}{g^2}$	$d\left[\dfrac{f}{g}\right] = \dfrac{g\,df - f\,dg}{g^2}$

For example,

$$\begin{aligned} d[x^2 \sin x] &= (x^2 \cos x + 2x \sin x)\,dx \\ &= x^2(\cos x\,dx) + (2x\,dx)\sin x \\ &= x^2 d[\sin x] + (\sin x)d[x^2] \end{aligned}$$

illustrates the differential version of the product rule.

EXERCISE SET 3.8 Graphing Calculator

1. (a) Use Formula (1) to obtain the local linear approximation of x^3 at $x_0 = 1$.
   (b) Use Formula (2) to rewrite the approximation obtained in part (a) in terms of Δx.
   (c) Use the result obtained in part (a) to approximate $(1.02)^3$, and confirm that the formula obtained in part (b) produces the same result.
2. (a) Use Formula (1) to obtain the local linear approximation of $1/x$ at $x_0 = 2$.
   (b) Use Formula (2) to rewrite the approximation obtained in part (a) in terms of Δx.
   (c) Use the result obtained in part (a) to approximate $1/2.05$, and confirm that the formula obtained in part (b) produces the same result.
3. (a) Find the local linear approximation of $f(x) = \sqrt{1+x}$ at $x_0 = 0$, and use it to approximate $\sqrt{0.9}$ and $\sqrt{1.1}$.
   (b) Graph f and its tangent line at x_0 together, and use the graphs to illustrate the relationship between the exact values and the approximations of $\sqrt{0.9}$ and $\sqrt{1.1}$.
4. (a) Find the local linear approximation of $f(x) = 1/\sqrt{x}$ at $x_0 = 4$, and use it to approximate $1/\sqrt{3.9}$ and $1/\sqrt{4.1}$.
   (b) Graph f and its tangent line at x_0 together, and use the graphs to illustrate the relationship between the exact values and the approximations of $1/\sqrt{3.9}$ and $1/\sqrt{4.1}$.

In Exercises 5–8, confirm that the stated formula is the local linear approximation at $x_0 = 0$.

5. $(1+x)^{15} \approx 1 + 15x$
6. $\dfrac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x$
7. $\tan x \approx x$
8. $\dfrac{1}{1+x} \approx 1 - x$

In Exercises 9–12, confirm that the stated formula is the local linear approximation of f at $x_0 = 1$, where $\Delta x = x - 1$.

9. $f(x) = x^4$; $(1+\Delta x)^4 \approx 1 + 4\Delta x$
10. $f(x) = \sqrt{x}$; $\sqrt{1+\Delta x} \approx 1 + \frac{1}{2}\Delta x$
11. $f(x) = \dfrac{1}{2+x}$; $\dfrac{1}{3+\Delta x} \approx \dfrac{1}{3} - \dfrac{1}{9}\Delta x$
12. $f(x) = (4+x)^3$; $(5+\Delta x)^3 \approx 125 + 75\Delta x$

In Exercises 13–16, confirm that the formula is the local linear approximation at $x_0 = 0$, and use a graphing utility to estimate an interval of x-values on which the error is at most ± 0.1.

13. $\sqrt{x+3} \approx \sqrt{3} + \dfrac{1}{2\sqrt{3}}x$
14. $\dfrac{1}{\sqrt{9-x}} \approx \dfrac{1}{3} + \dfrac{1}{54}x$
15. $\tan 2x \approx 2x$
16. $\dfrac{1}{(1+2x)^5} \approx 1 - 10x$

17. (a) Use the local linear approximation of $\sin x$ at $x_0 = 0$ obtained in Example 2 to approximate $\sin 1^\circ$, and compare the approximation to the result produced directly by your calculating device.
(b) How would you choose x_0 to approximate $\sin 44^\circ$?
(c) Approximate $\sin 44^\circ$; compare the approximation to the result produced directly by your calculating device.

18. (a) Use the local linear approximation of $\tan x$ at $x_0 = 0$ to approximate $\tan 2^\circ$, and compare the approximation to the result produced directly by your calculating device.
(b) How would you choose x_0 to approximate $\tan 61^\circ$?
(c) Approximate $\tan 61^\circ$; compare the approximation to the result produced directly by your calculating device.

In Exercises 19–27, use an appropriate local linear approximation to estimate the value of the given quantity.

19. $(3.02)^4$ **20.** $(1.97)^3$ **21.** $\sqrt{65}$
22. $\sqrt{24}$ **23.** $\sqrt{80.9}$ **24.** $\sqrt{36.03}$
25. $\sin 0.1$ **26.** $\tan 0.2$ **27.** $\cos 31^\circ$

28. The approximation $(1 + x)^k \approx 1 + kx$ is commonly used by engineers for quick calculations.
(a) Derive this result, and use it to make a rough estimate of $(1.001)^{37}$.
(b) Compare your estimate to that produced directly by your calculating device.
(c) Show that this formula produces a very bad estimate of $(1.1)^{37}$, and explain why.

29. (a) Let $y = x^2$. Find dy and Δy at $x = 2$ with $dx = \Delta x = 1$.
(b) Sketch the graph of $y = x^2$, showing dy and Δy in the picture.

30. (a) Let $y = x^3$. Find dy and Δy at $x = 1$ with $dx = \Delta x = 1$.
(b) Sketch the graph of $y = x^3$, showing dy and Δy in the picture.

31. (a) Let $y = 1/x$. Find dy and Δy at $x = 1$ with $dx = \Delta x = -0.5$.
(b) Sketch the graph of $y = 1/x$, showing dy and Δy in the picture.

32. (a) Let $y = \sqrt{x}$. Find dy and Δy at $x = 9$ with $dx = \Delta x = -1$.
(b) Sketch the graph of $y = \sqrt{x}$, showing dy and Δy in the picture.

In Exercises 33–36, find formulas for dy and Δy.

33. $y = x^3$ **34.** $y = 8x - 4$
35. $y = x^2 - 2x + 1$ **36.** $y = \sin x$

In Exercises 37–40, find the differential dy.

37. (a) $y = 4x^3 - 7x^2$ (b) $y = x \cos x$
38. (a) $y = 1/x$ (b) $y = 5 \tan x$
39. (a) $y = x\sqrt{1-x}$ (b) $y = (1 + x)^{-17}$
40. (a) $y = \dfrac{1}{x^3 - 1}$ (b) $y = \dfrac{1 - x^3}{2 - x}$

In Exercises 41–44, use dy to approximate Δy when x changes as indicated.

41. $y = \sqrt{3x - 2}$; from $x = 2$ to $x = 2.03$
42. $y = \sqrt{x^2 + 8}$; from $x = 1$ to $x = 0.97$
43. $y = \dfrac{x}{x^2 + 1}$; from $x = 2$ to $x = 1.96$
44. $y = x\sqrt{8x + 1}$; from $x = 3$ to $x = 3.05$

45. The side of a square is measured to be 10 ft, with a possible error of ± 0.1 ft.
(a) Use differentials to estimate the error in the calculated area.
(b) Estimate the percentage errors in the side and the area.

46. The side of a cube is measured to be 25 cm, with a possible error of ± 1 cm.
(a) Use differentials to estimate the error in the calculated volume.
(b) Estimate the percentage errors in the side and volume.

47. The hypotenuse of a right triangle is known to be 10 in exactly, and one of the acute angles is measured to be 30°, with a possible error of $\pm 1^\circ$.
(a) Use differentials to estimate the errors in the sides opposite and adjacent to the measured angle.
(b) Estimate the percentage errors in the sides.

48. One side of a right triangle is known to be 25 cm exactly. The angle opposite to this side is measured to be 60°, with a possible error of $\pm 0.5^\circ$.
(a) Use differentials to estimate the errors in the adjacent side and the hypotenuse.
(b) Estimate the percentage errors in the adjacent side and hypotenuse.

49. The electrical resistance R of a certain wire is given by $R = k/r^2$, where k is a constant and r is the radius of the wire. Assuming that the radius r has a possible error of $\pm 5\%$, use differentials to estimate the percentage error in R. (Assume k is exact.)

50. A 12-foot ladder leaning against a wall makes an angle θ with the floor. If the top of the ladder is h feet up the wall, express h in terms of θ and then use dh to estimate the change in h if θ changes from 60° to 59°.

51. The area of a right triangle with a hypotenuse of H is calculated using the formula $A = \frac{1}{4}H^2 \sin 2\theta$, where θ is one of the acute angles. Use differentials to approximate the error in calculating A if $H = 4$ cm (exactly) and θ is measured to be 30°, with a possible error of $\pm 15'$.

52. The side of a square is measured with a possible percentage error of $\pm 1\%$. Use differentials to estimate the percentage error in the area.

53. The side of a cube is measured with a possible percentage error of ±2%. Use differentials to estimate the percentage error in the volume.

54. The volume of a sphere is to be computed from a measured value of its radius. Estimate the maximum permissible percentage error in the measurement if the percentage error in the volume must be kept within ±3%. ($V = \frac{4}{3}\pi r^3$ is the volume of a sphere of radius r.)

55. The area of a circle is to be computed from a measured value of its diameter. Estimate the maximum permissible percentage error in the measurement if the percentage error in the area must be kept within ±1%.

56. A steel cube with 1-in sides is coated with 0.01 in of copper. Use differentials to estimate the volume of copper in the coating. [*Hint:* Let ΔV be the change in the volume of the cube.]

57. A metal rod 15 cm long and 5 cm in diameter is to be covered (except for the ends) with insulation that is 0.001 cm thick. Use differentials to estimate the volume of insulation. [*Hint:* Let ΔV be the change in volume of the rod.]

58. The time required for one complete oscillation of a pendulum is called its ***period***. If L is the length of the pendulum, then the period is given by $P = 2\pi\sqrt{L/g}$, where g is a constant called *the acceleration due to gravity*. Use differentials to show that the percentage error in P is approximately half the percentage error in L.

59. If the temperature T of a metal rod of length L is changed by an amount ΔT, then the length will change by the amount $\Delta L = \alpha L\,\Delta T$, where α is called the ***coefficient of linear expansion***. For moderate changes in temperature α is taken as constant.

(a) Suppose that a rod 40 cm long at 20°C is found to be 40.006 cm long when the temperature is raised to 30°C. Find α.

(b) If an aluminum pole is 180 cm long at 15°C, how long is the pole if the temperature is raised to 40°C? [Take $\alpha = 2.3 \times 10^{-5}/^\circ\text{C}$.]

60. If the temperature T of a solid or liquid of volume V is changed by an amount ΔT, then the volume will change by the amount $\Delta V = \beta V\,\Delta T$, where β is called the ***coefficient of volume expansion***. For moderate changes in temperature β is taken as constant. Suppose that a tank truck loads 4000 gallons of ethyl alcohol at a temperature of 35°C and delivers its load sometime later at a temperature of 15°C. Using $\beta = 7.5 \times 10^{-4}/^\circ\text{C}$ for ethyl alcohol, find the number of gallons delivered.

SUPPLEMENTARY EXERCISES

Graphing Calculator   C CAS

1. State the definition of a derivative, and give two interpretations of it.

2. Explain the difference between average and instantaneous rate of change, and discuss how they are calculated.

3. Given that $y = f(x)$, explain the difference between dy and Δy. Draw a picture that illustrates the relationship between these quantities.

4. Use the definition of a derivative to find dy/dx, and check your answer by calculating the derivative using appropriate derivative formulas.

(a) $y = \sqrt{9-4x}$ (b) $y = \dfrac{x}{x+1}$

In Exercises 5–8, find the values of x at which the curve $y = f(x)$ has a horizontal tangent line.

5. $f(x) = (2x+7)^6(x-2)^5$ **6.** $f(x) = \dfrac{(x-3)^4}{x^2+2x}$

7. $f(x) = \sqrt{3x+1}(x-1)^2$ **8.** $f(x) = \left(\dfrac{3x+1}{x^2}\right)^3$

9. The accompanying figure shows the graph of $y = f'(x)$ for an unspecified function f.

(a) For what values of x does the curve $y = f(x)$ have a horizontal tangent line?

(b) Over what intervals does the curve $y = f(x)$ have tangent lines with positive slope?

(c) Over what intervals does the curve $y = f(x)$ have tangent lines with negative slope?

(d) Given that $g(x) = f(x)\sin x$, and $f(0) = -1$, find $g''(0)$.

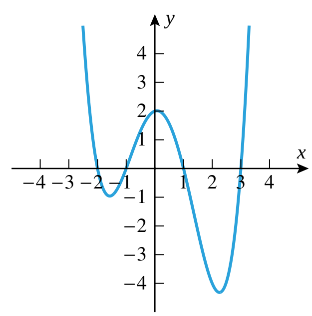


Figure Ex-9

10. In each part, evaluate the expression given that $f(1) = 1$, $g(1) = -2$, $f'(1) = 3$, and $g'(1) = -1$.

(a) $\left.\dfrac{d}{dx}[f(x)g(x)]\right|_{x=1}$ (b) $\left.\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]\right|_{x=1}$

(c) $\left.\dfrac{d}{dx}[\sqrt{f(x)}]\right|_{x=1}$ (d) $\dfrac{d}{dx}[f(1)g'(1)]$

11. Find the equations of all lines through the origin that are tangent to the curve $y = x^3 - 9x^2 - 16x$.

12. Find all values of x for which the tangent line to $y = 2x^3 - x^2$ is perpendicular to the line $x + 4y = 10$.

13. Find all values of x for which the line that is tangent to $y = 3x - \tan x$ is parallel to the line $y - x = 2$.

14. Suppose that $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ k(x-1), & x > 1. \end{cases}$

For what values of k is f
(a) continuous? (b) differentiable?

15. Let $f(x) = x^2$. Show that for any distinct values of a and b, the slope of the tangent line to $y = f(x)$ at $x = \frac{1}{2}(a+b)$ is equal to the slope of the secant line through the points (a, a^2) and (b, b^2). Draw a picture to illustrate this result.

16. A car is traveling on a straight road that is 120 mi long. For the first 100 mi the car travels at an average velocity of 50 mi/h. Show that no matter how fast the car travels for the final 20 mi it cannot bring the average velocity up to 60 mi/h for the entire trip.

17. In each part, use the given information to find Δx, Δy, and dy.
(a) $y = 1/(x-1)$; x decreases from 2 to 1.5.
(b) $y = \tan x$; x increases from $-\pi/4$ to 0.
(c) $y = \sqrt{25 - x^2}$; x increases from 0 to 3.

18. Use the formula $V = l^3$ for the volume of a cube of side l to find
(a) the average rate at which the volume of a cube changes with l as l increases from $l = 2$ to $l = 4$
(b) the instantaneous rate at which the volume of a cube changes with l when $l = 5$.

19. The amount of water in a tank t minutes after it has started to drain is given by $W = 100(t - 15)^2$ gal.
(a) At what rate is the water running out at the end of 5 min?
(b) What is the average rate at which the water flows out during the first 5 min?

20. Use an appropriate local linear approximation to estimate the value of $\cot 46^\circ$, and compare your answer to the value obtained with a calculating device.

21. The base of the Great Pyramid at Giza is a square that is 230 m on each side.
(a) As illustrated in the accompanying figure, suppose that an archaeologist standing at the center of a side measures the angle of elevation of the apex to be $\phi = 51^\circ$ with an error of $\pm 0.5^\circ$. What can the archaeologist reasonably say about the height of the pyramid?
(b) Use differentials to estimate the allowable error in the elevation angle that will ensure an error in the height is at most ± 5 m.

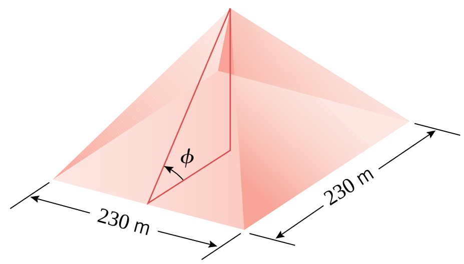


Figure Ex-21

22. The period T of a clock pendulum (i.e., the time required for one back-and-forth movement) is given in terms of its length L by $T = 2\pi\sqrt{L/g}$, where g is the gravitational constant.
(a) Assuming that the length of a clock pendulum can vary (say, due to temperature changes), find the rate of change of the period T with respect to the length L.
(b) If L is in meters (m) and T is in seconds (s), what are the units for the rate of change in part (a)?
(c) If a pendulum clock is running slow, should the length of the pendulum be increased or decreased to correct the problem?
(d) The constant g generally decreases with altitude. If you move a pendulum clock from sea level to a higher elevation, will it run faster or slower?
(e) Assuming the length of the pendulum to be constant, find the rate of change of the period T with respect to g.
(f) Assuming that T is in seconds (s) and g is in meters per second per second (m/s^2), find the units for the rate of change in part (e).

In Exercises 23 and 24, zoom in on the graph of f on an interval containing $x = x_0$ until the graph looks like a straight line. Estimate the slope of this line and then check your answer by finding the exact value of $f'(x_0)$.

23. (a) $f(x) = x^2 - 1$, $x_0 = 1.8$
(b) $f(x) = \dfrac{x^2}{x-2}$, $x_0 = 3.5$

24. (a) $f(x) = x^3 - x^2 + 1$, $x_0 = 2.3$
(b) $f(x) = \dfrac{x}{x^2+1}$, $x_0 = -0.5$

In Exercises 25 and 26, approximate $f'(2)$ by considering the difference quotients

$$\frac{f(x_1) - f(2)}{x_1 - 2}$$

for values of x_1 near 2. If you have a CAS, see if it can find the exact value of the limit of these difference quotients as $x_1 \to 2$.

25. $f(x) = 2^x$

26. $f(x) = x^{\sin x}$

27. At time $t = 0$ a car moves into the passing lane to pass a slow-moving truck. The average velocity of the car from $t = 1$ to $t = 1 + h$ is

$$v_{\text{ave}} = \frac{3(h+1)^{2.5} + 580h - 3}{10h}$$

Estimate the instantaneous velocity of the car at $t = 1$, where time is in seconds and distance is in feet.

28. A sky diver jumps from an airplane. Suppose that the distance she falls during the first t seconds before her parachute opens is $s(t) = 986((0.835)^t - 1) + 176t$, where s is in feet and $t \geq 1$. Graph s versus t for $1 \leq t \leq 20$, and use your graph to estimate the instantaneous velocity at $t = 15$.

29. Approximate the values of x at which the tangent line to the graph of $y = x^3 - \sin x$ is horizontal.

30. Use a graphing utility to graph the function

$$f(x) = |x^4 - x - 1| - x$$

and find the values of x where the derivative of this function does not exist.

31. Use a CAS to find the derivative of f from the definition

$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

and check the result by finding the derivative by hand.

(a) $f(x) = x^5$ (b) $f(x) = 1/x$

(c) $f(x) = 1/\sqrt{x}$ (d) $f(x) = \dfrac{2x+1}{x-1}$

(e) $f(x) = \sqrt{3x^2 + 5}$ (f) $f(x) = \sin 3x$

In Exercises 32–37: (a) Use a CAS to find $f'(x)$ via Definition 3.2.3; (b) use the CAS to find $f''(x)$.

32. $f(x) = x^2 \sin x$

33. $f(x) = \sqrt{x} + \cos^2 x$

34. $f(x) = \dfrac{2x^2 - x + 5}{3x + 2}$

35. $f(x) = \dfrac{\tan x}{1 + x^2}$

36. $f(x) = \dfrac{1}{x} \sin \sqrt{x}$

37. $f(x) = \dfrac{\sqrt{x^4 - 3x + 2}}{x(2 - \cos x)}$

In Exercises 38 and 39, find the equation of the tangent line at the specified point.

38. $x^{2/3} - y^{2/3} - y = 1$; $(1, -1)$

39. $\sin xy = y$; $(\pi/2, 1)$

40. The hypotenuse of a right triangle is growing at a constant rate of a centimeters per second and one leg is decreasing at a constant rate of b centimeters per second. How fast is the acute angle between the hypotenuse and the other leg changing at the instant when both legs are 1 cm?

EXPANDING THE CALCULUS HORIZON

Robotics

Robin designs and sells room dividers to defray college expenses. She is soon overwhelmed with orders and decides to build a robot to spray paint her dividers. As in most engineering projects, Robin begins with a simplified model that she will eventually refine to be more realistic. However, Robin quickly discovers that robotics (the design and control of robots) involves a considerable amount of mathematics, some of which we will discuss in this module.

The Design Plan

Robin's plan is to develop a two-dimensional version of the robot arm in Figure 1. As shown in Figure 2, Robin's robot arm will consist of two links of fixed length, each of which will rotate independently about a pivot point. A paint sprayer will be attached to the end of the second link, and a computer will vary the angles θ_1 and θ_2, thereby allowing the robot to paint a region of the xy-plane.

The Mathematical Analysis

To analyze the motion of the robot arm, Robin denotes the coordinates of the paint sprayer by (x, y), as in Figure 3, and she derives the following equations that express x and y in terms of the angles θ_1 and θ_2 and the lengths l_1 and l_2 of the links:

$$\begin{aligned} x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{aligned} \qquad (1)$$

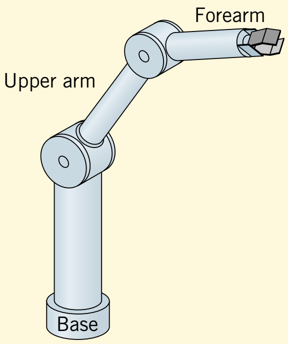


Figure 1

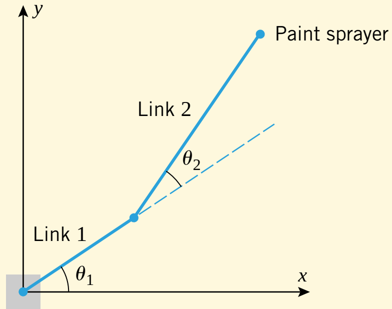


Figure 2

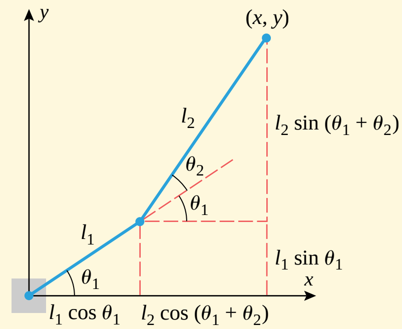


Figure 3

Exercise 1 Use Figure 3 to confirm the equations in (1).

In the language of robotics, θ_1 and θ_2 are called the ***control angles***, the point (x, y) is called the ***end effector***, and the equations in (1) are called the ***forward kinematic equations*** (from the Greek word *kinema*, meaning "motion").

Exercise 2 What is the region of the plane that can be reached by the end effector if: (a) $l_1 = l_2$, (b) $l_1 > l_2$, and (c) $l_1 < l_2$?

Exercise 3 What are the coordinates of the end effector if $l_1 = 2$, $l_2 = 3$, $\theta_1 = \pi/4$, and $\theta_2 = \pi/6$?

Simulating Paint Patterns

Robin recognizes that if θ_1 and θ_2 are regarded as functions of time, then the forward kinematic equations can be expressed as

$$x = l_1 \cos\theta_1(t) + l_2 \cos(\theta_1(t) + \theta_2(t))$$
$$y = l_1 \sin\theta_1(t) + l_2 \sin(\theta_1(t) + \theta_2(t))$$

which are parametric equations for the curve traced by the end effector. For example, if the arms extend horizontally along the positive x-axis at time $t = 0$, and if links 1 and 2 rotate at the constant rates of ω_1 and ω_2 radians per second (rad/s), respectively, then

$$\theta_1(t) = \omega_1 t \quad \text{and} \quad \theta_2(t) = \omega_2 t$$

and the parametric equations of motion for the end effector become

$$x = l_1 \cos\omega_1 t + l_2 \cos(\omega_1 t + \omega_2 t)$$
$$y = l_1 \sin\omega_1 t + l_2 \sin(\omega_1 t + \omega_2 t)$$

Exercise 4 Show that if $l_1 = l_2 = 1$, and if $\omega_1 = 2$ rad/s and $\omega_2 = 3$ rad/s, then the parametric equations of motion are

$$x = \cos 2t + \cos 5t$$
$$y = \sin 2t + \sin 5t$$

Use a graphing utility to show that the curve traced by the end effector over the time interval $0 \le t \le 2\pi$ is as shown in Figure 4. This would be the painting pattern of Robin's paint sprayer.

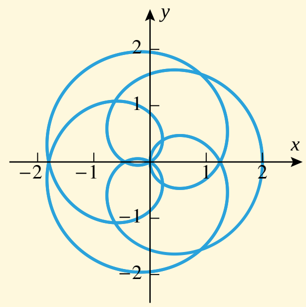


Figure 4

Exercise 5 Use a graphing utility to explore how the rotation rates of the links affect the spray patterns of a robot arm for which $l_1 = l_2 = 1$.

Exercise 6 Suppose that $l_1 = l_2 = 1$, and a malfunction in the robot arm causes the second link to lock at $\theta_2 = 0$, while the first link rotates at a constant rate of 1 rad/s. Make a conjecture about the path of the end effector, and confirm your conjecture by finding parametric equations for its motion.

Controlling the Position of the End Effector

Robin's plan is to make the robot paint the dividers in vertical strips, sweeping from the bottom up. After a strip is painted, she will have the arm return to the bottom of the divider and then move horizontally to position itself for the next upward sweep. Since the sections of her dividers will be 3 ft wide by 5 ft high, Robin decides on a robot with two 3-ft links whose base is positioned near the lower left corner of a divider section, as in Figure 5*a*. Since the fully extended links span a radius of 6 ft, she feels that this arrangement will work.

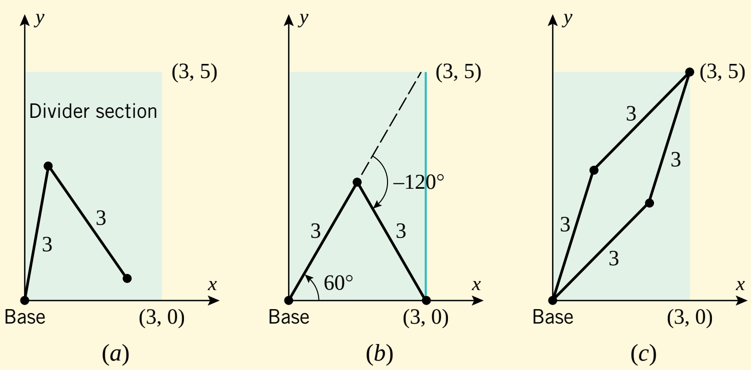


Figure 5

Robin starts with the problem of painting the far right edge from (3, 0) to (3, 5). With the help of some basic geometry (Figure 5*b*), she determines that the end effector can be placed at the point (3, 0) by taking the control angles to be $\theta_1 = \pi/3\ (= 60^\circ)$ and $\theta_2 = -2\pi/3\ (= -120^\circ)$ (verify). However, the problem of finding the control angles that correspond to the point (3, 5) is more complicated, so she starts by substituting the link lengths $l_1 = l_2 = 3$ into the forward kinematic equations in (1) to obtain

$$\begin{aligned} x &= 3\cos\theta_1 + 3\cos(\theta_1 + \theta_2) \\ y &= 3\sin\theta_1 + 3\sin(\theta_1 + \theta_2) \end{aligned} \qquad (2)$$

Thus, to put the end effector at the point (3, 5), the control angles must satisfy the equations

$$\begin{aligned}\cos\theta_1 + \cos(\theta_1+\theta_2) &= 1\\ 3\sin\theta_1 + 3\sin(\theta_1+\theta_2) &= 5\end{aligned} \tag{3}$$

Solving these equations for θ_1 and θ_2 challenges Robin's algebra and trigonometry skills, but she manages to do it using the procedure in the following exercise.

Exercise 7

(a) Use the equations in (3) and the identity

$$\sin^2(\theta_1+\theta_2) + \cos^2(\theta_1+\theta_2) = 1$$

to show that

$$15\sin\theta_1 + 9\cos\theta_1 = 17$$

(b) Solve the last equation for $\sin\theta_1$ in terms of $\cos\theta_1$ and substitute in the identity

$$\sin^2\theta_1 + \cos^2\theta_1 = 1$$

to obtain

$$153\cos^2\theta_1 - 153\cos\theta_1 + 32 = 0$$

(c) Treat this as a quadratic equation in $\cos\theta_1$, and use the quadratic formula to obtain

$$\cos\theta_1 = \frac{1}{2} \pm \frac{5\sqrt{17}}{102}$$

(d) Use the arccosine (inverse cosine) operation of a calculating utility to solve the equations in part (c) to obtain

$$\theta_1 \approx 0.792436 \text{ rad} \approx 45.4032^\circ \quad \text{and} \quad \theta_1 \approx 1.26832 \text{ rad} \approx 72.6693^\circ$$

(e) Substitute each of these angles into the first equation in (3), and solve for the corresponding values of θ_2.

At first, Robin was surprised that the solutions for θ_1 and θ_2 were not unique, but her sketch in Figure 5*c* quickly made it clear that there will ordinarily be two ways of positioning the links to put the end effector at a specified point.

Controlling the Motion of the End Effector

Now that Robin has figured out how to place the end effector at the points (3, 0) and (3, 5), she turns to the problem of making the robot paint the vertical line segment between those points. She recognizes that not only must she make the end effector move on a vertical line, but she must control its velocity—if the end effector moves too quickly, the paint will be too thin, and if it moves too slowly, the paint will be too thick.

After some experimentation, she decides that the end effector should have a constant velocity of 1 ft/s. Thus, Robin's mathematical problem is to determine the rotation rates $d\theta_1/dt$ and $d\theta_2/dt$ (in rad/s) that will make $dx/dt = 0$ and $dy/dt = 1$. The first condition will ensure that the end effector moves vertically (no horizontal velocity), and the second condition will ensure that it moves upward at 1 ft/s.

To find formulas for dx/dt and dy/dt, Robin uses the chain rule to differentiate the forward kinematic equations in (2) and obtains

$$\begin{aligned}\frac{dx}{dt} &= -3\sin\theta_1\frac{d\theta_1}{dt} - [3\sin(\theta_1+\theta_2)]\left(\frac{d\theta_1}{dt}+\frac{d\theta_2}{dt}\right)\\ \frac{dy}{dt} &= 3\cos\theta_1\frac{d\theta_1}{dt} + [3\cos(\theta_1+\theta_2)]\left(\frac{d\theta_1}{dt}+\frac{d\theta_2}{dt}\right)\end{aligned}$$

She uses the forward kinematic equations again to simplify these formulas and she then substitutes $dx/dt = 0$ and $dy/dt = 1$ to obtain

$$\begin{aligned} -y\frac{d\theta_1}{dt} - 3\sin(\theta_1+\theta_2)\frac{d\theta_2}{dt} &= 0 \\ x\frac{d\theta_1}{dt} + 3\cos(\theta_1+\theta_2)\frac{d\theta_2}{dt} &= 1 \end{aligned} \tag{4}$$

Exercise 8 Confirm Robin's computations.

The equations in (4) will be used in the following way: At a given time t, the robot will report the control angles θ_1 and θ_2 of its links to the computer, the computer will use the forward kinematic equations in (2) to calculate the x- and y-coordinates of the end effector, and then the values of θ_1, θ_2, x, and y will be substituted into (4) to produce two equations in the two unknowns $d\theta_1/dt$ and $d\theta_2/dt$. The computer will solve these equations to determine the required rotation rates for the links.

Exercise 9 In each part, use the given information to sketch the position of the links, and then calculate the rotation rates for the links in rad/s that will make the end effector of Robin's robot move upward with a velocity of 1 ft/s from that position.

(a) $\theta_1 = \pi/3,\ \theta_2 = -2\pi/3$ (b) $\theta_1 = \pi/2,\ \theta_2 = -\pi/2$

Module by Mary Ann Connors, USMA, West Point, and Howard Anton, Drexel University, and based on the article "Moving a Planar Robot Arm" by Walter Meyer, MAA Notes Number 29, The Mathematical Association of America, 1993.