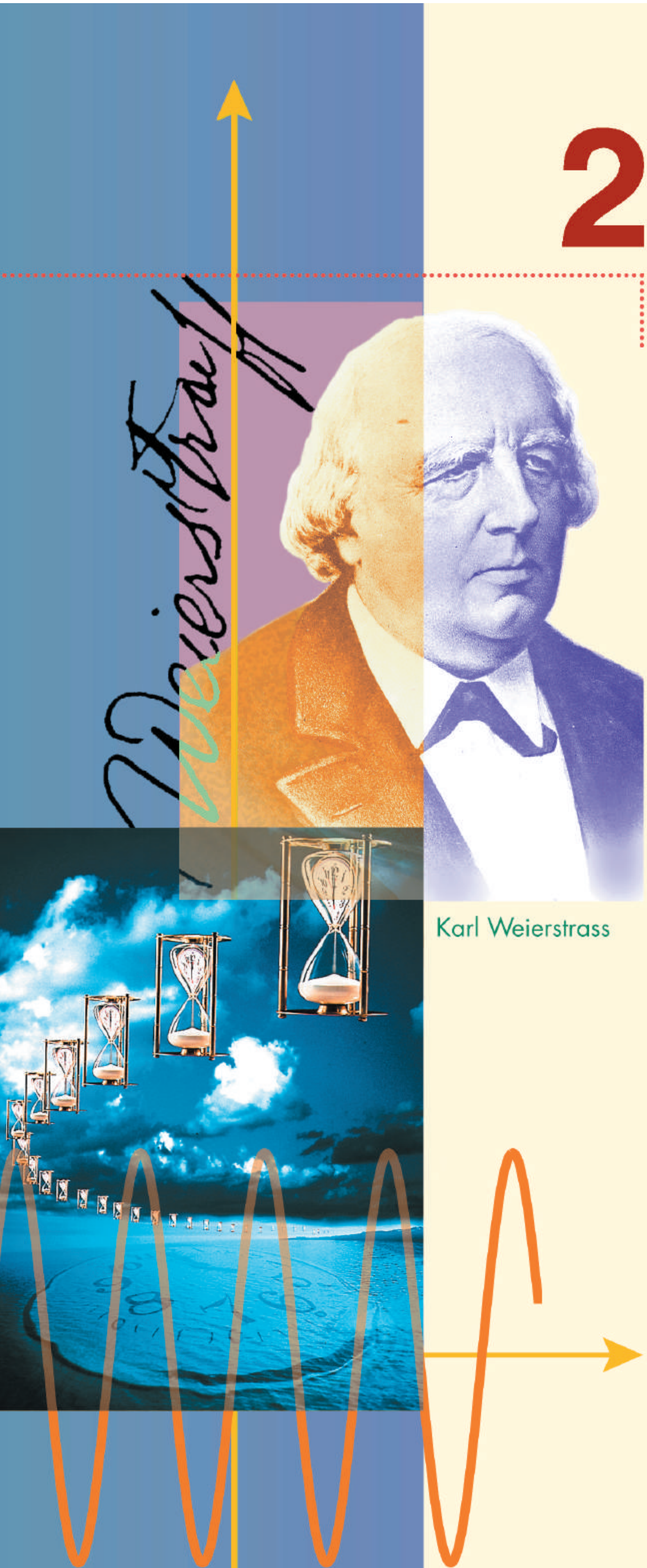


Karl Weierstrass

2

LIMITS AND CONTINUITY

The problem of defining and calculating instantaneous rates such as speed and acceleration attracted almost all the mathematicians of the seventeenth century.

—Morris Kline

The development of calculus in the seventeenth century by Newton and Leibniz provided scientists with their first real understanding of what is meant by an "instantaneous rate of change" such as velocity and acceleration. Once the idea was understood conceptually, efficient computational methods followed, and science took a quantum leap forward. The fundamental building block on which rates of change rest is the concept of a "limit," an idea that is so important that all other calculus concepts are now based on it.

In this chapter we will develop the concept of a limit in stages, proceeding from an informal, intuitive notion to a precise mathematical definition. We will also develop theorems and procedures for calculating limits, and we will conclude the chapter by using the limits to study "continuous" curves.

2.1 LIMITS (AN INTUITIVE APPROACH)

The concept of a limit is the fundamental building block on which all other calculus concepts are based. In this section we will study limits informally, with the goal of developing an "intuitive feel" for the basic ideas. In the following three sections we will focus on the computational methods and precise definitions.

INSTANTANEOUS VELOCITY AND THE SLOPE OF A CURVE

Recall from Formula (11) of Section 1.5 that if a particle moves along an s-axis, then the average velocity v_{ave} over the time interval from t_0 to t_1 is defined as

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{s_1 - s_0}{t_1 - t_0} \tag{1}$$

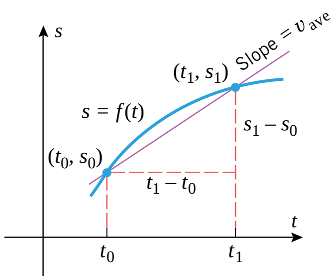


Figure 2.1.1

where s_0 and s_1 are the s-coordinates of the particle at times t_0 and t_1, respectively. Geometrically, v_{ave} is the slope of the line joining the points (t_0, s_0) and (t_1, s_1) on the position versus time curve for the particle (Figure 2.1.1).

Suppose, however, that we are not interested in average velocity over a time interval, but rather the velocity v_{inst} at a specific instant in time. It is not a simple matter of applying Formula (1), since the displacement and the elapsed time in an instant are both zero. However, intuition suggests that the velocity at an instant $t = t_0$ can be approximated by finding the position of the particle at a time t_1 just before, or just after, time t_0 and computing the average velocity over the brief time interval between the two moments. That is,

$$v_{\text{inst}} \approx v_{\text{ave}} = \frac{s_1 - s_0}{t_1 - t_0} \tag{2}$$

provided $\Delta t = t_1 - t_0$ is small. Moreover, if we are able to make very precise measurements, the closer t_1 is to t_0, the better v_{ave} approximates v_{inst}. That is, as we sample at times t_1, closer and closer to t_0, v_{ave} approaches a *limiting value* that we understand to be v_{inst}.

Example 1 Suppose that a ball is thrown vertically upward and the height in feet of the ball t seconds after its release is modeled by the function

$$s(t) = -16t^2 + 29t + 6, \qquad 0 \le t \le 2$$

What is a reasonable estimate for the instantaneous velocity of the ball at time $t = 0.5$ s?

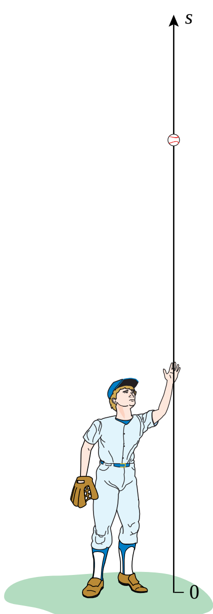


Figure 2.1.2

Solution. At any time $0 \le t \le 2$ we may envision the height $s(t)$ of the ball as a position on a (vertical) s-axis, where $s = 0$ corresponds to ground level (Figure 2.1.2). The height of the ball at time $t = 0.5$ s is $s(0.5) = 16.5$ ft, and the height of the ball 0.01 s later is $s(0.51) = 16.6284$ ft. Therefore, the average velocity of the ball over the time interval from $t = 0.5$ to $t = 0.51$ is

$$v_{\text{ave}} = \frac{16.6284 - 16.5}{0.51 - 0.5} = \frac{0.1284}{0.01} = 12.84 \text{ ft/s}$$

Similarly, the height of the ball 0.49 s after its release is $s(0.49) = 16.3684$ ft, and the average velocity of the ball over the time interval from $t = 0.49$ to $t = 0.5$ is

$$v_{\text{ave}} = \frac{16.3684 - 16.5}{0.49 - 0.5} = \frac{-0.1316}{-0.01} = 13.16 \text{ ft/s}$$

Consequently, we would expect the instantaneous velocity of the ball at time $t = 0.5$ to be between 12.84 ft/s and 13.16 ft/s. To improve our estimate of this instantaneous velocity, we can compute the average velocity

$$v_{\text{ave}}(t_1) = \frac{s(t_1) - 16.5}{t_1 - 0.5} = \frac{-16t_1^2 + 29t_1 + 6 - 16.5}{t_1 - 0.5} = \frac{-16t_1^2 + 29t_1 - 10.5}{t_1 - 0.5}$$

for values of t_1 even closer to 0.5. Table 2.1.1 displays the results of several such computa-

Table 2.1.1

TIME t_1 (s)	$v_{\text{ave}}(t_1) = \dfrac{-16t_1^2 + 29t_1 - 10.5}{t_1 - 0.5}$ (ft/s)
0.5010	12.9840
0.5005	12.9920
0.5001	12.9984
0.4999	13.0016
0.4995	13.0080
0.4990	13.0160

tions. It appears from these computations that a reasonable estimate for the instantaneous velocity of the ball at time $t = 0.5$ s is 13 ft/s. ◀

FOR THE READER. The domain of the height function $s(t) = -16t^2 + 29t + 6$ in Example 1 is the closed interval $[0, 2]$. Why do we not consider values of t less than 0 or greater than 2 for this function? In Table 2.1.1, why is there not a value of $v_{\text{ave}}(t_1)$ for $t_1 = 0.5$?

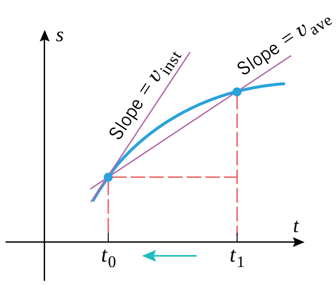


Figure 2.1.3

We can interpret v_{inst} geometrically from the interpretation of v_{ave} as the slope of the line joining the points (t_0, s_0) and (t_1, s_1) on the position versus time curve for the particle. When $\Delta t = t_1 - t_0$ is small, the points (t_0, s_0) and (t_1, s_1) are very close to each other on the curve. As the sampling point (t_1, s_1) is selected closer to our anchoring point (t_0, s_0), the slope v_{ave} more nearly approximates what we might reasonably call the *slope* of the position curve at time $t = t_0$. Thus, v_{inst} can be viewed as the slope of the position curve at time $t = t_0$ (Figure 2.1.3). We will explore this connection more fully in Section 3.1.

LIMITS

In Example 1 it appeared that choosing values of t_1 close to (but not equal to) 0.5 resulted in values of $v_{\text{ave}}(t_1)$ that were close to 13. One way of describing this behavior is to say that the *limiting value* of $v_{\text{ave}}(t_1)$ as t_1 approaches 0.5 is 13 or, equivalently, that 13 is the *limit* of $v_{\text{ave}}(t_1)$ as t_1 approaches 0.5. More generally, we will see that the concept of the limit of a function provides a foundation for the tools of calculus. Thus, it is appropriate to start a study of calculus by focusing on the limit concept itself.

The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value. For example, let us examine the behavior of the function

$$f(x) = x^2 - x + 1$$

for x-values closer and closer to 2. It is evident from the graph and table in Figure 2.1.4 that the values of $f(x)$ get closer and closer to 3 as values of x are selected closer and closer to 2 on either the left or the right side of 2. We describe this by saying that the "limit of $x^2 - x + 1$ is 3 as x approaches 2 from either side," and we write

$$\lim_{x\to 2} (x^2 - x + 1) = 3 \tag{3}$$

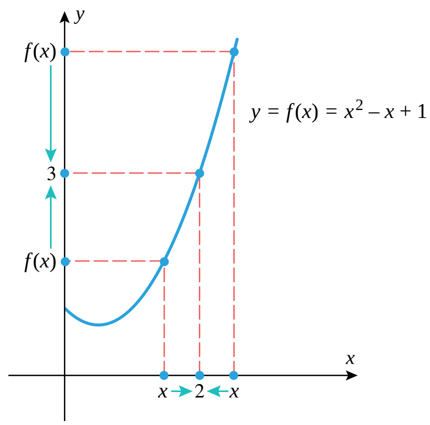
Observe that in our investigation of $\lim_{x\to 2} (x^2 - x + 1)$ we are only concerned with the values of $f(x)$ *near* $x = 2$ and not the value of $f(x)$ *at* $x = 2$.

This leads us to the following general idea.

2.1.1 LIMITS (AN INFORMAL VIEW). If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x\to a} f(x) = L \tag{4}$$

which is read "the limit of $f(x)$ as x approaches a is L."



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

Figure 2.1.4

Equation (4) is also commonly written as

$$f(x) \to L \quad \text{as} \quad x \to a$$

With this notation we can express (3) as

$$x^2 - x + 1 \to 3 \quad \text{as} \quad x \to 2$$

In order to investigate $\lim_{x\to a} f(x)$, we ask ourselves the question, "If x is close to, but different from, a, is there a particular number to which $f(x)$ is close?" This question presumes that the function f is defined "everywhere near a," in other words, that f is defined at all points x in some open interval containing a, except possibly at $x = a$. The value of f at a, if it exists at all, is not relevant to the determination of $\lim_{x\to a} f(x)$. Many important applications of the limit concept involve contexts in which the domain of the function excludes a. Indeed, our discussion of instantaneous velocity concluded that v_{inst} could be interpreted as a limit of the average velocities, even though the average velocity at an instant is not defined.

The process of determining a limit generally involves a *discovery* phase, followed by a *verification* phase. The discovery phase begins with sampled x-values, and ends with a conjecture for the limit. Figure 2.1.4 illustrates the discovery phase for the problem of finding the value of $\lim_{x\to 2}(x^2 - x + 1)$. We sampled values for x near 2 and found that the corresponding values of $f(x)$ were close to 3. Indeed, values of x nearer to 2 produced values of $f(x)$ closer to 3. Our conjecture that $\lim_{x\to 2}(x^2 - x + 1) = 3$ concluded the discovery phase for this limit. However, a complete treatment of any limit also involves a verification phase in which it is shown that the conjectured limit is actually correct. For example, consider our conjecture that $\lim_{x\to 2}(x^2 - x + 1) = 3$. We can only sample a relatively few values of x near 2, even by using a graphing utility. We cannot sample *all* values of x near 2, for no matter how close to 2 we take an x-value, there are infinitely many values of x nearer yet to 2. To verify that $\lim_{x\to 2}(x^2 - x + 1)$ is indeed 3, we need to resort to an analysis that can overcome this dilemma. This analysis will require a more mathematically precise definition of limit and is the focus of Section 2.4. In this section, we concentrate on the discovery phase for limit problems.

Example 2 Make a conjecture about the value of the limit

$$\lim_{x\to 0} \frac{x}{\sqrt{x+1} - 1} \tag{5}$$

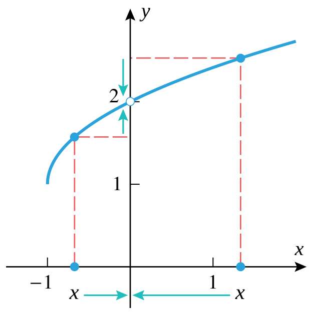


Figure 2.1.5

Solution. Observe that the function

$$f(x) = \frac{x}{\sqrt{x+1}-1}$$

is not defined at $x = 0$. However, f is defined for $x > -1$, $x \neq 0$, so the domain of f contains values of x "everywhere near 0." Table 2.1.2 shows samples of x-values approaching 0 from the left side and from the right side. In both cases the values of $f(x)$, calculated to six decimal places, appear to get closer and closer to 2, and hence we conjecture that

$$\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1} = 2 \tag{6}$$

A graphing utility could be used to produce Figure 2.1.5, providing more evidence in support of our conjecture. In the next section we will see that the graph of $f(x)$ is identical to that of $y = \sqrt{x+1}+1$, except for a hole at $(0, 2)$. ◀

Table 2.1.2

x	−0.01	−0.001	−0.0001	−0.00001	0	0.00001	0.0001	0.001	0.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988

Left side → ← Right side

FOR THE READER. Using a graphing utility, find a window about $x = 0$ in which all values of $f(x)$ are within 0.5 of $y = 2$. Find a window in which all values of $f(x)$ are within 0.1 of $y = 2$.

Example 3 Make a conjecture about the value of the limit

$$\lim_{x\to 0} \frac{\sin x}{x} \tag{7}$$

Solution. The function $f(x) = (\sin x)/x$ is not defined at $x = 0$, but, as discussed previously, this has no bearing on the limit. With the help of a calculating utility set in radian mode, we obtain the table in Figure 2.1.6.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{8}$$

The result is consistent with the graph of $f(x) = (\sin x)/x$ shown in the figure. Later in this chapter we will give a geometric argument to prove that our conjecture is correct. ◀

x (RADIANS)	$y = \dfrac{\sin x}{x}$
±1.0	0.84147
±0.9	0.87036
±0.8	0.89670
±0.7	0.92031
±0.6	0.94107
±0.5	0.95885
±0.4	0.97355
±0.3	0.98507
±0.2	0.99335
±0.1	0.99833
±0.01	0.99998

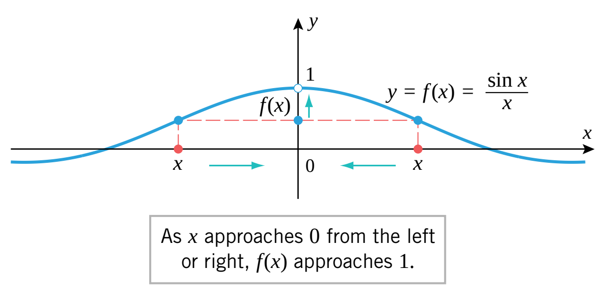


Figure 2.1.6

FOR THE READER. Use a calculating utility to sample x-values closer to 0 than in Table **??**. Does the limit change if x is in degrees?

SAMPLING PITFALLS

Although numerical and graphical evidence is helpful for guessing at limits, we can be misled by an insufficient or poorly selected sample. For example, the table in Figure 2.1.7 shows values of $f(x) = \sin(\pi/x)$ at selected values of x on both sides of 0. The data incorrectly suggest that

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 0$$

The fact that this is incorrect is evidenced by the graph of f shown in the figure. This graph indicates that as $x \to 0$, the values of f oscillate between -1 and 1 with increasing rapidity, and hence do not approach a limit. The data are deceiving because the table consists only of sample values of x that are x-intercepts for $f(x)$.

x (RADIANS)	$\frac{\pi}{x}$	$f(x) = \sin\left(\frac{\pi}{x}\right)$
$x = \pm 1$	$\pm\pi$	$\sin(\pm\pi) = 0$
$x = \pm 0.1$	$\pm 10\pi$	$\sin(\pm 10\pi) = 0$
$x = \pm 0.01$	$\pm 100\pi$	$\sin(\pm 100\pi) = 0$
$x = \pm 0.001$	$\pm 1000\pi$	$\sin(\pm 1000\pi) = 0$
$x = \pm 0.0001$	$\pm 10{,}000\pi$	$\sin(\pm 10{,}000\pi) = 0$
$\vdots$	$\vdots$	$\vdots$

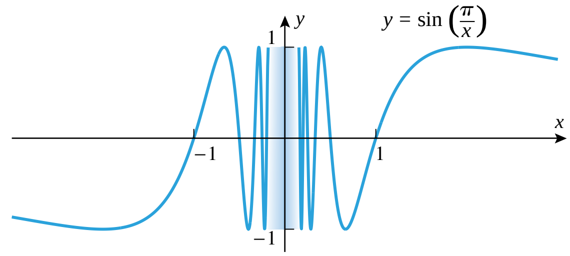


Figure 2.1.7

Numerical evidence can lead to incorrect conclusions about limits because of roundoff error or because the sample of values used is not extensive enough to give a good indication of the behavior of the function. Thus, when a limit is conjectured from a table of values, it is important to look for corroborating evidence to support the conjecture.

ONE-SIDED LIMITS

The limit in (4) is commonly called a ***two-sided limit*** because it requires the values of $f(x)$ to get closer and closer to L as values of x are taken from *either* side of $x = a$. However, some functions exhibit different behaviors on the two sides of an x-value a, in which case it is necessary to distinguish whether values of x near a are on the left side or on the right side of a for purposes of investigating limiting behavior. For example, consider the function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

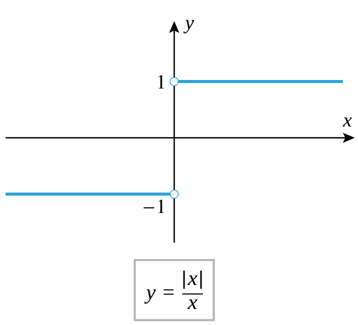


Figure 2.1.8

(Figure 2.1.8). Note that x-values approaching 0 and to the *right* of 0 produce $f(x)$ values that approach 1 (in fact, they are exactly 1 for all such values of x). On the other hand, x-values approaching 0 and to the *left* of 0 produce $f(x)$ values that approach -1. We describe these two statements by saying that "the limit of $f(x) = |x|/x$ is 1 as x approaches 0 from the right" and that "the limit of $f(x) = |x|/x$ is -1 as x approaches 0 from the left." We denote these limits by writing

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{|x|}{x} = -1 \qquad (9\text{–}10)$$

With this notation, the superscript "+" indicates a limit from the right and the superscript "−" indicates a limit from the left.

This leads to the following general idea.

2.1.2 ONE-SIDED LIMITS (AN INFORMAL VIEW). If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L \tag{11}$$

which is read "the limit of $f(x)$ as x approaches a from the right is L." Similarly, if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L \tag{12}$$

which is read "the limit of $f(x)$ as x approaches a from the left is L."

Expressions (11) and (12), which are called ***one-sided limits***, are also commonly written as

$$f(x) \to L \text{ as } x \to a^+ \quad \text{and} \quad f(x) \to L \text{ as } x \to a^-$$

respectively. With this notation (9) and (10) can be expressed as

$$\frac{|x|}{x} \to 1 \text{ as } x \to 0^+ \quad \text{and} \quad \frac{|x|}{x} \to -1 \text{ as } x \to 0^-$$

THE RELATIONSHIP BETWEEN ONE-SIDED LIMITS AND TWO-SIDED LIMITS

In general, there is no guarantee that a function will have a limit at a specified location. If the values of $f(x)$ do not get closer and closer to some *single* number L as $x \to a$, then we say that the limit of $f(x)$ as x approaches a ***does not exist*** (and similarly for one-sided limits). For example, the two-sided limit $\lim_{x \to 0} |x|/x$ does not exist because the values of $f(x)$ do not approach a single number as $x \to 0$; the values approach -1 from the left and 1 from the right.

In general, the following condition must be satisfied for the two-sided limit of a function to exist.

2.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS. The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

REMARK. Sometimes, one or both of the one-sided limits may fail to exist (which, in turn, implies that the two-sided limit does not exist). For example, we saw earlier that the one-sided limits of $f(x) = \sin(\pi/x)$ do not exist as x approaches 0 because the function keeps oscillating between -1 and 1, failing to settle on a single value. This implies that the two-sided limit does not exist as x approaches 0.

Example 4 For the functions in Figure 2.1.9, find the one-sided and two-sided limits at $x = a$ if they exist.

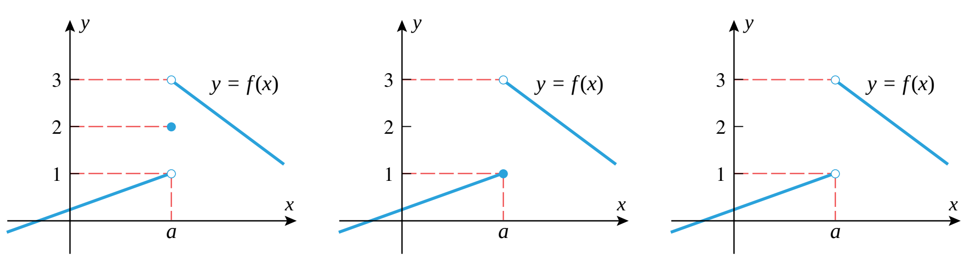


Figure 2.1.9

Solution. The functions in all three figures have the same one-sided limits as $x \to a$, since the functions are identical, except at $x = a$. These limits are

$$\lim_{x \to a^+} f(x) = 3 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 1$$

In all three cases the two-sided limit does not exist as $x \to a$ because the one-sided limits are not equal. ◀

Example 5 For the functions in Figure 2.1.10, find the one-sided and two-sided limits at $x = a$ if they exist.

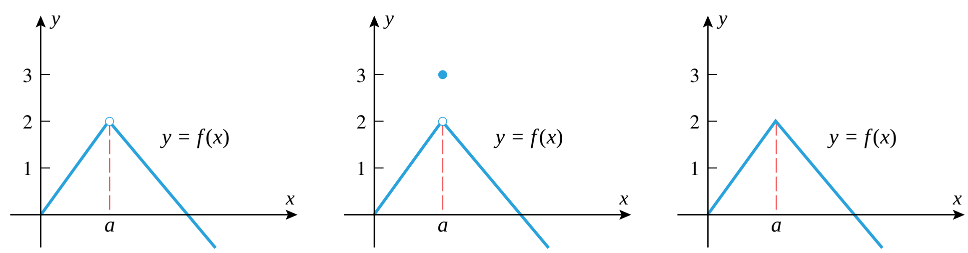


Figure 2.1.10

Solution. As in the preceding example, the value of f at $x = a$ has no bearing on the limits as $x \to a$, so that in all three cases we have

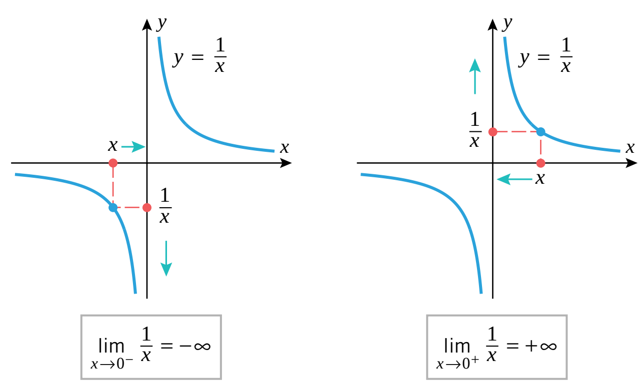
$$\lim_{x \to a^+} f(x) = 2 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 2$$

Since the one-sided limits are equal, the two-sided limit exists and

$$\lim_{x \to a} f(x) = 2$$ ◀

INFINITE LIMITS AND VERTICAL ASYMPTOTES

Sometimes one-sided or two-sided limits will fail to exist because the values of the function increase or decrease indefinitely. For example, consider the behavior of the function $f(x) = 1/x$ for values of x near 0. It is evident from the table and graph in Figure 2.1.11 that as x-values are taken closer and closer to 0 from the right, the values of $f(x) = 1/x$ are positive and increase indefinitely; and as x-values are taken closer and closer to 0 from the left, the values of $f(x) = 1/x$ are negative and decrease indefinitely. We describe these



x	–1	–0.1	–0.01	–0.001	–0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	–1	–10	–100	–1000	–10,000		10,000	1000	100	10	1

Left side → ← Right side

Figure 2.1.11

limiting behaviors by writing

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

More generally:

2.1.4 INFINITE LIMITS (AN INFORMAL VIEW). If the values of $f(x)$ increase indefinitely as x approaches a from the right or left, then we write

$$\lim_{x\to a^+} f(x) = +\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = +\infty$$

as appropriate, and we say that $f(x)$ ***increases without bound***, or $f(x)$ ***approaches*** $+\infty$, as $x \to a^+$ or as $x \to a^-$. Similarly, if the values of $f(x)$ decrease indefinitely as x approaches a from the right or left, then we write

$$\lim_{x\to a^+} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = -\infty$$

as appropriate, and say that $f(x)$ ***decreases without bound***, or $f(x)$ ***approaches*** $-\infty$, as $x \to a^+$ or as $x \to a^-$. Moreover, if both one-sided limits are $+\infty$, then we write

$$\lim_{x\to a} f(x) = +\infty$$

and if both one-sided limits are $-\infty$, then we write

$$\lim_{x\to a} f(x) = -\infty$$

REMARK. It should be emphasized that the symbols $+\infty$ and $-\infty$ are *not* real numbers. The phrase "$f(x)$ approaches $+\infty$" is akin to saying that "$f(x)$ approaches the unapproachable"; it is a colloquialism for "$f(x)$ increases without bound." The symbols $+\infty$ and $-\infty$ are used here to encapsulate a particular way in which limits fail to exist. To say, for example, that $f(x) \to +\infty$ as $x \to a^+$ is to indicate that $\lim_{x\to a^+} f(x)$ *does not exist*, and to say further that this limit fails to exist because values of $f(x)$ increase without bound as x approaches a from the right. Furthermore, since $+\infty$ and $-\infty$ are not numbers, it is inappropriate to manipulate these symbols using rules of algebra. For example, it is *not* correct to write $(+\infty) - (+\infty) = 0$.

Example 6 For the functions in Figure 2.1.12, describe the limits at $x = a$ in appropriate limit notation.

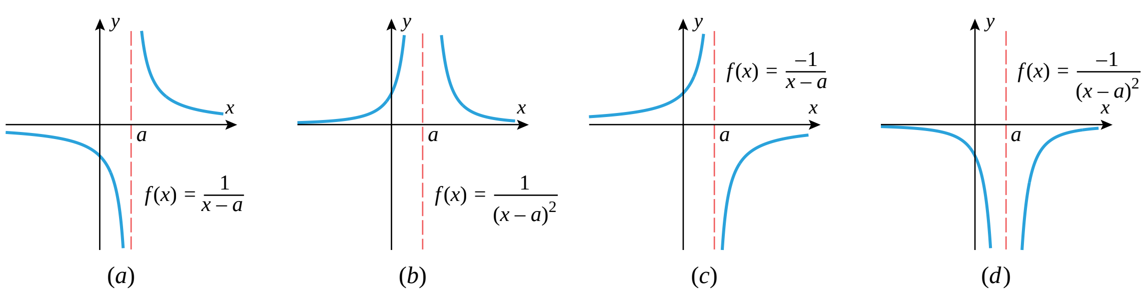


Figure 2.1.12

Solution (a). In Figure 2.1.12*a*, the function increases indefinitely as x approaches a from the right and decreases indefinitely as x approaches a from the left. Thus,

$$\lim_{x\to a^+} \frac{1}{x-a} = +\infty \quad \text{and} \quad \lim_{x\to a^-} \frac{1}{x-a} = -\infty$$

Solution (b). In Figure 2.1.12*b*, the function increases indefinitely as x approaches a from both the left and right. Thus,

$$\lim_{x\to a}\frac{1}{(x-a)^2}=\lim_{x\to a^+}\frac{1}{(x-a)^2}=\lim_{x\to a^-}\frac{1}{(x-a)^2}=+\infty$$

Solution (c). In Figure 2.1.12*c*, the function decreases indefinitely as x approaches a from the right and increases indefinitely as x approaches a from the left. Thus,

$$\lim_{x\to a^+}\frac{-1}{x-a}=-\infty \quad \text{and} \quad \lim_{x\to a^-}\frac{-1}{x-a}=+\infty$$

Solution (d). In Figure 2.1.12*d*, the function decreases indefinitely as x approaches a from both the left and right. Thus,

$$\lim_{x\to a}\frac{-1}{(x-a)^2}=\lim_{x\to a^+}\frac{-1}{(x-a)^2}=\lim_{x\to a^-}\frac{-1}{(x-a)^2}=-\infty$$ ◀

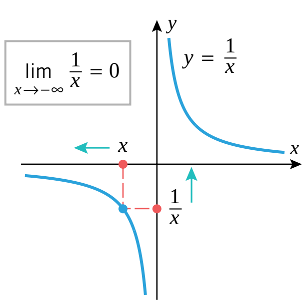
Geometrically, if $f(x)\to+\infty$ as $x\to a^-$ or $x\to a^+$, then the graph of $y=f(x)$ rises without bound and squeezes closer to the vertical line $x=a$ on the indicated side of $x=a$. If $f(x)\to-\infty$ as $x\to a^-$ or $x\to a^+$, then the graph of $y=f(x)$ falls without bound and squeezes closer to the vertical line $x=a$ on the indicated side of $x=a$. In these cases, we call the line $x=a$ a *vertical asymptote*. ("Asymptote" comes from the Greek *asymptotos*, meaning "nonintersecting." We will soon see that taking "asymptote" to be synonymous with "nonintersecting" is a bit misleading.)

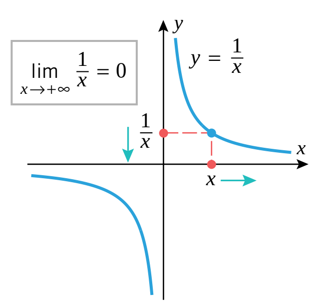
2.1.5 DEFINITION. A line $x=a$ is called a ***vertical asymptote*** of the graph of a function f if $f(x)\to+\infty$ or $f(x)\to-\infty$ as x approaches a from the left or right.

Example 7 The four functions graphed in Figure 2.1.12 all have a vertical asymptote at $x=a$, which is indicated by the dashed vertical lines in the figure. ◀

LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

Thus far, we have used limits to describe the behavior of $f(x)$ as x approaches a. However, sometimes we will not be concerned with the behavior of $f(x)$ near a specific x-value, but rather with how the values of $f(x)$ behave as x increases without bound or decreases without bound. This is sometimes called the ***end behavior*** of the function because it describes how the function behaves for values of x that are far from the origin. For example, it is evident from the table and graph in Figure 2.1.13 that as x increases without bound, the values of





x	···	−10,000	−1000	−100	−10	−1
$f(x)$	···	−0.0001	−0.001	−0.01	−0.1	−1

← x decreasing without bound

x	1	10	100	1000	10,000	···
$f(x)$	1	0.1	0.01	0.001	0.0001	···

x increasing without bound →

Figure 2.1.13

$f(x) = 1/x$ are positive, but get closer and closer to 0; and as x decreases without bound, the values of $f(x) = 1/x$ are negative, and also get closer and closer to 0. We indicate these limiting behaviors by writing

$$\lim_{x\to+\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0$$

More generally:

2.1.6 LIMITS AT INFINITY (AN INFORMAL VIEW). If the values of $f(x)$ eventually get closer and closer to a number L as x increases without bound, then we write

$$\lim_{x\to+\infty} f(x) = L \quad \text{or} \quad f(x)\to L \text{ as } x\to+\infty \tag{13}$$

Similarly, if the values of $f(x)$ eventually get closer and closer to a number L as x decreases without bound, then we write

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad f(x)\to L \text{ as } x\to-\infty \tag{14}$$

Geometrically, if $f(x)\to L$ as $x\to+\infty$, then the graph of $y = f(x)$ eventually gets closer and closer to the line $y = L$ as the graph is traversed in the positive direction (Figure 2.1.14*a*); and if $f(x)\to L$ as $x\to-\infty$, then the graph of $y = f(x)$ eventually gets closer and closer to the line $y = L$ as the graph is traversed in the negative x-direction (Figure 2.1.14*b*). In either case we call the line $y = L$ a *horizontal asymptote* of the graph of f. For example, the function in Figure 2.1.13 all have $y = 0$ as a horizontal asymptote.

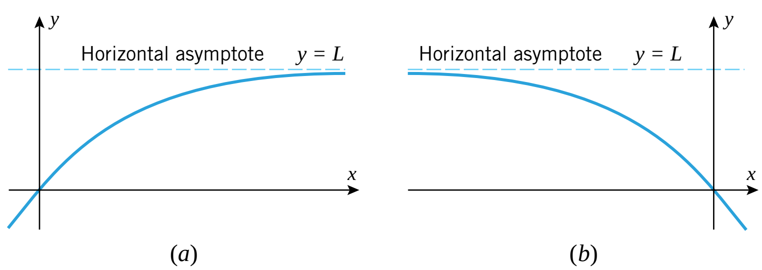


Figure 2.1.14

2.1.7 DEFINITION. A line $y = L$ is called a ***horizontal asymptote*** of the graph of a function f if

$$\lim_{x\to+\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

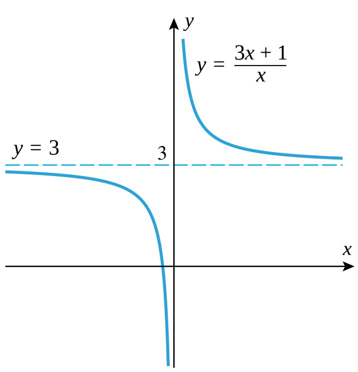


Figure 2.1.15

Sometimes the existence of a horizontal asymptote of a function f will be readily apparent from the formula for f. For example, it is evident that the function

$$f(x) = \frac{3x+1}{x} = 3 + \frac{1}{x}$$

has a horizontal asymptote at $y = 3$ (Figure 2.1.15), since the value of $1/x$ approaches 0 as $x\to+\infty$ or $x\to-\infty$. For more complicated functions, algebraic manipulations or special techniques that we will study in the next section may have to be applied to confirm the existence of horizontal asymptotes.

HOW LIMITS AT INFINITY CAN FAIL TO EXIST

Limits at infinity can fail to exist for various reasons. One possibility is that the values of $f(x)$ may increase or decrease without bound as $x\to+\infty$ or as $x\to-\infty$. For example, the values of $f(x) = x^3$ increase without bound as $x\to+\infty$ and decrease without bound as

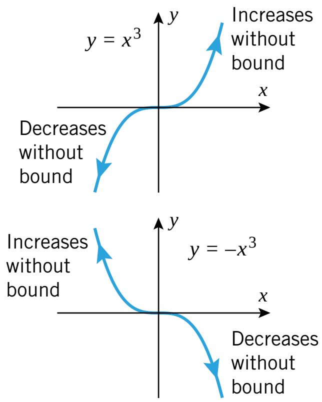


Figure 2.1.16

$x \to -\infty$; and for $f(x) = -x^3$ the values decrease without bound as $x \to +\infty$ and increase without bound as $x \to -\infty$ (Figure 2.1.16). We denote this by writing

$$\lim_{x \to +\infty} x^3 = +\infty, \quad \lim_{x \to -\infty} x^3 = -\infty, \quad \lim_{x \to +\infty} (-x^3) = -\infty, \quad \lim_{x \to -\infty} (-x^3) = +\infty$$

More generally:

2.1.8 INFINITE LIMITS AT INFINITY (AN INFORMAL VIEW). If the values of $f(x)$ increase without bound as $x \to +\infty$ or as $x \to -\infty$, then we write

$$\lim_{x \to +\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) = +\infty$$

as appropriate; and if the values of $f(x)$ decrease without bound as $x \to +\infty$ or as $x \to -\infty$, then we write

$$\lim_{x \to +\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) = -\infty$$

as appropriate.

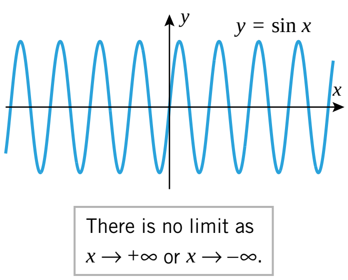


Figure 2.1.17

Limits at infinity can also fail to exist because the graph of the function oscillates indefinitely in such a way that the values of the function do not approach a fixed number and do not increase or decrease without bound; the trigonometric functions $\sin x$ and $\cos x$ have this property, for example (Figure 2.1.17). In such cases we say that the limit ***fails to exist because of oscillation***.

EXERCISE SET 2.1 Graphing Calculator C CAS

1. For the function f graphed in the accompanying figure, find
(a) $\lim_{x \to 3^-} f(x)$ (b) $\lim_{x \to 3^+} f(x)$ (c) $\lim_{x \to 3} f(x)$
(d) $f(3)$ (e) $\lim_{x \to -\infty} f(x)$ (f) $\lim_{x \to +\infty} f(x)$.

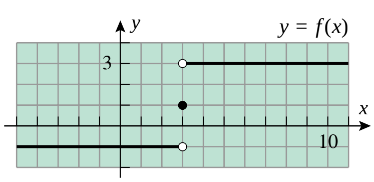


Figure Ex-1

2. For the function f graphed in the accompanying figure, find
(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$ (c) $\lim_{x \to 2} f(x)$
(d) $f(2)$ (e) $\lim_{x \to -\infty} f(x)$ (f) $\lim_{x \to +\infty} f(x)$.

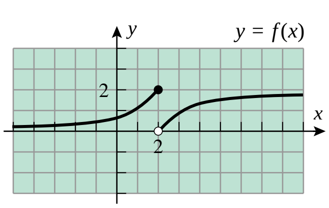


Figure Ex-2

3. For the function g graphed in the accompanying figure, find
(a) $\lim_{x \to 4^-} g(x)$ (b) $\lim_{x \to 4^+} g(x)$ (c) $\lim_{x \to 4} g(x)$
(d) $g(4)$ (e) $\lim_{x \to -\infty} g(x)$ (f) $\lim_{x \to +\infty} g(x)$.

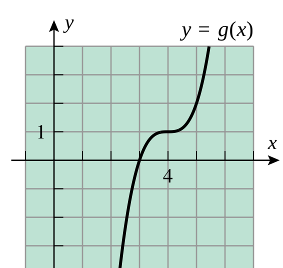


Figure Ex-3

4. For the function g graphed in the accompanying figure, find
(a) $\lim_{x \to 0^-} g(x)$ (b) $\lim_{x \to 0^+} g(x)$ (c) $\lim_{x \to 0} g(x)$
(d) $g(0)$ (e) $\lim_{x \to -\infty} g(x)$ (f) $\lim_{x \to +\infty} g(x)$.

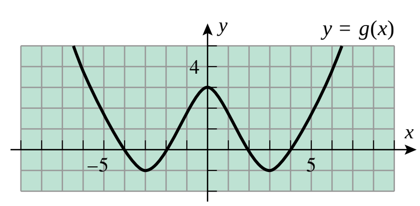


Figure Ex-4

5. For the function F graphed in the accompanying figure, find
(a) $\lim_{x\to -2^-} F(x)$ (b) $\lim_{x\to -2^+} F(x)$ (c) $\lim_{x\to -2} F(x)$
(d) $F(-2)$ (e) $\lim_{x\to -\infty} F(x)$ (f) $\lim_{x\to +\infty} F(x)$.

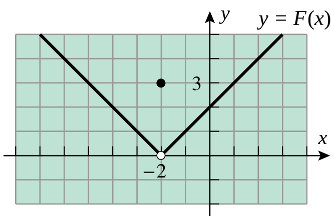


Figure Ex-5

6. For the function F graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} F(x)$ (b) $\lim_{x\to 3^+} F(x)$ (c) $\lim_{x\to 3} F(x)$
(d) $F(3)$ (e) $\lim_{x\to -\infty} F(x)$ (f) $\lim_{x\to +\infty} F(x)$.

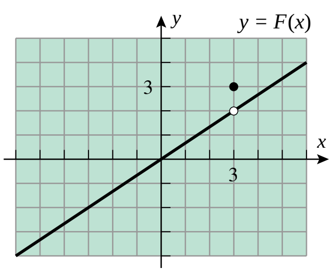


Figure Ex-6

7. For the function ϕ graphed in the accompanying figure, find
(a) $\lim_{x\to -2^-} \phi(x)$ (b) $\lim_{x\to -2^+} \phi(x)$ (c) $\lim_{x\to -2} \phi(x)$
(d) $\phi(-2)$ (e) $\lim_{x\to -\infty} \phi(x)$ (f) $\lim_{x\to +\infty} \phi(x)$.

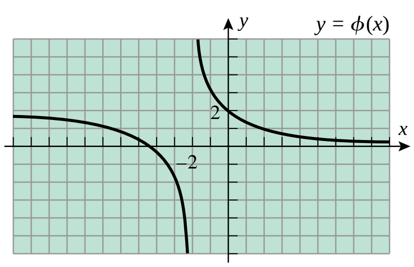


Figure Ex-7

8. For the function ϕ graphed in the accompanying figure, find
(a) $\lim_{x\to 4^-} \phi(x)$ (b) $\lim_{x\to 4^+} \phi(x)$ (c) $\lim_{x\to 4} \phi(x)$
(d) $\phi(4)$ (e) $\lim_{x\to -\infty} \phi(x)$ (f) $\lim_{x\to +\infty} \phi(x)$.

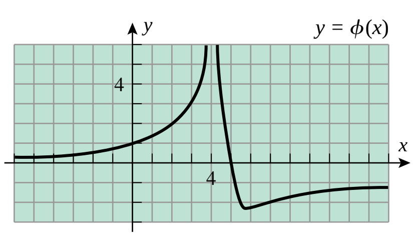


Figure Ex-8

9. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$ (c) $\lim_{x\to 3} f(x)$
(d) $f(3)$ (e) $\lim_{x\to -\infty} f(x)$ (f) $\lim_{x\to +\infty} f(x)$.

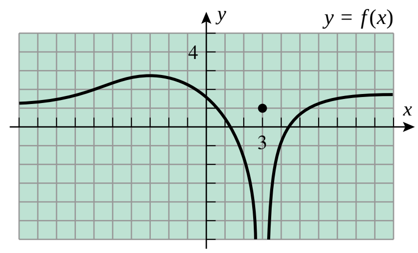


Figure Ex-9

10. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$
(d) $f(0)$ (e) $\lim_{x\to -\infty} f(x)$ (f) $\lim_{x\to +\infty} f(x)$.

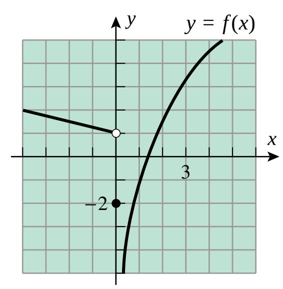


Figure Ex-10

11. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$ (c) $\lim_{x\to 0} G(x)$
(d) $G(0)$ (e) $\lim_{x\to -\infty} G(x)$ (f) $\lim_{x\to +\infty} G(x)$.

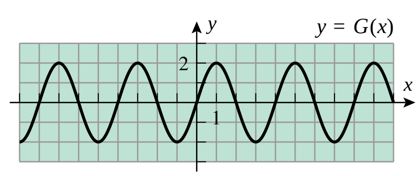


Figure Ex-11

12. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$ (c) $\lim_{x\to 0} G(x)$
(d) $G(0)$ (e) $\lim_{x\to -\infty} G(x)$ (f) $\lim_{x\to +\infty} G(x)$.

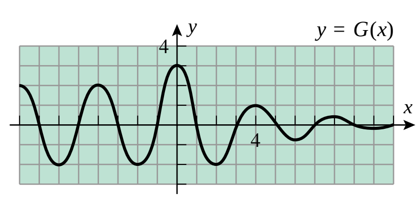


Figure Ex-12

13. Consider the function g graphed in the accompanying figure. For what values of x_0 does $\lim_{x\to x_0} g(x)$ exist?

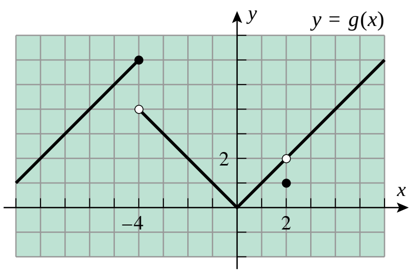


Figure Ex-13

14. Consider the function f graphed in the accompanying figure. For what values of x_0 does $\lim_{x\to x_0} f(x)$ exist?

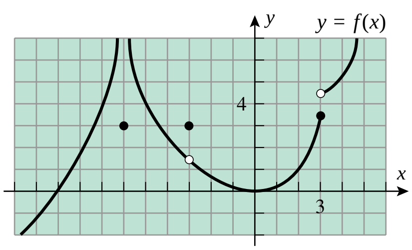


Figure Ex-14

In Exercises 15–18, sketch a possible graph for a function f with the specified properties. (Many different solutions are possible.)

15. (i) $f(0) = 2$ and $f(2) = 1$
(ii) $\lim_{x\to 1^-} f(x) = +\infty$ and $\lim_{x\to 1^+} f(x) = -\infty$
(iii) $\lim_{x\to +\infty} f(x) = 0$ and $\lim_{x\to -\infty} f(x) = +\infty$

16. (i) $f(0) = f(2) = 1$
(ii) $\lim_{x\to 2^-} f(x) = +\infty$ and $\lim_{x\to 2^+} f(x) = 0$
(iii) $\lim_{x\to -1^-} f(x) = -\infty$ and $\lim_{x\to -1^+} f(x) = +\infty$
(iv) $\lim_{x\to +\infty} f(x) = 2$ and $\lim_{x\to -\infty} f(x) = +\infty$

17. (i) $f(x) = 0$ if x is an integer and $f(x) \neq 0$ if x is not an integer
(ii) $\lim_{x\to +\infty} f(x) = 0$ and $\lim_{x\to -\infty} f(x) = 0$

18. (i) $f(x) = 1$ if x is a positive integer and $f(x) \neq 1$ if $x > 0$ is not a positive integer
(ii) $f(x) = -1$ if x is a negative integer and $f(x) \neq -1$ if $x < 0$ is not a negative integer
(iii) $\lim_{x\to +\infty} f(x) = 1$ and $\lim_{x\to -\infty} f(x) = -1$

In Exercises 19–22: (i) Make a guess at the limit (if it exists) by evaluating the function at the specified x-values. (ii) Confirm your conclusions about the limit by graphing the function over an appropriate interval. (iii) If you have a CAS, then use it to find the limit. [*Note:* For the trigonometric functions, be sure to set your calculating and graphing utilities to the radian mode.]

C **19.** (a) $\lim_{x\to 1} \dfrac{x-1}{x^3-1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 0, 0.5, 0.9, 0.99, 0.999$
(b) $\lim_{x\to 1^+} \dfrac{x+1}{x^3-1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 1.0001$
(c) $\lim_{x\to 1^-} \dfrac{x+1}{x^3-1}$; $x = 0, 0.5, 0.9, 0.99, 0.999, 0.9999$

C **20.** (a) $\lim_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$
(b) $\lim_{x\to 0^+} \dfrac{\sqrt{x+1}+1}{x}$; $x = 0.25, 0.1, 0.001, 0.0001$
(c) $\lim_{x\to 0^-} \dfrac{\sqrt{x+1}+1}{x}$; $x = -0.25, -0.1, -0.001, -0.0001$

C **21.** (a) $\lim_{x\to 0} \dfrac{\sin 3x}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$
(b) $\lim_{x\to -1} \dfrac{\cos x}{x+1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$

C **22.** (a) $\lim_{x\to -1} \dfrac{\tan(x+1)}{x+1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$
(b) $\lim_{x\to 0} \dfrac{\sin(5x)}{\sin(2x)}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

23. Consider the motion of the ball described in Example 1. By interpreting instantaneous velocity as a limit of average velocity, make a conjecture for the value of the instantaneous velocity of the ball 0.25 s after its release.

24. Consider the motion of the ball described in Example 1. By interpreting instantaneous velocity as a limit of average velocity, make a conjecture for the value of the instantaneous velocity of the ball 0.75 s after its release.

In Exercises 25 and 26: (i) Approximate the y-coordinates of all horizontal asymptotes of $y = f(x)$ by evaluating f at the x-values $\pm 10, \pm 100, \pm 1000, \pm 100{,}000$, and $\pm 100{,}000{,}000$. (ii) Confirm your conclusions by graphing $y = f(x)$ over an appropriate interval. (iii) If you have a CAS, then use it to find the horizontal asymptotes.

C **25.** (a) $f(x) = \dfrac{2x+3}{x+4}$ (b) $f(x) = \left(1 + \dfrac{3}{x}\right)^x$
(c) $f(x) = \dfrac{x^2+1}{x+1}$

C **26.** (a) $f(x) = \dfrac{x^2 - 1}{5x^2 + 1}$ (b) $f(x) = \left(2 + \dfrac{1}{x}\right)^x$

(c) $f(x) = \dfrac{\sin x}{x}$

27. Assume that a particle is accelerated by a constant force. The two curves $v = n(t)$ and $v = e(t)$ in the accompanying figure provide velocity versus time curves for the particle as predicted by classical physics and by the special theory of relativity, respectively. The parameter c designates the speed of light. Using the language of limits, describe the differences in the long-term predictions of the two theories.

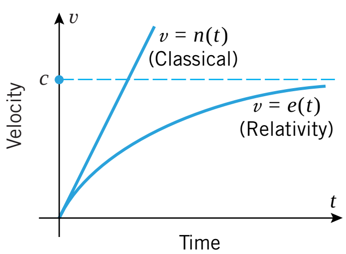


Figure Ex-27

28. Let $T = f(t)$ denote the temperature of a baked potato t minutes after it has been removed from a hot oven. The accompanying figure shows the temperature versus time curve for the potato, where r is the temperature of the room.

(a) What is the physical significance of $\lim_{t\to 0^+} f(t)$?

(b) What is the physical significance of $\lim_{t\to +\infty} f(t)$?

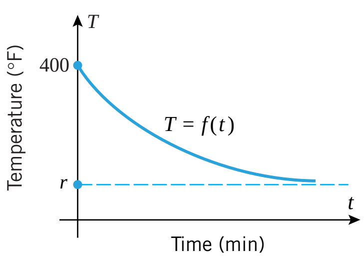


Figure Ex-28

In Exercises 29 and 30: (i) Conjecture a limit from numerical evidence. (ii) Use the substitution $t = 1/x$ to express the limit as an equivalent limit in which $t \to 0^+$ or $t \to 0^-$, as appropriate. (iii) Use a graphing utility to make a conjecture about your limit in (ii).

29. (a) $\lim_{x\to +\infty} x \sin\left(\dfrac{1}{x}\right)$ (b) $\lim_{x\to +\infty} \dfrac{1 - x}{1 + x}$

(c) $\lim_{x\to -\infty} \left(1 + \dfrac{2}{x}\right)^x$

30. (a) $\lim_{x\to +\infty} \dfrac{\cos(\pi/x)}{\pi/x}$ (b) $\lim_{x\to +\infty} \dfrac{x}{1 + x}$

(c) $\lim_{x\to -\infty} (1 - 2x)^{1/x}$

31. Suppose that $f(x)$ denotes a function such that

$$\lim_{t\to 0} f(1/t) = L$$

What can be said about

$$\lim_{x\to +\infty} f(x) \quad \text{and} \quad \lim_{x\to -\infty} f(x)?$$

32. (a) Do any of the trigonometric functions, $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$, have horizontal asymptotes?

(b) Do any of them have vertical asymptotes? Where?

33. (a) Let

$$f(x) = (1 + x^2)^{1.1/x^2}$$

Graph f in the window $[-1, 1] \times [2.5, 3.5]$ and use the calculator's trace feature to make a conjecture about the limit of f as $x \to 0$.

(b) Graph f in the window $[-0.001, 0.001] \times [2.5, 3.5]$ and use the calculator's trace feature to make a conjecture about the limit of f as $x \to 0$.

(c) Graph f in the window $[-0.000001, 0.000001] \times [2.5, 3.5]$ and use the calculator's trace feature to make a conjecture about the limit of f as $x \to 0$.

(d) Later we will be able to show that

$$\lim_{x\to 0} (1 + x^2)^{1.1/x^2} \approx 3.00416602$$

What flaw do your graphs reveal about using numerical evidence (as revealed by the graphs you obtained) to make conjectures about limits?

Roundoff error is one source of inaccuracy in calculator and computer computations. Another source of error, called ***catastrophic subtraction***, occurs when two nearly equal numbers are subtracted, and the result is used as part of another calculation. For example, by hand calculation we have

$$(0.123456789012345 - 0.123456789012344) \times 10^{15} = 1$$

However, a calculator that can only store 14 decimal digits produces a value of 0 for this computation, since the numbers being subtracted are identical in the first 14 digits. Catastrophic subtraction can sometimes be avoided by rearranging formulas algebraically, but your best defense is to be aware that it can occur. Watch out for it in the next exercise.

C **34.** (a) Let

$$f(x) = \frac{x - \sin x}{x^3}$$

Make a conjecture about the limit of f as $x \to 0^+$ by evaluating $f(x)$ at $x = 0.1, 0.01, 0.001, 0.0001$.

(b) Evaluate $f(x)$ at $x = 0.000001, 0.0000001, 0.00000001, 0.000000001, 0.0000000001$, and make another conjecture.

(c) What flaw does this reveal about using numerical evidence to make conjectures about limits?

(d) If you have a CAS, use it to show that the exact value of the limit is $\frac{1}{6}$.

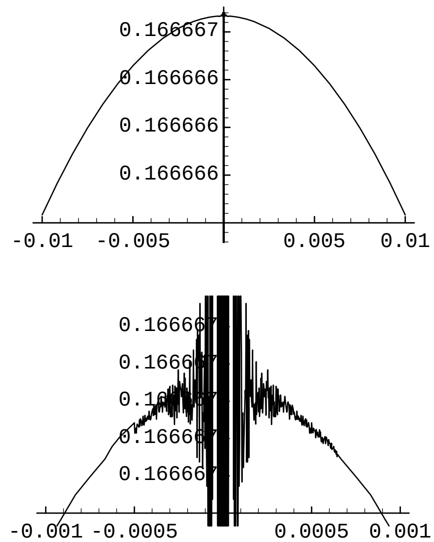
35. (a) The accompanying figure shows two different views of the graph of the function in Exercise 34, as generated by *Mathematica*. What is happening?

(b) Use your graphing utility to generate the graphs, and see whether the same problem occurs.

(c) Would you expect a similar problem to occur in the vicinity of $x = 0$ for the function

$$f(x) = \frac{1 - \cos x}{x}?$$

See if it does.



Erratic graph generated by Mathematica

Figure Ex-35

2.2 COMPUTING LIMITS

In this section we will discuss algebraic techniques for computing limits of many functions. We base these results on the informal development of the limit concept discussed in the preceding section. A more formal derivation of these results is possible after Section 2.4.

SOME BASIC LIMITS

Our strategy for finding limits algebraically has two parts:

- First we will obtain the limits of some simple functions.
- Then we will develop a repertoire of theorems that will enable us to use the limits of those simple functions as building blocks for finding limits of more complicated functions.

We start with the cases of a constant function $f(x) = k$, the identity function $f(x) = x$, and the reciprocal function $f(x) = 1/x$.

2.2.1 THEOREM. *Let a and k be real numbers.*

$$\lim_{x\to a} k = k \qquad \lim_{x\to a} x = a$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x\to 0^+} \frac{1}{x} = +\infty$$

The four limits in Theorem 2.2.1 should be evident from inspection of the function graphs shown in Figure 2.2.1.

In the case of the constant function $f(x) = k$, the values of $f(x)$ do not change as x varies, so the limit of $f(x)$ is k, regardless of at which number a the limit is taken. For example,

$$\lim_{x\to -25} 3 = 3, \qquad \lim_{x\to 0} 3 = 3, \qquad \lim_{x\to \pi} 3 = 3$$

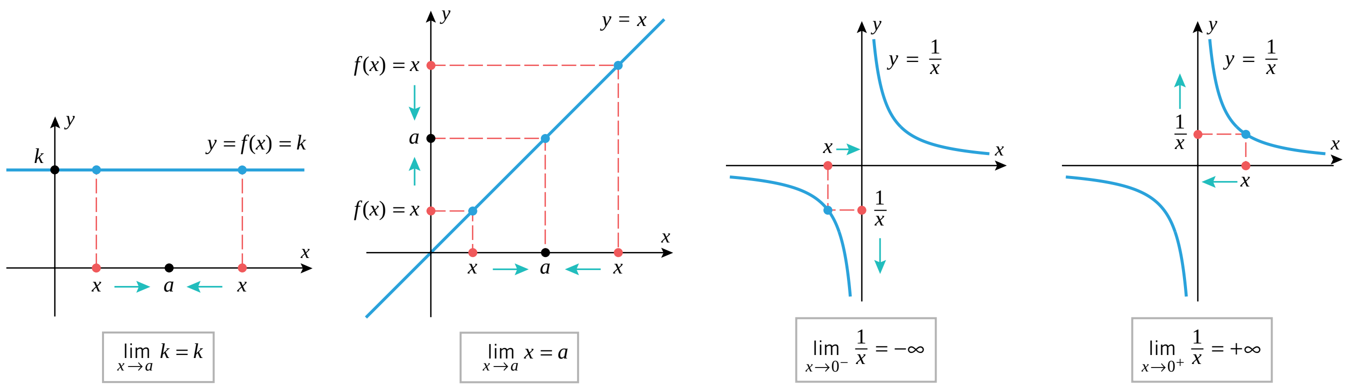


Figure 2.2.1

Since the identity function $f(x) = x$ just echoes its input, it is clear that $f(x) = x \to a$ as $x \to a$. In terms of our informal definition of limits (2.1.1), if we decide just how close to a we would like the value of $f(x) = x$ to be, we need only restrict its input x to be just as close to a.

The one-sided limits of the reciprocal function $f(x) = 1/x$ about 0 should conform with your experience with fractions: making the denominator closer to zero increases the magnitude of the fraction (i.e., increases its absolute value). This is illustrated in Table 2.2.1.

Table 2.2.1

	VALUES						CONCLUSION
x	-1	-0.1	-0.01	-0.001	-0.0001	$\cdots$	As $x \to 0^-$ the value of $1/x$
$1/x$	-1	-10	-100	-1000	$-10{,}000$	$\cdots$	decreases without bound.
x	1	0.1	0.01	0.001	0.0001	$\cdots$	As $x \to 0^+$ the value of $1/x$
$1/x$	1	10	100	1000	10,000	$\cdots$	increases without bound.

The following theorem, parts of which are proved in Appendix G, will be our basic tool for finding limits algebraically.

2.2.2 THEOREM. *Let a be a real number, and suppose that*

$$\lim_{x\to a} f(x) = L_1 \quad \textit{and} \quad \lim_{x\to a} g(x) = L_2$$

That is, the limits exist and have values L_1 and L_2, respectively. Then,

(*a*) $\displaystyle \lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$

(*b*) $\displaystyle \lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$

(*c*) $\displaystyle \lim_{x\to a} [f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1 L_2$

(*d*) $\displaystyle \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2},$ *provided* $L_2 \neq 0$

(*e*) $\displaystyle \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1},$ *provided* $L_1 > 0$ *if n is even.*

Moreover, these statements are also true for one-sided limits.

A casual restatement of this theorem is as follows:

(a) *The limit of a sum is the sum of the limits.*

(b) *The limit of a difference is the difference of the limits.*

(c) *The limit of a product is the product of the limits.*

(d) *The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero.*

(e) *The limit of an nth root is the nth root of the limit.*

REMARK. Although results (a) and (c) in Theorem 2.2.2 are stated for two functions, they hold for any finite number of functions. For example, if the limits of $f(x)$, $g(x)$, and $h(x)$ exist as $x \to a$, then the limit of their sum and the limit of their product also exist as $x \to a$ and are given by the formulas

$$\lim_{x \to a} [f(x) + g(x) + h(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) + \lim_{x \to a} h(x)$$

$$\lim_{x \to a} [f(x)g(x)h(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)\left(\lim_{x \to a} h(x)\right)$$

In particular, if $f(x) = g(x) = h(x)$, then this yields

$$\lim_{x \to a} [f(x)]^3 = \left(\lim_{x \to a} f(x)\right)^3$$

More generally, if n is a positive integer, then the limit of the nth power of a function is the nth power of the function's limit. Thus,

$$\lim_{x \to a} x^n = \left(\lim_{x \to a} x\right)^n = a^n \tag{1}$$

For example,

$$\lim_{x \to 3} x^4 = 3^4 = 81$$

Another useful result follows from part (c) of Theorem 2.2.2 in the special case when one of the factors is a constant k:

$$\lim_{x \to a} (k \cdot f(x)) = \left(\lim_{x \to a} k\right) \cdot \left(\lim_{x \to a} f(x)\right) = k \cdot \left(\lim_{x \to a} f(x)\right) \tag{2}$$

and similarly for $\lim_{x \to a}$ replaced by a one-sided limit, $\lim_{x \to a^+}$ or $\lim_{x \to a^-}$. Rephrased, this last statement says:

A constant factor can be moved through a limit symbol.

LIMITS OF POLYNOMIALS AND RATIONAL FUNCTIONS AS $x \to a$

Example 1 Find $\lim_{x \to 5} (x^2 - 4x + 3)$ and justify each step.

Solution. First note that $\lim_{x \to 5} x^2 = 5^2 = 25$ by Equation (1). Also, from Equation (2), $\lim_{x \to 5} 4x = 4(\lim_{x \to 5} x) = 4(5) = 20$. Since $\lim_{x \to 5} 3 = 3$ by Theorem 2.2.1, we may appeal to Theorem 2.2.2(a) and (b) to write

$$\lim_{x \to 5} (x^2 - 4x + 3) = \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3 = 25 - 20 + 3 = 8$$

However, for conciseness, it is common to reverse the order of this argument and simply write

$$
\begin{aligned}
\lim_{x\to 5}(x^2-4x+3) &= \lim_{x\to 5} x^2 - \lim_{x\to 5} 4x + \lim_{x\to 5} 3 && \boxed{\text{Theorem 2.2.2}(a), (b)}\\
&= \Big(\lim_{x\to 5} x\Big)^2 - 4\lim_{x\to 5} x + \lim_{x\to 5} 3 && \boxed{\text{Equations (1), (2)}}\\
&= 5^2 - 4(5) + 3 && \boxed{\text{Theorem 2.2.1}}\\
&= 8
\end{aligned}
$$

◀

REMARK. In our presentation of limit arguments, we will adopt the convention of providing just a concise, reverse argument, bearing in mind that the validity of each equality may be conditional upon the successful resolution of the remaining limits.

Our next result will show that the limit of a polynomial $p(x)$ at $x = a$ is the same as the value of the polynomial at $x = a$. This greatly simplifies the computation of limits of polynomials by allowing us to simply evaluate the polynomial.

2.2.3 THEOREM. *For any polynomial*

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

and any real number a,

$$\lim_{x\to a} p(x) = c_0 + c_1 a + \cdots + c_n a^n = p(a)$$

Proof.

$$
\begin{aligned}
\lim_{x\to a} p(x) &= \lim_{x\to a}\left(c_0 + c_1 x + \cdots + c_n x^n\right)\\
&= \lim_{x\to a} c_0 + \lim_{x\to a} c_1 x + \cdots + \lim_{x\to a} c_n x^n\\
&= \lim_{x\to a} c_0 + c_1 \lim_{x\to a} x + \cdots + c_n \lim_{x\to a} x^n\\
&= c_0 + c_1 a + \cdots + c_n a^n = p(a)
\end{aligned}
$$

■

Recall that a rational function is a ratio of two polynomials. Theorem 2.2.3 and Theorem 2.2.2(d) can often be used in combination to compute limits of rational functions.

Example 2 Find $\displaystyle\lim_{x\to 2}\frac{5x^3+4}{x-3}$.

Solution.

$$
\begin{aligned}
\lim_{x\to 2}\frac{5x^3+4}{x-3} &= \frac{\lim_{x\to 2}(5x^3+4)}{\lim_{x\to 2}(x-3)} && \boxed{\text{Theorem 2.2.2}(d)}\\
&= \frac{5\cdot 2^3+4}{2-3} = -44 && \boxed{\text{Theorem 2.2.3}}
\end{aligned}
$$

◀

2.2.4 THEOREM. *Consider the rational function*

$$f(x) = \frac{n(x)}{d(x)}$$

where $n(x)$ and $d(x)$ are polynomials. For any real number a,

(a) if $d(a) \neq 0$, then $\lim_{x\to a} f(x) = f(a)$.

(b) if $d(a) = 0$ but $n(a) \neq 0$, then $\lim_{x\to a} f(x)$ does not exist.

Proof. If $d(a) \neq 0$, then

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} \frac{n(x)}{d(x)} \\
&= \frac{\lim_{x\to a} n(x)}{\lim_{x\to a} d(x)} && \boxed{\text{Theorem 2.2.2}(d)} \\
&= \frac{n(a)}{d(a)} = f(a) && \boxed{\text{Theorem 2.2.3}}
\end{aligned}$$

If $d(a) = 0$ and $n(a) \neq 0$, then we again appeal to your experience with fractions. For values of x sufficiently near a, the value of $n(x)$ will be near $n(a)$ and not zero. Thus, since $0 = d(a) = \lim_{x\to a} d(x)$, as values of x approach a, the magnitude (absolute value) of the fraction $n(x)/d(x)$ will increase without bound, so $\lim_{x\to a} f(x)$ does not exist. ∎

As an illustration of part (*b*) of Theorem 2.2.4, consider

$$\lim_{x\to 3} \frac{5x^3+4}{x-3}$$

Note that $\lim_{x\to 3}(5x^3+4) = 5\cdot 3^3 + 4 = 139$ and $\lim_{x\to 3}(x-3) = 3-3 = 0$. It is evident from Table 2.2.2 that

$$\lim_{x\to 3} \frac{5x^3+4}{x-3}$$

does not exist.

Table 2.2.2

	VALUES				CONCLUSION
x	2.99	2.999	2.9999	. . .	The value of $\frac{5x^3+4}{x-3}$ decreases without bound as $x \to 3^-$.
$\frac{5x^3+4}{x-3}$	–13,765.45	–138,865.04	–1,389,865.00	. . .	
x	3.01	3.001	3.0001	. . .	The value of $\frac{5x^3+4}{x-3}$ increases without bound as $x \to 3^+$.
$\frac{5x^3+4}{x-3}$	14,035.45	139,135.05	1,390,135.00	. . .	

In Theorem 2.2.4(*b*), where the limit of the denominator is zero but the limit of the numerator is not zero, the response "does not exist" can be elaborated upon in one of the following three ways.

- The limit may be $-\infty$.
- The limit may be $+\infty$.
- The limit may be $-\infty$ from one side and $+\infty$ from the other.

Figure 2.2.2 illustrates these three possibilities graphically for rational functions of the form $1/(x-a)$, $1/(x-a)^2$, and $-1/(x-a)^2$.

Example 3 Find

(a) $\displaystyle\lim_{x\to 4^-} \frac{2-x}{(x-4)(x+2)}$ (b) $\displaystyle\lim_{x\to 4^+} \frac{2-x}{(x-4)(x+2)}$ (c) $\displaystyle\lim_{x\to 4} \frac{2-x}{(x-4)(x+2)}$

Solution. With $n(x) = 2-x$ and $d(x) = (x-4)(x+2)$, we see that $n(4) = -2$ and $d(4) = 0$. By Theorem 2.2.4(*b*), each of the limits does not exist. To be more specific, we

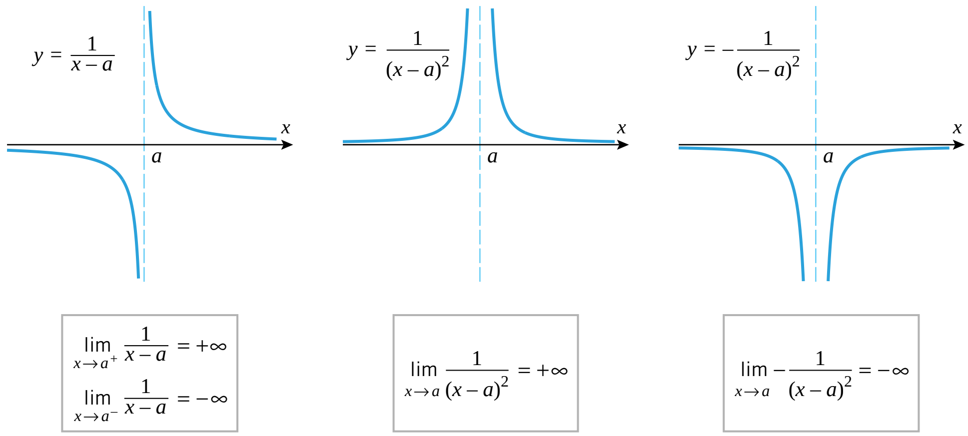


Figure 2.2.2

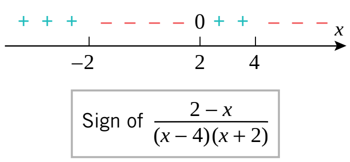


Figure 2.2.3

analyze the sign of the ratio $n(x)/d(x)$ near $x = 4$. The sign of the ratio, which is given in Figure 2.2.3, is determined by the signs of $2 - x$, $x - 4$, and $x + 2$. (The method of test values, discussed in Appendix A, provides a simple way of finding the sign of the ratio here.) It follows from this figure that as x approaches 4 from the left, the ratio is always positive; and as x approaches 4 from the right, the ratio is always negative. Thus,

$$\lim_{x\to 4^-} \frac{2-x}{(x-4)(x+2)} = +\infty \quad \text{and} \quad \lim_{x\to 4^+} \frac{2-x}{(x-4)(x+2)} = -\infty$$

Because the one-sided limits have opposite signs, all we can say about the two-sided limit is that it does not exist. ◀

INDETERMINATE FORMS OF TYPE 0/0

The missing case in Theorem 2.2.4 is when *both* the numerator and the denominator of a rational function $f(x) = n(x)/d(x)$ have a zero at $x = a$. In this case, $n(x)$ and $d(x)$ will each have a factor of $x - a$, and canceling this factor *may* result in a rational function to which Theorem 2.2.4 applies.

Example 4 Find $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$.

Solution. Since 2 is a zero of both the numerator and denominator, they share a common factor of $x - 2$. The limit can be obtained as follows:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$$ ◀

REMARK. Although correct, the second equality in the preceding computation needs some justification, since canceling the factor $x - 2$ alters the function by expanding its domain. However, as discussed in Example 5 of Section 1.2, the two functions are identical, except at $x = 2$ (Figure 1.2.9). From our discussions in the last section, we know that this difference has no effect on the limit as x approaches 2.

Example 5 Find

(a) $\displaystyle\lim_{x\to 3} \frac{x^2-6x+9}{x-3}$ (b) $\displaystyle\lim_{x\to -4} \frac{2x+8}{x^2+x-12}$ (c) $\displaystyle\lim_{x\to 5} \frac{x^2-3x-10}{x^2-10x+25}$

Solution (a). The numerator and the denominator both have a zero at $x = 3$, so there is a common factor of $x - 3$. Then,

$$\lim_{x\to 3} \frac{x^2-6x+9}{x-3} = \lim_{x\to 3} \frac{(x-3)^2}{x-3} = \lim_{x\to 3}(x-3) = 0$$

Solution (b). The numerator and the denominator both have a zero at $x = -4$, so there is a common factor of $x - (-4) = x + 4$. Then,

$$\lim_{x \to -4} \frac{2x + 8}{x^2 + x - 12} = \lim_{x \to -4} \frac{2(x + 4)}{(x + 4)(x - 3)} = \lim_{x \to -4} \frac{2}{x - 3} = -\frac{2}{7}$$

Solution (c). The numerator and the denominator both have a zero at $x = 5$, so there is a common factor of $x - 5$. Then,

$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5} \frac{(x - 5)(x + 2)}{(x - 5)(x - 5)} = \lim_{x \to 5} \frac{x + 2}{x - 5}$$

However,

$$\lim_{x \to 5} (x + 2) = 7 \neq 0 \quad \text{and} \quad \lim_{x \to 5} (x - 5) = 0$$

By Theorem 2.2.4(*b*),

$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5} \frac{x + 2}{x - 5}$$

does not exist. ◀

The case of a limit of a quotient,

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, is called an ***indeterminate form of type 0/0***. Note that the limits in Examples 4 and 5 produced a variety of answers. The word "indeterminate" here refers to the fact that the limiting behavior of the quotient cannot be determined without further study. The expression "0/0" is just a mnemonic device to describe the circumstance of a limit of a quotient in which both the numerator and denominator approach 0.

LIMITS INVOLVING RADICALS

Example 6 Find $\displaystyle\lim_{x \to 0} \frac{x}{\sqrt{x + 1} - 1}$.

Solution. Recall that in Example 2 of Section 2.1 we conjectured this limit to be 2. Note that this limit expression is an indeterminate form of type 0/0, so Theorem 2.2.2(*d*) does not apply. One strategy for resolving this limit is to first rationalize the denominator of the function. This yields

$$\frac{x}{\sqrt{x + 1} - 1} = \frac{x(\sqrt{x + 1} + 1)}{(x + 1) - 1} = \sqrt{x + 1} + 1, \quad x \neq 0$$

Therefore,

$$\lim_{x \to 0} \frac{x}{\sqrt{x + 1} - 1} = \lim_{x \to 0} (\sqrt{x + 1} + 1) = 2$$

◀

LIMITS OF PIECEWISE-DEFINED FUNCTIONS

For functions that are defined piecewise, a two-sided limit at an x-value where the formula changes is best obtained by first finding the one-sided limits at that number.

Example 7 Let

$$f(x) = \begin{cases} 1/(x + 2), & x < -2 \\ x^2 - 5, & -2 < x \leq 3 \\ \sqrt{x + 13}, & x > 3 \end{cases}$$

Find

(a) $\displaystyle\lim_{x \to -2} f(x)$ (b) $\displaystyle\lim_{x \to 0} f(x)$ (c) $\displaystyle\lim_{x \to 3} f(x)$

Solution (a). As x approaches -2 from the left, the formula for f is

$$f(x) = \frac{1}{x+2}$$

so that

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{1}{x+2} = -\infty$$

As x approaches -2 from the right, the formula for f is

$$f(x) = x^2 - 5$$

so that

$$\lim_{x\to -2^+} f(x) = \lim_{x\to 2^+} (x^2-5) = (-2)^2 - 5 = -1$$

Thus, $\lim_{x\to -2} f(x)$ does not exist.

Solution (b). As x approaches 0 from either the left or the right, the formula for f is

$$f(x) = x^2 - 5$$

Thus,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(x^2-5) = 0^2 - 5 = -5$$

Solution (c). As x approaches 3 from the left, the formula for f is

$$f(x) = x^2 - 5$$

so that

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2-5) = 3^2 - 5 = 4$$

As x approaches 3 from the right, the formula for f is

$$f(x) = \sqrt{x+13}$$

so that

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x+13} = \sqrt{\lim_{x\to 3^+}(x+13)} = \sqrt{3+13} = 4$$

Since the one-sided limits are equal, we have

$$\lim_{x\to 3} f(x) = 4$$ ◀

EXERCISE SET 2.2

1. In each part, find the limit by inspection.

(a) $\lim_{x\to 8} 7$ (b) $\lim_{x\to 0^+} \pi$

(c) $\lim_{x\to -2} 3x$ (d) $\lim_{y\to 3^+} 12y$

2. In each part, find the stated limit of $f(x) = x/|x|$ by inspection.

(a) $\lim_{x\to 5} f(x)$ (b) $\lim_{x\to -5} f(x)$

(c) $\lim_{x\to 0^+} f(x)$ (d) $\lim_{x\to 0^-} f(x)$

3. Given that

$$\lim_{x\to a} f(x) = 2, \quad \lim_{x\to a} g(x) = -4, \quad \lim_{x\to a} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to a} [f(x) + 2g(x)]$ (b) $\lim_{x\to a} [h(x) - 3g(x) + 1]$

(c) $\lim_{x\to a} [f(x)g(x)]$ (d) $\lim_{x\to a} [g(x)]^2$

(e) $\lim_{x\to a} \sqrt[3]{6 + f(x)}$ (f) $\lim_{x\to a} \dfrac{2}{g(x)}$

(g) $\lim_{x\to a} \dfrac{3f(x) - 8g(x)}{h(x)}$ (h) $\lim_{x\to a} \dfrac{7g(x)}{2f(x) + g(x)}$

4. Use the graphs of f and g in the accompanying figure to find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to 2}[f(x)+g(x)]$ (b) $\lim_{x\to 0}[f(x)+g(x)]$

(c) $\lim_{x\to 0^+}[f(x)+g(x)]$ (d) $\lim_{x\to 0^-}[f(x)+g(x)]$

(e) $\lim_{x\to 2}\dfrac{f(x)}{1+g(x)}$ (f) $\lim_{x\to 2}\dfrac{1+g(x)}{f(x)}$

(g) $\lim_{x\to 0^+}\sqrt{f(x)}$ (h) $\lim_{x\to 0^-}\sqrt{f(x)}$

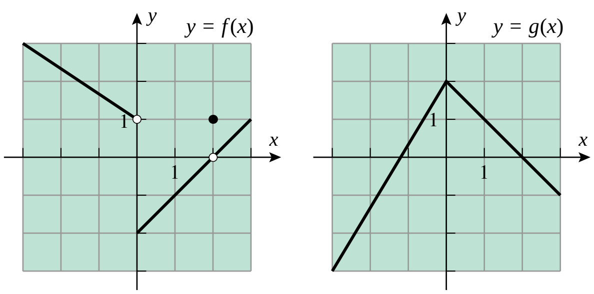


Figure Ex-4

In Exercises 5–30, find the limits.

5. $\lim_{y\to 2^-}\dfrac{(y-1)(y-2)}{y+1}$ **6.** $\lim_{x\to 3}\dfrac{x^2-2x}{x+1}$

7. $\lim_{x\to 4}\dfrac{x^2-16}{x-4}$ **8.** $\lim_{x\to 0}\dfrac{6x-9}{x^3-12x+3}$

9. $\lim_{x\to 1^+}\dfrac{x^4-1}{x-1}$ **10.** $\lim_{t\to -2}\dfrac{t^3+8}{t+2}$

11. $\lim_{x\to -1}\dfrac{x^2+6x+5}{x^2-3x-4}$ **12.** $\lim_{x\to 2}\dfrac{x^2-4x+4}{x^2+x-6}$

13. $\lim_{t\to 2}\dfrac{t^3+3t^2-12t+4}{t^3-4t}$ **14.** $\lim_{t\to 1}\dfrac{t^3+t^2-5t+3}{t^3-3t+2}$

15. $\lim_{x\to 3^+}\dfrac{x}{x-3}$ **16.** $\lim_{x\to 3^-}\dfrac{x}{x-3}$

17. $\lim_{x\to 3}\dfrac{x}{x-3}$ **18.** $\lim_{x\to 2^+}\dfrac{x}{x^2-4}$

19. $\lim_{x\to 2^-}\dfrac{x}{x^2-4}$ **20.** $\lim_{x\to 2}\dfrac{x}{x^2-4}$

21. $\lim_{y\to 6^+}\dfrac{y+6}{y^2-36}$ **22.** $\lim_{y\to 6^-}\dfrac{y+6}{y^2-36}$

23. $\lim_{y\to 6}\dfrac{y+6}{y^2-36}$ **24.** $\lim_{x\to 4^+}\dfrac{3-x}{x^2-2x-8}$

25. $\lim_{x\to 4^-}\dfrac{3-x}{x^2-2x-8}$ **26.** $\lim_{x\to 4}\dfrac{3-x}{x^2-2x-8}$

27. $\lim_{x\to 2^+}\dfrac{1}{|2-x|}$ **28.** $\lim_{x\to 3^-}\dfrac{1}{|x-3|}$

29. $\lim_{x\to 9}\dfrac{x-9}{\sqrt{x}-3}$ **30.** $\lim_{y\to 4}\dfrac{4-y}{2-\sqrt{y}}$

31. Verify the limit in Example 1 of Section 2.1. That is, find

$$\lim_{t_1\to 0.5}\frac{-16t_1^2+29t_1-10.5}{t_1-0.5}$$

32. Let $s(t)=-16t^2+29t+6$. Find

$$\lim_{t\to 1.5}\frac{s(t)-s(1.5)}{t-1.5}$$

33. Let

$$f(x)=\begin{cases} x-1, & x\le 3\\ 3x-7, & x>3\end{cases}$$

Find

(a) $\lim_{x\to 3^-}f(x)$ (b) $\lim_{x\to 3^+}f(x)$ (c) $\lim_{x\to 3}f(x)$.

34. Let

$$g(t)=\begin{cases} t^2, & t\ge 0\\ t-2, & t<0\end{cases}$$

Find

(a) $\lim_{t\to 0^-}g(t)$ (b) $\lim_{t\to 0^+}g(t)$ (c) $\lim_{t\to 0}g(t)$.

35. Let $f(x)=\dfrac{x^3-1}{x-1}$.

(a) Find $\lim_{x\to 1}f(x)$.

(b) Sketch the graph of $y=f(x)$.

36. Let

$$f(x)=\begin{cases} \dfrac{x^2-9}{x+3}, & x\ne -3\\ k, & x=-3\end{cases}$$

(a) Find k so that $f(-3)=\lim_{x\to -3}f(x)$.

(b) With k assigned the value $\lim_{x\to -3}f(x)$, show that $f(x)$ can be expressed as a polynomial.

37. (a) Explain why the following calculation is incorrect.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=\lim_{x\to 0^+}\frac{1}{x}-\lim_{x\to 0^+}\frac{1}{x^2}$$
$$=+\infty-(+\infty)=0$$

(b) Show that $\lim_{x\to 0^+}\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)=-\infty$.

38. Find $\lim_{x\to 0^-}\left(\dfrac{1}{x}+\dfrac{1}{x^2}\right)$.

In Exercises 39 and 40, first rationalize the numerator, then find the limit.

39. $\lim_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}$ **40.** $\lim_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x}$

41. Let $p(x)$ and $q(x)$ be polynomials, and suppose $q(x_0)=0$. Discuss the behavior of the graph of $y=p(x)/q(x)$ in the vicinity of $x=x_0$. Give examples to support your conclusions.

2.3 COMPUTING LIMITS: END BEHAVIOR

In this section we will discuss algebraic techniques for computing limits at $\pm\infty$ for many functions. We base these results on the informal development of the limit concept discussed in Section 2.1. A more formal development of these results is possible after Section 2.4.

SOME BASIC LIMITS

The behavior of a function toward the extremes of its domain is sometimes called its ***end behavior***. Here we will use limits to investigate the end behavior of a function as $x \to -\infty$ or as $x \to +\infty$. As in the last section, we will begin by obtaining limits of some simple functions and then use these as building blocks for finding limits of more complicated functions.

2.3.1 THEOREM. *Let k be a real number.*

$$\lim_{x\to-\infty} k = k \qquad \lim_{x\to+\infty} k = k$$

$$\lim_{x\to-\infty} x = -\infty \qquad \lim_{x\to+\infty} x = +\infty$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0 \qquad \lim_{x\to+\infty} \frac{1}{x} = 0$$

The six limits in Theorem 2.3.1 should be evident from inspection of the function graphs in Figure 2.3.1.

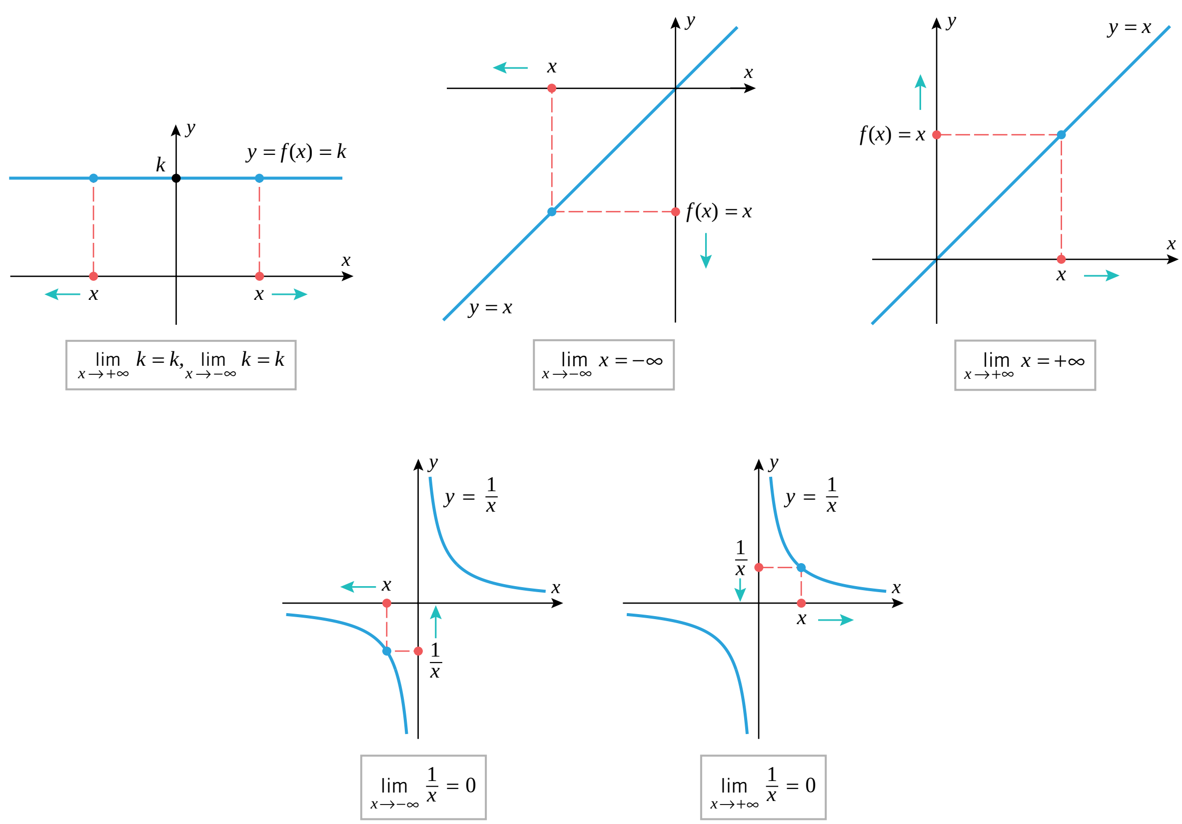


Figure 2.3.1

The limits of the reciprocal function $f(x) = 1/x$ should make sense to you intuitively, based on your experience with fractions: increasing the magnitude of x makes its reciprocal closer to zero. This is illustrated in Table 2.3.1.

Table 2.3.1

	VALUES						CONCLUSION
x	–1	–10	–100	–1000	–10,000	$\cdots$	As $x \to -\infty$ the value of $1/x$
$1/x$	–1	–0.1	–0.01	–0.001	–0.0001	$\cdots$	increases toward zero.
x	1	10	100	1000	10,000	$\cdots$	As $x \to +\infty$ the value of $1/x$
$1/x$	1	0.1	0.01	0.001	0.0001	$\cdots$	decreases toward zero.

The following theorem mirrors Theorem 2.2.2 as our tool for finding limits at $\pm\infty$ algebraically. (The proof is similar to that of the portions of Theorem 2.2.2 that are proved in Appendix G.)

2.3.2 THEOREM. *Suppose that*

$$\lim_{x\to+\infty} f(x) = L_1 \quad \textit{and} \quad \lim_{x\to+\infty} g(x) = L_2$$

That is, the limits exist and have values L_1 and L_2, respectively. Then,

(*a*) $\displaystyle \lim_{x\to+\infty}[f(x)+g(x)] = \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} g(x) = L_1 + L_2$

(*b*) $\displaystyle \lim_{x\to+\infty}[f(x)-g(x)] = \lim_{x\to+\infty} f(x) - \lim_{x\to+\infty} g(x) = L_1 - L_2$

(*c*) $\displaystyle \lim_{x\to+\infty}[f(x)g(x)] = \left(\lim_{x\to+\infty} f(x)\right)\left(\lim_{x\to+\infty} g(x)\right) = L_1L_2$

(*d*) $\displaystyle \lim_{x\to+\infty}\frac{f(x)}{g(x)} = \frac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} g(x)} = \frac{L_1}{L_2},$ *provided* $L_2 \neq 0$

(*e*) $\displaystyle \lim_{x\to+\infty}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to+\infty} f(x)} = \sqrt[n]{L_1},$ *provided* $L_1 > 0$ *if* n *is even.*

Moreover, these statements are also true if $x \to -\infty$.

REMARK. As in the remark following Theorem 2.2.2, results (*a*) and (*c*) can be extended to sums or products of any finite number of functions. In particular, for any positive integer n,

$$\lim_{x\to+\infty}(f(x))^n = \left(\lim_{x\to+\infty} f(x)\right)^n \qquad \lim_{x\to-\infty}(f(x))^n = \left(\lim_{x\to-\infty} f(x)\right)^n$$

Also, since $\lim_{x\to+\infty}(1/x) = 0$, if n is a positive integer, then

$$\lim_{x\to+\infty}\frac{1}{x^n} = \left(\lim_{x\to+\infty}\frac{1}{x}\right)^n = 0 \qquad \lim_{x\to-\infty}\frac{1}{x^n} = \left(\lim_{x\to-\infty}\frac{1}{x}\right)^n = 0 \tag{1}$$

For example,

$$\lim_{x\to+\infty}\frac{1}{x^4} = 0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{x^4} = 0$$

Another useful result follows from part (*c*) of Theorem 2.3.2 in the special case where one of the factors is a constant k:

$$\lim_{x\to+\infty}(k\cdot f(x)) = \left(\lim_{x\to+\infty} k\right)\cdot\left(\lim_{x\to+\infty} f(x)\right) = k\cdot\left(\lim_{x\to+\infty} f(x)\right) \tag{2}$$

and similarly, for $\lim_{x\to+\infty}$ replaced by $\lim_{x\to-\infty}$. Rephrased, this last statement says:

A constant factor can be moved through a limit symbol.

LIMITS OF x^n AS $x \to \pm\infty$

In Figure 2.3.2 we have graphed the polynomials of the form x^n for $n = 1, 2, 3$, and 4. Below each figure we have indicated the limits as $x \to +\infty$ and as $x \to -\infty$. The results in the figure are special cases of the following general results:

$$\lim_{x\to+\infty} x^n = +\infty, \quad n = 1, 2, 3, \ldots \tag{3}$$

$$\lim_{x\to-\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \ldots \\ +\infty, & n = 2, 4, 6, \ldots \end{cases} \tag{4}$$

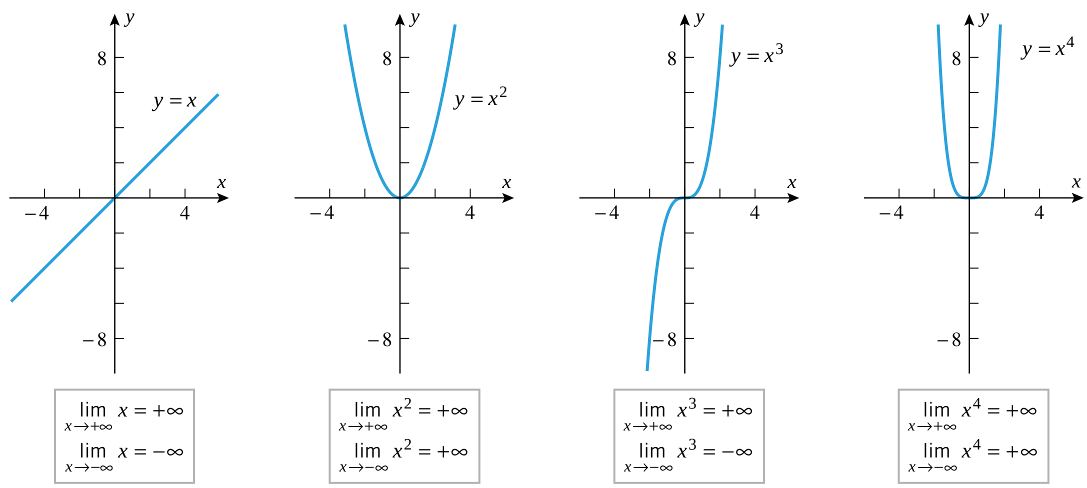


Figure 2.3.2

Multiplying x^n by a positive real number does not affect limits (3) and (4), but multiplying by a negative real number reverses the sign.

Example 1

$$\lim_{x\to+\infty} 2x^5 = +\infty, \qquad \lim_{x\to-\infty} 2x^5 = -\infty$$

$$\lim_{x\to+\infty} -7x^6 = -\infty, \qquad \lim_{x\to-\infty} -7x^6 = -\infty$$

◀

LIMITS OF POLYNOMIALS AS $x \to \pm\infty$

There is a useful principle about polynomials which, expressed informally, states that:

The end behavior of a polynomial matches the end behavior of its highest degree term.

More precisely, if $c_n \neq 0$ then

$$\lim_{x\to-\infty} \left(c_0 + c_1x + \cdots + c_nx^n\right) = \lim_{x\to-\infty} c_nx^n \tag{5}$$

$$\lim_{x\to+\infty} \left(c_0 + c_1x + \cdots + c_nx^n\right) = \lim_{x\to+\infty} c_nx^n \tag{6}$$

We can motivate these results by factoring out the highest power of x from the polynomial and examining the limit of the factored expression. Thus,

$$c_0 + c_1 x + \cdots + c_n x^n = x^n \left(\frac{c_0}{x^n} + \frac{c_1}{x^{n-1}} + \cdots + c_n \right)$$

As $x \to -\infty$ or $x \to +\infty$, it follows from (1) that all of the terms with positive powers of x in the denominator approach 0, so (5) and (6) are certainly plausible.

Example 2

$$\lim_{x\to-\infty} (7x^5 - 4x^3 + 2x - 9) = \lim_{x\to-\infty} 7x^5 = -\infty$$

$$\lim_{x\to-\infty} (-4x^8 + 17x^3 - 5x + 1) = \lim_{x\to-\infty} -4x^8 = -\infty$$

◀

LIMITS OF RATIONAL FUNCTIONS AS $x \to \pm\infty$

A useful technique for determining the end behavior of a rational function $f(x) = n(x)/d(x)$ is to factor and cancel the highest power of x that occurs in the denominator $d(x)$ from both $n(x)$ and $d(x)$. The denominator of the resulting fraction then has a (nonzero) limit equal to the leading coefficient of $d(x)$, so the limit of the resulting fraction can be quickly determined using (1), (5), and (6). The following examples illustrate this technique.

Example 3 Find $\displaystyle\lim_{x\to+\infty} \frac{3x+5}{6x-8}$.

Solution. Divide the numerator and denominator by the highest power of x that occurs in the denominator; that is, $x^1 = x$. We obtain

$$\lim_{x\to+\infty} \frac{3x+5}{6x-8} = \lim_{x\to+\infty} \frac{x(3+5/x)}{x(6-8/x)} = \lim_{x\to+\infty} \frac{3+5/x}{6-8/x} = \frac{\displaystyle\lim_{x\to+\infty}(3+5/x)}{\displaystyle\lim_{x\to+\infty}(6-8/x)}$$

$$= \frac{\displaystyle\lim_{x\to+\infty} 3 + \lim_{x\to+\infty} 5/x}{\displaystyle\lim_{x\to+\infty} 6 - \lim_{x\to+\infty} 8/x} = \frac{3 + 5\displaystyle\lim_{x\to+\infty} 1/x}{6 - 8\displaystyle\lim_{x\to+\infty} 1/x}$$

$$= \frac{3 + (5\cdot 0)}{6 - (8 \cdot 0)} = \frac{1}{2}$$

◀

Example 4 Find

(a) $\displaystyle\lim_{x\to-\infty} \frac{4x^2 - x}{2x^3 - 5}$ (b) $\displaystyle\lim_{x\to-\infty} \frac{5x^3 - 2x^2 + 1}{3x + 5}$

Solution (a). Divide the numerator and denominator by the highest power of x that occurs in the denominator, namely x^3. We obtain

$$\lim_{x\to-\infty} \frac{4x^2 - x}{2x^3 - 5} = \lim_{x\to-\infty} \frac{x^3(4/x - 1/x^2)}{x^3(2 - 5/x^3)} = \lim_{x\to-\infty} \frac{4/x - 1/x^2}{2 - 5/x^3}$$

$$= \frac{\displaystyle\lim_{x\to-\infty} (4/x - 1/x^2)}{\displaystyle\lim_{x\to-\infty} (2 - 5/x^3)} = \frac{(4\cdot 0) - 0}{2 - (5 \cdot 0)} = \frac{0}{2} = 0$$

Solution (b). Divide the numerator and denominator by x to obtain

$$\lim_{x\to-\infty} \frac{5x^3 - 2x^2 + 1}{3x + 5} = \lim_{x\to-\infty} \frac{5x^2 - 2x + 1/x}{3 + 5/x} = +\infty$$

where the final step is justified by the fact that

$$5x^2 - 2x \to +\infty, \quad \frac{1}{x} \to 0, \quad \text{and} \quad 3 + \frac{5}{x} \to 3$$

as $x \to -\infty$. ◀

LIMITS INVOLVING RADICALS

Example 5 Find $\lim_{x\to+\infty} \sqrt[3]{\dfrac{3x+5}{6x-8}}$.

Solution.

$$\lim_{x\to+\infty} \sqrt[3]{\frac{3x+5}{6x-8}} = \sqrt[3]{\lim_{x\to+\infty} \frac{3x+5}{6x-8}} \qquad \text{Theorem 2.3.2}(e)$$
$$= \sqrt[3]{\frac{1}{2}} \qquad \text{Example 3}$$ ◀

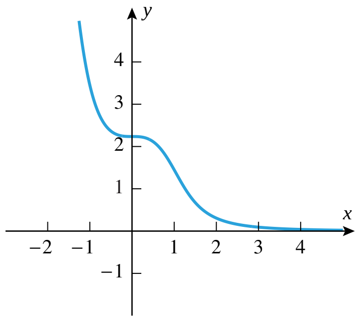
Example 6 Find

(a) $\lim_{x\to+\infty} \dfrac{\sqrt{x^2+2}}{3x-6}$ (b) $\lim_{x\to-\infty} \dfrac{\sqrt{x^2+2}}{3x-6}$

In both parts it would be helpful to manipulate the function so that the powers of x are transformed to powers of $1/x$. This can be achieved in both cases by dividing the numerator and denominator by $|x|$ and using the fact that $\sqrt{x^2} = |x|$.

Solution (a). As $x \to +\infty$, the values of x under consideration are positive, so we can replace $|x|$ by x where helpful. We obtain

$$\lim_{x\to+\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x\to+\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} = \lim_{x\to+\infty} \frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/x}$$
$$= \lim_{x\to+\infty} \frac{\sqrt{1+2/x^2}}{3-6/x} = \frac{\lim_{x\to+\infty} \sqrt{1+2/x^2}}{\lim_{x\to+\infty} (3-6/x)}$$
$$= \frac{\sqrt{\lim_{x\to+\infty} (1+2/x^2)}}{\lim_{x\to+\infty} (3-6/x)} = \frac{\sqrt{\left(\lim_{x\to+\infty} 1\right) + \left(2 \lim_{x\to+\infty} 1/x^2\right)}}{\left(\lim_{x\to+\infty} 3\right) - \left(6 \lim_{x\to+\infty} 1/x\right)}$$
$$= \frac{\sqrt{1+(2\cdot 0)}}{3-(6\cdot 0)} = \frac{1}{3}$$

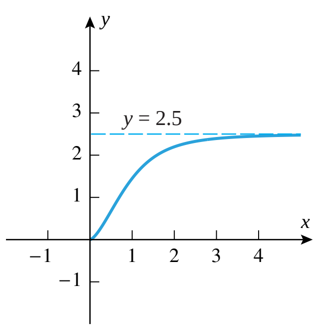


$y = \sqrt{x^6+5} - x^3$

(*a*)

Solution (b). As $x \to -\infty$, the values of x under consideration are negative, so we can replace $|x|$ by $-x$ where helpful. We obtain

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x\to-\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} = \lim_{x\to-\infty} \frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/(-x)}$$
$$= \lim_{x\to-\infty} \frac{\sqrt{1+2/x^2}}{-3+6/x} = -\frac{1}{3}$$ ◀



$y = \sqrt{x^6+5x^3} - x^3, x \geq 0$

(*b*)

Figure 2.3.3

FOR THE READER. Use a graphing utility to explore the end behavior of

$$f(x) = \frac{\sqrt{x^2+2}}{3x-6}$$

Your investigation should support the results of Example 6.

Example 7 Find

(a) $\lim_{x\to+\infty} (\sqrt{x^6+5} - x^3)$ (b) $\lim_{x\to+\infty} (\sqrt{x^6+5x^3} - x^3)$

Solution. Graphs of the functions $f(x) = \sqrt{x^6+5} - x^3$ and $g(x) = \sqrt{x^6+5x^3} - x^3$ for $x \geq 0$ are shown in Figure 2.3.3. From the graphs we might conjecture that the limits are 0 and 2.5, respectively. To confirm this, we treat each function as a fraction with denominator

1 and rationalize the numerator.

$$\lim_{x\to+\infty}(\sqrt{x^6+5}-x^3) = \lim_{x\to+\infty}(\sqrt{x^6+5}-x^3)\left(\frac{\sqrt{x^6+5}+x^3}{\sqrt{x^6+5}+x^3}\right)$$

$$= \lim_{x\to+\infty}\frac{(x^6+5)-x^6}{\sqrt{x^6+5}+x^3} = \lim_{x\to+\infty}\frac{5}{\sqrt{x^6+5}+x^3}$$

$$= \lim_{x\to+\infty}\frac{5/x^3}{\sqrt{1+5/x^6}+1} \qquad \boxed{\sqrt{x^6}=x^3 \text{ for } x>0}$$

$$= \frac{0}{\sqrt{1+0}+1} = 0$$

$$\lim_{x\to+\infty}(\sqrt{x^6+5x^3}-x^3) = \lim_{x\to+\infty}(\sqrt{x^6+5x^3}-x^3)\left(\frac{\sqrt{x^6+5x^3}+x^3}{\sqrt{x^6+5x^3}+x^3}\right)$$

$$= \lim_{x\to+\infty}\frac{(x^6+5x^3)-x^6}{\sqrt{x^6+5x^3}+x^3} = \lim_{x\to+\infty}\frac{5x^3}{\sqrt{x^6+5x^3}+x^3}$$

$$= \lim_{x\to+\infty}\frac{5}{\sqrt{1+5/x^3}+1} \qquad \boxed{\sqrt{x^6}=x^3 \text{ for } x>0}$$

$$= \frac{5}{\sqrt{1+0}+1} = \frac{5}{2}$$

◀

REMARK. Example 7 illustrates an ***indeterminate form of type*** $\infty - \infty$. Exercises 31–34 explore more examples of this type.

EXERCISE SET 2.3 Graphing Calculator

1. In each part, find the limit by inspection.
(a) $\lim_{x\to-\infty}(-3)$ (b) $\lim_{h\to+\infty}(-2h)$

2. In each part, find the stated limit of $f(x) = x/|x|$ by inspection.
(a) $\lim_{x\to+\infty} f(x)$ (b) $\lim_{x\to-\infty} f(x)$

3. Given that

$$\lim_{x\to+\infty} f(x) = 3, \quad \lim_{x\to+\infty} g(x) = -5, \quad \lim_{x\to+\infty} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.
(a) $\lim_{x\to+\infty}[f(x)+3g(x)]$ (b) $\lim_{x\to+\infty}[h(x)-4g(x)+1]$
(c) $\lim_{x\to+\infty}[f(x)g(x)]$ (d) $\lim_{x\to+\infty}[g(x)]^2$
(e) $\lim_{x\to+\infty}\sqrt[3]{5+f(x)}$ (f) $\lim_{x\to+\infty}\dfrac{3}{g(x)}$
(g) $\lim_{x\to+\infty}\dfrac{3h(x)+4}{x^2}$ (h) $\lim_{x\to+\infty}\dfrac{6f(x)}{5f(x)+3g(x)}$

4. Given that

$$\lim_{x\to-\infty} f(x) = 7, \quad \lim_{x\to-\infty} g(x) = -6$$

find the limits that exist. If the limit does not exist, explain why.
(a) $\lim_{x\to-\infty}[2f(x)-g(x)]$ (b) $\lim_{x\to-\infty}[6f(x)+7g(x)]$
(c) $\lim_{x\to-\infty}[x^2+g(x)]$ (d) $\lim_{x\to-\infty}[x^2g(x)]$
(e) $\lim_{x\to-\infty}\sqrt[3]{f(x)g(x)}$ (f) $\lim_{x\to-\infty}\dfrac{g(x)}{f(x)}$
(g) $\lim_{x\to-\infty}\left[f(x)+\dfrac{g(x)}{x}\right]$ (h) $\lim_{x\to-\infty}\dfrac{xf(x)}{(2x+3)g(x)}$

In Exercises 5–28, find the limits.

5. $\lim_{x\to-\infty}(3-x)$ **6.** $\lim_{x\to-\infty}\left(5-\dfrac{1}{x}\right)$

7. $\lim_{x\to+\infty}(1+2x-3x^5)$ **8.** $\lim_{x\to+\infty}(2x^3-100x+5)$

9. $\lim_{x\to+\infty}\sqrt{x}$ **10.** $\lim_{x\to-\infty}\sqrt{5-x}$

11. $\lim_{x\to+\infty}\dfrac{3x+1}{2x-5}$ **12.** $\lim_{x\to+\infty}\dfrac{5x^2-4x}{2x^2+3}$

13. $\lim_{y\to-\infty}\dfrac{3}{y+4}$ **14.** $\lim_{x\to+\infty}\dfrac{1}{x-12}$

15. $\lim_{x\to-\infty}\dfrac{x-2}{x^2+2x+1}$ **16.** $\lim_{x\to+\infty}\dfrac{5x^2+7}{3x^2-x}$

17. $\lim_{x\to+\infty}\sqrt[3]{\dfrac{2+3x-5x^2}{1+8x^2}}$ **18.** $\lim_{s\to+\infty}\sqrt[3]{\dfrac{3s^7-4s^5}{2s^7+1}}$

19. $\lim_{x\to-\infty} \dfrac{\sqrt{5x^2-2}}{x+3}$ **20.** $\lim_{x\to+\infty} \dfrac{\sqrt{5x^2-2}}{x+3}$

21. $\lim_{y\to-\infty} \dfrac{2-y}{\sqrt{7+6y^2}}$ **22.** $\lim_{y\to+\infty} \dfrac{2-y}{\sqrt{7+6y^2}}$

23. $\lim_{x\to-\infty} \dfrac{\sqrt{3x^4+x}}{x^2-8}$ **24.** $\lim_{x\to+\infty} \dfrac{\sqrt{3x^4+x}}{x^2-8}$

25. $\lim_{x\to+\infty} \dfrac{7-6x^5}{x+3}$ **26.** $\lim_{t\to-\infty} \dfrac{5-2t^3}{t^2+1}$

27. $\lim_{t\to+\infty} \dfrac{6-t^3}{7t^3+3}$ **28.** $\lim_{x\to-\infty} \dfrac{x+4x^3}{1-x^2+7x^3}$

29. Let
$$f(x) = \begin{cases} 2x^2+5, & x<0 \\ \dfrac{3-5x^3}{1+4x+x^3}, & x\geq 0 \end{cases}$$
Find
(a) $\lim_{x\to-\infty} f(x)$ (b) $\lim_{x\to+\infty} f(x)$.

30. Let
$$g(t) = \begin{cases} \dfrac{2+3t}{5t^2+6}, & t<1{,}000{,}000 \\ \dfrac{\sqrt{36t^2-100}}{5-t}, & t>1{,}000{,}000 \end{cases}$$
Find
(a) $\lim_{t\to-\infty} g(t)$ (b) $\lim_{t\to+\infty} g(t)$.

In Exercises 31–34, find the limits.

31. $\lim_{x\to+\infty} (\sqrt{x^2+3}-x)$ **32.** $\lim_{x\to+\infty} (\sqrt{x^2-3x}-x)$

33. $\lim_{x\to+\infty} (\sqrt{x^2+ax}-x)$

34. $\lim_{x\to+\infty} (\sqrt{x^2+ax}-\sqrt{x^2+bx})$

35. Discuss the limits of $p(x)=(1-x)^n$ as $x\to+\infty$ and $x\to-\infty$ for positive integer values of n.

36. Let $p(x)=(1-x)^n$ and $q(x)=(1-x)^m$. Discuss the limits of $p(x)/q(x)$ as $x\to+\infty$ and $x\to-\infty$ for positive integer values of m and n.

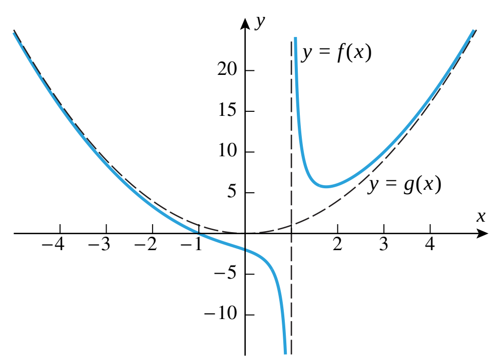
37. Let $p(x)$ be a polynomial of degree n. Discuss the limits of $p(x)/x^m$ as $x\to+\infty$ and $x\to-\infty$ for positive integer values of m.

38. In each part, find examples of polynomials $p(x)$ and $q(x)$ that satisfy the stated condition and such that $p(x)\to+\infty$ and $q(x)\to+\infty$ as $x\to+\infty$.
(a) $\lim_{x\to+\infty} \dfrac{p(x)}{q(x)}=1$ (b) $\lim_{x\to+\infty} \dfrac{p(x)}{q(x)}=0$
(c) $\lim_{x\to+\infty} \dfrac{p(x)}{q(x)}=+\infty$ (d) $\lim_{x\to+\infty} [p(x)-q(x)]=3$

39. Assuming that m and n are positive integers, find
$$\lim_{x\to-\infty} \frac{2+3x^n}{1-x^m}$$
[*Hint:* Your answer will depend on whether $m<n$, $m=n$, or $m>n$.]

40. Find
$$\lim_{x\to+\infty} \frac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_mx^m}$$
where $c_n\neq 0$ and $d_m\neq 0$. [*Hint:* Your answer will depend on whether $m<n$, $m=n$, or $m>n$.]

The notion of an asymptote can be extended to include curves as well as lines. Specifically, we say that ***f(x) is asymptotic to g(x) as x → +∞*** if
$$\lim_{x\to+\infty} [f(x)-g(x)]=0$$
and that ***f(x) is asymptotic to g(x) as x → −∞*** if
$$\lim_{x\to-\infty} [f(x)-g(x)]=0$$
Informally stated, if $f(x)$ is asymptotic to $g(x)$ as $x\to+\infty$, then the graph of $y=f(x)$ gets closer and closer to the graph of $y=g(x)$ as $x\to+\infty$, and if $f(x)$ is asymptotic to $g(x)$ as $x\to-\infty$, then the graph of $y=f(x)$ gets closer and closer to the graph of $y=g(x)$ as $x\to-\infty$. For example, if
$$f(x)=x^2+\frac{2}{x-1} \quad\text{and}\quad g(x)=x^2$$
then $f(x)$ is asymptotic to $g(x)$ as $x\to+\infty$ and as $x\to-\infty$ since
$$\lim_{x\to+\infty} [f(x)-g(x)] = \lim_{x\to+\infty} \frac{1}{x-1}=0$$
$$\lim_{x\to-\infty} [f(x)-g(x)] = \lim_{x\to-\infty} \frac{1}{x-1}=0$$
This asymptotic behavior is illustrated in the following figure, which also shows the vertical asymptote of $f(x)$ at $x=1$.



In Exercises 41–46, determine a function $g(x)$ to which $f(x)$ is asymptotic as $x\to+\infty$ or $x\to-\infty$. Use a graphing utility to generate the graphs of $y=f(x)$ and $y=g(x)$ and identify all vertical asymptotes.

41. $f(x)=\dfrac{x^2-2}{x-2}$ **42.** $f(x)=\dfrac{x^3-x+3}{x}$

43. $f(x)=\dfrac{-x^3+3x^2+x-1}{x-3}$

44. $f(x)=\dfrac{x^5-x^3+3}{x^2-1}$

45. $f(x)=\sin x+\dfrac{1}{x-1}$ **46.** $f(x)=\sqrt{\dfrac{x^3-x^2+2}{x-1}}$

2.4 LIMITS (DISCUSSED MORE RIGOROUSLY)

Thus far, our discussion of limits has been based on our intuitive feeling of what it means for the values of a function to get closer and closer to a limiting value. However, this level of informality can only take us so far, so our goal in this section is to define limits precisely. From a purely mathematical point of view these definitions are needed to establish limits with certainty and to prove theorems about them. However, they will also provide us with a deeper understanding of the limit concept, making it possible for us to visualize some of the more subtle properties of functions.

In Sections 2.1 to 2.3 our emphasis was on the discovery of values of limits, either through the sampling of selected x-values or through the application of limit theorems. In the preceding sections we interpreted $\lim_{x\to a} f(x) = L$ to mean that the values of $f(x)$ can be made as close as we like to L by selecting x-values sufficiently close to a (but not equal to a). Although this informal definition is sufficient for many purposes, we need a more precise definition to verify that a conjectured limit is actually correct, or to prove the limit theorems in Sections 2.2 and 2.3. One of our goals in this section is to give the informal phrases "as close as we like to L" and "sufficiently close to a" a precise mathematical interpretation. This will enable us to replace the informal definition of limit given in Definition 2.1.1 with a more fully developed version that may be used in proofs.

To start, consider the function f graphed in Figure 2.4.1a for which $f(x)\to L$ as $x\to a$. We have intentionally placed a hole in the graph at $x = a$ to emphasize that the function f need not be defined at $x = a$ to have a limit there. Also, to simplify the discussion, we have chosen a function that is increasing on an open interval containing a.

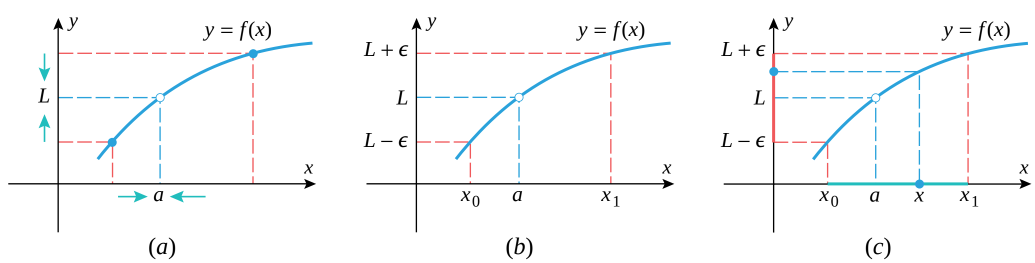


Figure 2.4.1

To motivate an appropriate definition for a two-sided limit, suppose that we choose *any* positive number, say ϵ, and draw horizontal lines from $L+\epsilon$ and $L-\epsilon$ on the y-axis to the curve $y = f(x)$ and then draw vertical lines from those points on the curve to the x-axis. As shown in Figure 2.4.1b, let x_0 and x_1 be points where the vertical lines intersect the x-axis.

Next, imagine that x gets closer and closer to a (from either side). Eventually, x will lie inside the interval (x_0, x_1), which is marked in green in Figure 2.4.1c; and when this happens, the value of $f(x)$ will fall between $L-\epsilon$ and $L+\epsilon$, marked in red in the figure. Thus, we conclude:

> *If $f(x)\to L$ as $x\to a$, then for any positive number ϵ, we can find an open interval (x_0, x_1) on the x-axis that contains a and has the property that for each x in that interval (except possibly for $x = a$), the value of $f(x)$ is between $L-\epsilon$ and $L+\epsilon$.*

FOR THE READER. Consider the limit, $\lim_{x\to 0}(\sin x)/x$, conjectured to be 1 in Example 3 of Section 2.1. Draw a figure similar to Figure 2.4.1 that illustrates the preceding analysis for this limit.

What is important about this result is that it holds no matter how small we make ϵ. However, making ϵ smaller and smaller forces $f(x)$ *closer and closer* to L—which is precisely the concept we were trying to capture mathematically.

Observe that in Figure 2.4.1*c* the interval (x_0, x_1) extends farther on the right side of a than on the left side. However, for many purposes it is preferable to have an interval that extends the same distance on both sides of a. For this purpose, let us choose any positive number δ that is smaller than both $x_1 - a$ and $a - x_0$, and consider the interval $(a-\delta, a+\delta)$. This interval extends the same distance δ on both sides of a and lies inside of the interval (x_0, x_1) (Figure 2.4.2). Moreover, the condition $L - \epsilon < f(x) < L + \epsilon$ holds for every x in this interval (except possibly $x = a$), since this condition holds on the larger interval (x_0, x_1). This is illustrated by graphing f in the window $(a-\delta, a+\delta) \times (L-\epsilon, L+\epsilon)$ and observing that the graph "exits" the window at the sides, not at the top or bottom (except possibly at $x = a$).

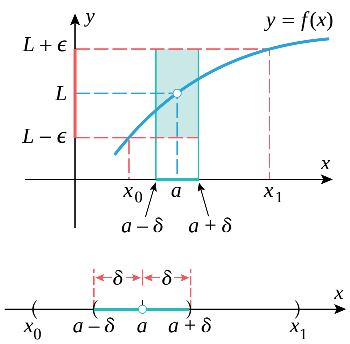


Figure 2.4.2

Example 1 Let $f(x) = \frac{1}{2}x + \frac{1}{4}\sin(\pi x/2)$. It can be shown that $\lim_{x\to 1} f(x) = L = 0.75$. Let $\epsilon = 0.05$.

(a) Use a graphing utility to find an open interval (x_0, x_1) containing $a = 1$ such that for each x in this interval, $f(x)$ is between $L - \epsilon = 0.75 - \epsilon = 0.75 - 0.05 = 0.70$ and $L + \epsilon = 0.75 + \epsilon = 0.75 + 0.05 = 0.80$.

(b) Find a value of δ such that $f(x)$ is between 0.70 and 0.80 for every x in the interval $(1-\delta, 1+\delta)$.

Solution (a). Figure 2.4.3 displays the graph of f. With a graphing utility, we discover that (to five decimal places) the points (0.90769, 0.70122) and (1.09231, 0.79353) are on the graph of f. Suppose that we take $x_0 = 0.908$ and $x_1 = 1.09$. Since the graph of f rises from left to right, we see that for $x_0 = 0.908 < x < 1.090 = x_1$, we have $0.90769 < x < 1.09231$ and therefore $0.7 < 0.70122 < f(x) < 0.79353 < 0.8$.

Solution (b). Since $x_1 - a = 1.09 - 1 = 0.09$ and $a - x_0 = 1 - 0.908 = 0.902$, any value or δ that is less than 0.09 will be acceptable. For example, for $\delta = 0.08$, if x belongs to the interval $(1-\delta, 1+\delta) = (0.92, 1.08)$, then $f(x)$ will lie between 0.70 and 0.80. ◀

Note that the condition $L - \epsilon < f(x) < L + \epsilon$ can be expressed as

$$|f(x) - L| < \epsilon$$

and the condition that x lies in the interval $(a-\delta, a+\delta)$, but $x \neq a$, can be expressed as

$$0 < |x - a| < \delta$$

Thus, we can summarize this discussion in the following definition.

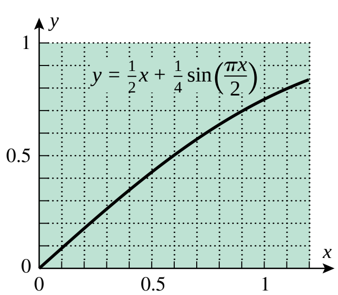


Figure 2.4.3

2.4.1 LIMIT DEFINITION. Let $f(x)$ be defined for all x in some open interval containing the number a, with the possible exception that $f(x)$ need not be defined at a. We will write

$$\lim_{x\to a} f(x) = L$$

if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta$$

REMARK. With this definition we have made the transition from informal to formal in the definition of a two-sided limit. The phrase "as close as we like to L" has been given quantitative meaning by the number $\epsilon > 0$, and the phrase "sufficiently close to a" has been made precise by the number $\delta > 0$. Commonly known as the "ϵ-δ definition" of a limit, Definition 2.4.1 was developed primarily by the German mathematician Karl Weierstrass* in the nineteenth century.

The definitions for one-sided limits are similar to Definition 2.4.1. For example, in the definition of $\lim_{x\to a^+} f(x)$ we assume that $f(x)$ is defined for all x in an interval of the form (a, b) and replace the condition $0 < |x - a| < \delta$ by the condition $a < x < a + \delta$. Comparable changes are made in the definition of $\lim_{x\to a^-} f(x)$.

In the preceding sections we illustrated various numerical and graphical methods for *guessing* at limits. Now that we have a precise definition to work with, we can actually confirm the validity of those guesses with mathematical proof. Here is a typical example of such a proof.

Example 2 Use Definition 2.4.1 to prove that $\lim\limits_{x\to 2}(3x - 5) = 1$.

Solution. We must show that given any positive number ϵ, we can find a positive number δ such that

$$|\underbrace{(3x-5)}_{f(x)} - \underbrace{1}_{L}| < \epsilon \quad \text{if} \quad 0 < |x - \underbrace{2}_{a}| < \delta \tag{1}$$

There are two things to do. First, we must *discover* a value of δ for which this statement holds, and then we must *prove* that the statement holds for that δ. For the discovery part we begin by simplifying (1) and writing it as

$$|3x - 6| < \epsilon \quad \text{if} \quad 0 < |x - 2| < \delta$$

Next, we will rewrite this statement in a form that will facilitate the discovery of an appropriate δ:

$$\begin{aligned} 3|x-2| &< \epsilon \quad &\text{if} \quad 0 < |x-2| < \delta \\ |x-2| &< \epsilon/3 \quad &\text{if} \quad 0 < |x-2| < \delta \end{aligned} \tag{2}$$

It should be self-evident that this last statement holds if $\delta = \epsilon/3$, which completes the discovery portion of our work. Now we need to prove that (1) holds for this choice of δ. However, statement (1) is equivalent to (2), and (2) holds with $\delta = \epsilon/3$, so (1) also holds with $\delta = \epsilon/3$. This proves that $\lim_{x\to 2}(3x - 5) = 1$. ◀

*KARL WEIERSTRASS (1815–1897). Weierstrass, the son of a customs officer, was born in Ostenfelde, Germany. As a youth Weierstrass showed outstanding skills in languages and mathematics. However, at the urging of his dominant father, Weierstrass entered the law and commerce program at the University of Bonn. To the chagrin of his family, the rugged and congenial young man concentrated instead on fencing and beer drinking. Four years later he returned home without a degree. In 1839 Weierstrass entered the Academy of Münster to study for a career in secondary education, and he met and studied under an excellent mathematician named Christof Gudermann. Gudermann's ideas greatly influenced the work of Weierstrass. After receiving his teaching certificate, Weierstrass spent the next 15 years in secondary education teaching German, geography, and mathematics. In addition, he taught handwriting to small children. During this period much of Weierstrass's mathematical work was ignored because he was a secondary schoolteacher and not a college professor. Then, in 1854, he published a paper of major importance that created a sensation in the mathematics world and catapulted him to international fame overnight. He was immediately given an honorary Doctorate at the University of Königsberg and began a new career in college teaching at the University of Berlin in 1856. In 1859 the strain of his mathematical research caused a temporary nervous breakdown and led to spells of dizziness that plagued him for the rest of his life. Weierstrass was a brilliant teacher and his classes overflowed with multitudes of auditors. In spite of his fame, he never lost his early beer-drinking congeniality and was always in the company of students, both ordinary and brilliant. Weierstrass was acknowledged as the leading mathematical analyst in the world. He and his students opened the door to the modern school of mathematical analysis.

REMARK. This example illustrates the general form of a limit proof: We *assume* that we are given a positive number ϵ, and we try to *prove* that we can find a positive number δ such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta \tag{3}$$

This is done by first discovering δ, and then proving that the discovered δ works. Since the argument has to be general enough to work for all positive values of ϵ, the quantity δ has to be expressed as a function of ϵ. In Example 2 we found the function $\delta = \epsilon/3$ by some simple algebra; however, most limit proofs require a little more algebraic and logical ingenuity. Thus, if you find our ensuing discussion of "ϵ-δ" proofs challenging, do not become discouraged; the concepts and techniques are intrinsically difficult. In fact, a precise understanding of limits evaded the finest mathematical minds for more than 150 years after the basic concepts of calculus were discovered.

Example 3 Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Solution. Note that the domain of $\sqrt{x}$ is $0 \le x$, so it is valid to discuss the limit as $x \to 0^+$. We must show that given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\sqrt{x} - 0| < \epsilon \quad \text{if} \quad 0 < x < 0 + \delta$$

or more simply,

$$\sqrt{x} < \epsilon \quad \text{if} \quad 0 < x < \delta \tag{4}$$

But, by squaring both sides of the inequality $\sqrt{x} < \epsilon$, we can rewrite (4) as

$$x < \epsilon^2 \quad \text{if} \quad 0 < x < \delta \tag{5}$$

It should be self-evident that (5) is true if $\delta = \epsilon^2$; and since (5) is a reformulation of (4), we have shown that (4) holds with $\delta = \epsilon^2$. This proves that $\lim_{x \to 0^+} \sqrt{x} = 0$. ◀

REMARK. In this example the limit from the left and the two-sided limit do not exist at $x = 0$ because the domain of $\sqrt{x}$ includes no numbers to the left of 0.

THE VALUE OF δ IS NOT UNIQUE

In preparation for our next example, we note that the value of δ in Definition 2.4.1 is not unique; once we have found a value of δ that fulfills the requirements of the definition, then any *smaller* positive number δ_1 will also fulfill those requirements. That is, if it is true that

$$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta$$

then it will also be true that

$$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta_1$$

This is because $\{x : 0 < |x - a| < \delta_1\}$ is a subset of $\{x : 0 < |x - a| < \delta\}$ (Figure 2.4.4), and hence if $|f(x) - L| < \epsilon$ is satisfied for all x in the larger set, then it will automatically be satisfied for all x in the subset. Thus, in Example 2, where we used $\delta = \epsilon/3$, we could have used any smaller value of δ such as $\delta = \epsilon/4$, $\delta = \epsilon/5$, or $\delta = \epsilon/6$.

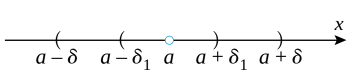


Figure 2.4.4

Example 4 Prove that $\lim_{x \to 3} x^2 = 9$.

Solution. We must show that given any positive number ϵ, we can find a positive number δ such that

$$|x^2 - 9| < \epsilon \quad \text{if} \quad 0 < |x - 3| < \delta \tag{6}$$

Because $|x - 3|$ occurs on the right side of this "if statement," it will be helpful to factor the left side to introduce a factor of $|x - 3|$. This yields the following alternative form of (6)

$$|x + 3||x - 3| < \epsilon \quad \text{if} \quad 0 < |x - 3| < \delta \tag{7}$$

Using the triangle inequality, we see that

$$|x+3| = |(x-3)+6| \leq |x-3| + 6$$

Therefore, if $0 < |x-3| < \delta$ then

$$|x+3||x-3| \leq (|x-3|+6)|x-3| < (\delta+6)\delta$$

It follows that (7) will be satisfied for any positive value of δ such that $(\delta+6)\delta \leq \epsilon$. Let us agree to restrict our attention to positive values of δ such that $\delta \leq 1$. (This is justified because of our earlier observation that once a value of δ is found, then any smaller positive value of δ can be used.) With this restriction, $(\delta+6)\delta \leq 7\delta$, so that (7) will be satisfied as long as it is also the case that $7\delta \leq \epsilon$. We can achieve this by taking δ to be the minimum of the numbers $\epsilon/7$ and 1, which is sometimes written as $\delta = \min(\epsilon/7, 1)$. This proves that $\lim_{x\to 3} x^2 = 9$. ◀

REMARK. You may have wondered how we knew to make the restriction $\delta \leq 1$ (as opposed to $\delta \leq \frac{1}{2}$ or $\delta \leq 5$, for example). Actually, it does not matter; any restriction of the form $\delta \leq c$ would work equally well.

LIMITS AS $x \to \pm\infty$

In Section 2.1 we discussed the limits

$$\lim_{x\to+\infty} f(x) = L \quad \text{and} \quad \lim_{x\to-\infty} f(x) = L$$

from an intuitive viewpoint. We interpreted the first statement to mean that the values of $f(x)$ eventually get closer and closer to L as x increases indefinitely, and we interpreted the second statement to mean that the values of $f(x)$ eventually get closer and closer to L as x decreases indefinitely. These ideas are captured more precisely in the following definitions and are illustrated in Figure 2.4.5.

2.4.2 DEFINITION. Let $f(x)$ be defined for all x in some infinite open interval extending in the positive x-direction. We will write

$$\lim_{x\to+\infty} f(x) = L$$

if given any number $\epsilon > 0$, there corresponds a positive number N such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad x > N$$

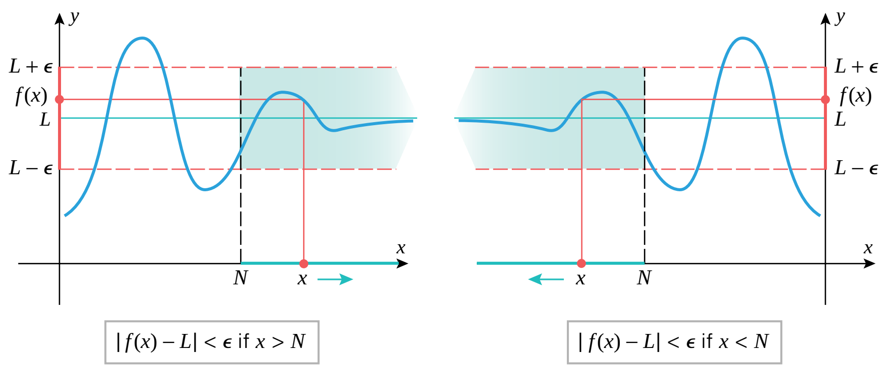
2.4.3 DEFINITION. Let $f(x)$ be defined for all x in some infinite open interval extending in the negative x-direction. We will write

$$\lim_{x\to-\infty} f(x) = L$$

if given any number $\epsilon > 0$, there corresponds a negative number N such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad x < N$$

To see how these definitions relate to our informal concepts of these limits, suppose that $f(x) \to L$ as $x \to +\infty$, and for a given ϵ let N be the positive number described in Definition 2.4.2. If x is allowed to increase indefinitely, then eventually x will lie in the interval $(N, +\infty)$, which is marked in green in Figure 2.4.5*a*; when this happens, the value of $f(x)$ will fall between $L-\epsilon$ and $L+\epsilon$, marked in red in the figure. Since this is true for all positive values of ϵ (no matter how small), we can force the values of $f(x)$ as close as we like to L by making N sufficiently large. This agrees with our informal concept of this limit. Similarly, Figure 2.4.5*b* illustrates Definition 2.4.3.



(*a*) (*b*)

Figure 2.4.5

Example 5 Prove that $\lim\limits_{x\to+\infty} \dfrac{1}{x} = 0.$

Solution. Applying Definition 2.4.2 with $f(x) = 1/x$ and $L = 0$, we must show that given $\epsilon > 0$, we can find a number $N > 0$ such that

$$\left|\frac{1}{x} - 0\right| < \epsilon \quad \text{if} \quad x > N \tag{8}$$

Because $x \to +\infty$ we can assume that $x > 0$. Thus, we can eliminate the absolute values in this statement and rewrite it as

$$\frac{1}{x} < \epsilon \quad \text{if} \quad x > N$$

or, on taking reciprocals,

$$x > \frac{1}{\epsilon} \quad \text{if} \quad x > N \tag{9}$$

It is self-evident that $N = 1/\epsilon$ satisfies this requirement, and since (9) is equivalent to (8) for $x > 0$, the proof is complete. ◀

INFINITE LIMITS

In Section 2.1 we discussed limits of the following type from an intuitive viewpoint:

$$\lim_{x\to a} f(x) = +\infty, \qquad \lim_{x\to a} f(x) = -\infty \tag{10}$$

$$\lim_{x\to a^+} f(x) = +\infty, \qquad \lim_{x\to a^+} f(x) = -\infty \tag{11}$$

$$\lim_{x\to a^-} f(x) = +\infty, \qquad \lim_{x\to a^-} f(x) = -\infty \tag{12}$$

Recall that each of these expressions describes a particular way in which the limit fails to exist. The $+\infty$ indicates that the limit fails to exist because $f(x)$ increases without bound, and the $-\infty$ indicates that the limit fails to exist because $f(x)$ decreases without bound. These ideas are captured more precisely in the following definitions and are illustrated in Figure 2.4.6.

2.4.4 DEFINITION. Let $f(x)$ be defined for all x in some open interval containing a, except that $f(x)$ need not be defined at a. We will write

$$\lim_{x\to a} f(x) = +\infty$$

if given any positive number M, we can find a number $\delta > 0$ such that $f(x)$ satisfies

$$f(x) > M \quad \text{if} \quad 0 < |x-a| < \delta$$

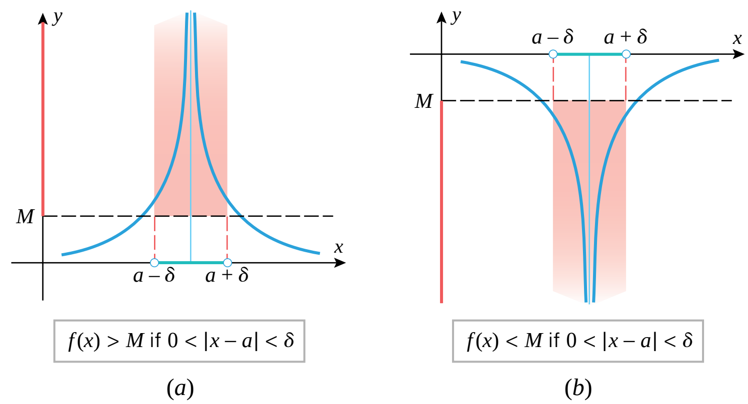


Figure 2.4.6

2.4.5 DEFINITION. Let $f(x)$ be defined for all x in some open interval containing a, except that $f(x)$ need not be defined at a. We will write

$$\lim_{x\to a} f(x) = -\infty$$

if given any negative number M, we can find a number $\delta > 0$ such that $f(x)$ satisfies

$$f(x) < M \quad \text{if} \quad 0 < |x-a| < \delta$$

To see how these definitions relate to our informal concepts of these limits, suppose that $f(x) \to +\infty$ as $x \to a$, and for a given M let δ be the corresponding positive number described in Definition 2.4.4. Next, imagine that x gets closer and closer to a (from either side). Eventually, x will lie in the interval $(a-\delta, a+\delta)$, which is marked in green in Figure 2.4.6*a*; when this happens the value of $f(x)$ will be greater than M, marked in red in the figure. Since this is true for any positive value of M (no matter how large), we can force the values of $f(x)$ to be as large as we like by making x sufficiently close to a. This agrees with our informal concept of this limit. Similarly, Figure 2.4.6*b* illustrates Definition 2.4.5.

REMARK. The definitions for the one-sided limits are similar. For example, in the definition of $\lim_{x\to a^-} f(x) = +\infty$ we assume that $f(x)$ is defined for all x in some interval of the form (c, a) and replace the condition $0 < |x-a| < \delta$ by the condition $a - \delta < x < a$.

Example 6 Prove that $\lim_{x\to 0} \dfrac{1}{x^2} = +\infty$.

Solution. Applying Definition 2.4.4 with $f(x) = 1/x^2$ and $a = 0$, we must show that given a number $M > 0$, we can find a number $\delta > 0$ such that

$$\frac{1}{x^2} > M \quad \text{if} \quad 0 < |x-0| < \delta \tag{13}$$

or, on taking reciprocals and simplifying,

$$x^2 < \frac{1}{M} \quad \text{if} \quad 0 < |x| < \delta \tag{14}$$

But $x^2 < 1/M$ if $|x| < 1/\sqrt{M}$, so that $\delta = 1/\sqrt{M}$ satisfies (14). Since (13) is equivalent to (14), the proof is complete. ◀

FOR THE READER. How would you define

$$\begin{aligned} &\lim_{x\to+\infty} f(x) = +\infty, \qquad \lim_{x\to+\infty} f(x) = -\infty \\ &\lim_{x\to-\infty} f(x) = +\infty, \qquad \lim_{x\to-\infty} f(x) = -\infty? \end{aligned} \tag{15}$$

EXERCISE SET 2.4 Graphing Calculator

1. (a) Find the largest open interval, centered at the origin on the x-axis, such that for each x in the interval the value of the function $f(x) = x + 2$ is within 0.1 unit of the number $f(0) = 2$.
(b) Find the largest open interval, centered at $x = 3$, such that for each x in the interval the value of the function $f(x) = 4x - 5$ is within 0.01 unit of the number $f(3) = 7$.
(c) Find the largest open interval, centered at $x = 4$, such that for each x in the interval the value of the function $f(x) = x^2$ is within 0.001 unit of the number $f(4) = 16$.

2. In each part, find the largest open interval, centered at $x = 0$, such that for each x in the interval the value of $f(x) = 2x + 3$ is within ϵ units of the number $f(0) = 3$.
(a) $\epsilon = 0.1$ (b) $\epsilon = 0.01$
(c) $\epsilon = 0.0012$

3. (a) Find the values of x_1 and x_2 in the accompanying figure.
(b) Find a positive number δ such that $|\sqrt{x} - 2| < 0.05$ if $0 < |x - 4| < \delta$.

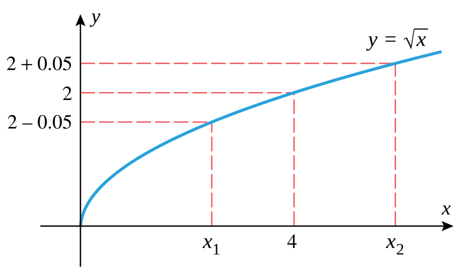


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Figure Ex-3

4. (a) Find the values of x_1 and x_2 in the accompanying figure.
(b) Find a positive number δ such that $|(1/x) - 1| < 0.1$ if $0 < |x - 1| < \delta$.

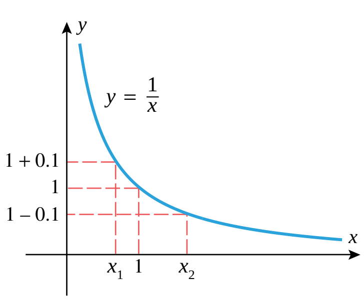

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Figure Ex-4

5. Generate the graph of $f(x) = x^3 - 4x + 5$ with a graphing utility, and use the graph to find a number δ such that $|f(x) - 2| < 0.05$ if $0 < |x - 1| < \delta$. [*Hint:* Show that the inequality $|f(x) - 2| < 0.05$ can be rewritten as $1.95 < x^3 - 4x + 5 < 2.05$, and estimate the values of x for which $x^3 - 4x + 5 = 1.95$ and $x^3 - 4x + 5 = 2.05$.]

6. Use the method of Exercise 5 to find a number δ such that $|\sqrt{5x+1} - 4| < 0.5$ if $0 < |x - 3| < \delta$.

7. Let $f(x) = x + \sqrt{x}$ with $L = \lim_{x\to 1} f(x)$ and let $\epsilon = 0.2$. Use a graphing utility and its trace feature to find a positive number δ such that $|f(x) - L| < \epsilon$ if $0 < |x - 1| < \delta$.

8. Let $f(x) = (\sin 2x)/x$ and use a graphing utility to conjecture the value of $L = \lim_{x\to 0} f(x)$. Then let $\epsilon = 0.1$ and use the graphing utility and its trace feature to find a positive number δ such that $|f(x) - L| < \epsilon$ if $0 < |x| < \delta$.

In Exercises 9–18, a positive number ϵ and the limit L of a function f at a are given. Find a number δ such that $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.

9. $\lim_{x\to 4} 2x = 8;\ \epsilon = 0.1$

10. $\lim_{x\to -2} \frac{1}{2}x = -1;\ \epsilon = 0.1$

11. $\lim_{x\to -1} (7x + 5) = -2;\ \epsilon = 0.01$

12. $\lim_{x\to 3} (5x - 2) = 13;\ \epsilon = 0.01$

13. $\lim_{x\to 2} \dfrac{x^2 - 4}{x - 2} = 4;\ \epsilon = 0.05$

14. $\lim_{x\to -1} \dfrac{x^2 - 1}{x + 1} = -2;\ \epsilon = 0.05$

15. $\lim_{x\to 4} x^2 = 16;\ \epsilon = 0.001$

16. $\lim_{x\to 9} \sqrt{x} = 3;\ \epsilon = 0.001$

17. $\lim_{x\to 5} \dfrac{1}{x} = \dfrac{1}{5};\ \epsilon = 0.05$

18. $\lim_{x\to 0} |x| = 0;\ \epsilon = 0.05$

In Exercises 19–32, use Definition 2.4.1 to prove that the stated limit is correct.

19. $\lim_{x\to 5} 3x = 15$

20. $\lim_{x\to 3} (4x - 5) = 7$

21. $\lim_{x\to 2} (2x - 7) = -3$

22. $\lim_{x\to -1} (2 - 3x) = 5$

23. $\lim_{x\to 0} \dfrac{x^2 + x}{x} = 1$

24. $\lim_{x\to -3} \dfrac{x^2 - 9}{x + 3} = -6$

25. $\lim_{x\to 1} 2x^2 = 2$

26. $\lim_{x\to 3} (x^2 - 5) = 4$

27. $\lim_{x\to 1/3} \dfrac{1}{x} = 3$

28. $\lim_{x\to -2} \dfrac{1}{x + 1} = -1$

29. $\lim_{x\to 4} \sqrt{x} = 2$

30. $\lim_{x\to 6} \sqrt{x + 3} = 3$

31. $\lim_{x\to 1} f(x) = 3$, where $f(x) = \begin{cases} x + 2, & x \neq 1 \\ 10, & x = 1 \end{cases}$

32. $\lim_{x\to 2} (x^2 + 3x - 1) = 9$

33. (a) Find the smallest positive number N such that for each x in the interval $(N, +\infty)$, the value of the function $f(x) = 1/x^2$ is within 0.1 unit of $L = 0$.

(b) Find the smallest positive number N such that for each x in the interval $(N, +\infty)$, the value of $f(x) = x/(x+1)$ is within 0.01 unit of $L = 1$.

(c) Find the largest negative number N such that for each x in the interval $(-\infty, N)$, the value of the function $f(x) = 1/x^3$ is within 0.001 unit of $L = 0$.

(d) Find the largest negative number N such that for each x in the interval $(-\infty, N)$, the value of the function $f(x) = x/(x+1)$ is within 0.01 unit of $L = 1$.

34. In each part, find the smallest positive value of N such that for each x in the interval $(N, +\infty)$, the function $f(x) = 1/x^3$ is within ϵ units of the number $L = 0$.
(a) $\epsilon = 0.1$ (b) $\epsilon = 0.01$ (c) $\epsilon = 0.001$

35. (a) Find the values of x_1 and x_2 in the accompanying figure.

(b) Find a positive number N such that

$$\left|\frac{x^2}{1+x^2} - 1\right| < \epsilon$$

for $x > N$.

(c) Find a negative number N such that

$$\left|\frac{x^2}{1+x^2} - 1\right| < \epsilon$$

for $x < N$.

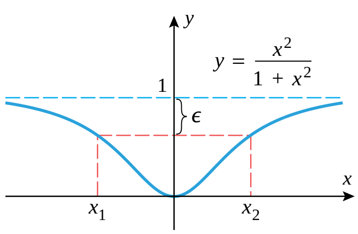

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Figure Ex-35

36. (a) Find the values of x_1 and x_2 in the accompanying figure.

(b) Find a positive number N such that

$$\left|\frac{1}{\sqrt[3]{x}} - 0\right| = \left|\frac{1}{\sqrt[3]{x}}\right| < \epsilon$$

for $x > N$.

(c) Find a negative number N such that

$$\left|\frac{1}{\sqrt[3]{x}} - 0\right| = \left|\frac{1}{\sqrt[3]{x}}\right| < \epsilon$$

for $x < N$.

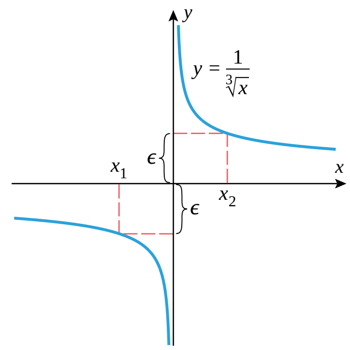


Figure Ex-36

In Exercises 37–40, a positive number ϵ and the limit L of a function f at $+\infty$ are given. Find a positive number N such that $|f(x) - L| < \epsilon$ if $x > N$.

37. $\lim\limits_{x\to+\infty} \frac{1}{x^2} = 0;\ \epsilon = 0.01$

38. $\lim\limits_{x\to+\infty} \frac{1}{x+2} = 0;\ \epsilon = 0.005$

39. $\lim\limits_{x\to+\infty} \frac{x}{x+1} = 1;\ \epsilon = 0.001$

40. $\lim\limits_{x\to+\infty} \frac{4x-1}{2x+5} = 2;\ \epsilon = 0.1$

In Exercises 41–44, a positive number ϵ and the limit L of a function f at $-\infty$ are given. Find a negative number N such that $|f(x) - L| < \epsilon$ if $x < N$.

41. $\lim\limits_{x\to-\infty} \frac{1}{x+2} = 0;\ \epsilon = 0.005$

42. $\lim\limits_{x\to-\infty} \frac{1}{x^2} = 0;\ \epsilon = 0.01$

43. $\lim\limits_{x\to-\infty} \frac{4x-1}{2x+5} = 2;\ \epsilon = 0.1$

44. $\lim\limits_{x\to-\infty} \frac{x}{x+1} = 1;\ \epsilon = 0.001$

In Exercises 45–52, use Definition 2.4.2 or 2.4.3 to prove that the stated limit is correct.

45. $\lim\limits_{x\to+\infty} \frac{1}{x^2} = 0$

46. $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$

47. $\lim\limits_{x\to-\infty} \frac{1}{x+2} = 0$

48. $\lim\limits_{x\to+\infty} \frac{1}{x+2} = 0$

49. $\lim\limits_{x\to+\infty} \frac{x}{x+1} = 1$

50. $\lim\limits_{x\to-\infty} \frac{x}{x+1} = 1$

51. $\lim\limits_{x\to-\infty} \frac{4x-1}{2x+5} = 2$

52. $\lim\limits_{x\to+\infty} \frac{4x-1}{2x+5} = 2$

53. (a) Find the largest open interval, centered at the origin on the x-axis, such that for each x in the interval, other than the center, the values of $f(x) = 1/x^2$ are greater than 100.

(b) Find the largest open interval, centered at $x = 1$, such that for each x in the interval, other than the center, the values of the function

$$f(x) = 1/|x - 1|$$

are greater than 1000.

(c) Find the largest open interval, centered at $x = 3$, such that for each x in the interval, other than the center, the values of the function

$$f(x) = -1/(x - 3)^2$$

are less than -1000.

(d) Find the largest open interval, centered at the origin on the x-axis, such that for each x in the interval, other than the center, the values of $f(x) = -1/x^4$ are less than $-10{,}000$.

54. In each part, find the largest open interval, centered at $x = 1$, such that for each x in the interval the value of $f(x) = 1/(x - 1)^2$ is greater than M.
(a) $M = 10$ (b) $M = 1000$ (c) $M = 100{,}000$

In Exercises 55–60, use Definition 2.4.4 or 2.4.5 to prove that the stated limit is correct.

55. $\lim\limits_{x\to 3} \dfrac{1}{(x-3)^2} = +\infty$ **56.** $\lim\limits_{x\to 3} \dfrac{-1}{(x-3)^2} = -\infty$

57. $\lim\limits_{x\to 0} \dfrac{1}{|x|} = +\infty$ **58.** $\lim\limits_{x\to 1} \dfrac{1}{|x-1|} = +\infty$

59. $\lim\limits_{x\to 0} \left(-\dfrac{1}{x^4}\right) = -\infty$ **60.** $\lim\limits_{x\to 0} \dfrac{1}{x^4} = +\infty$

In Exercises 61–66, use the remark following Definition 2.4.1 to prove that the stated limit is correct.

61. $\lim\limits_{x\to 2^+} (x+1) = 3$ **62.** $\lim\limits_{x\to 1^-} (3x+2) = 5$

63. $\lim\limits_{x\to 4^+} \sqrt{x-4} = 0$ **64.** $\lim\limits_{x\to 0^-} \sqrt{-x} = 0$

65. $\lim\limits_{x\to 2^+} f(x) = 2$, where $f(x) = \begin{cases} x, & x > 2 \\ 3x, & x \le 2 \end{cases}$

66. $\lim\limits_{x\to 2^-} f(x) = 6$, where $f(x) = \begin{cases} x, & x > 2 \\ 3x, & x \le 2 \end{cases}$

In Exercises 67 and 68, use the remark following Definitions 2.4.4 and 2.4.5 to prove that the stated limit is correct.

67. (a) $\lim\limits_{x\to 1^+} \dfrac{1}{1-x} = -\infty$ (b) $\lim\limits_{x\to 1^-} \dfrac{1}{1-x} = +\infty$

68. (a) $\lim\limits_{x\to 0^+} \dfrac{1}{x} = +\infty$ (b) $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$

For Exercises 69 and 70, write out definitions of the four limits in (18), and use your definitions to prove that the stated limits are correct.

69. (a) $\lim\limits_{x\to +\infty} (x+1) = +\infty$ (b) $\lim\limits_{x\to -\infty} (x+1) = -\infty$

70. (a) $\lim\limits_{x\to +\infty} (x^2-3) = +\infty$ (b) $\lim\limits_{x\to -\infty} (x^3+5) = -\infty$

71. Prove the result in Example 4 under the assumption that $\delta \le 2$ rather than $\delta \le 1$.

72. (a) In Definition 2.4.1 there is a condition requiring that $f(x)$ be defined for all x in some open interval containing a, except possibly at a itself. What is the purpose of this requirement?
(b) Why is $\lim\limits_{x\to 0} \sqrt{x} = 0$ an incorrect statement?
(c) Is $\lim\limits_{x\to 0.01} \sqrt{x} = 0.1$ a correct statement?

2.5 CONTINUITY

A moving object cannot vanish at some point and reappear someplace else to continue its motion. Thus, we perceive the path of a moving object as an unbroken curve, without gaps, breaks, or holes. In this section, we translate "unbroken curve" into a precise mathematical formulation called continuity, and develop some fundamental properties of continuous curves.

DEFINITION OF CONTINUITY

Recall from Theorem 2.2.3 that if $p(x)$ is a polynomial and c is a real number, then $\lim_{x\to c} p(x) = p(c)$ (see Figure 2.5.1). Together with Theorem 2.2.2, we are able to calculate limits of a variety of combinations of functions by evaluating the combination. That is, we saw many examples of functions $f(x)$ such that $\lim_{x\to c} f(x) = f(c)$ if $f(x)$ is defined on an interval containing a number c. In this case, function values $f(x)$ can be guaranteed to be near $f(c)$ for any x-value selected close enough to c. (See Exercise 53 for a precise formulation of this statement.)

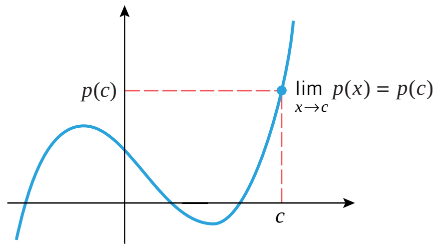


Figure 2.5.1

On the other hand, we have also seen functions for which this nice property is not true. For example,

$$f(x) = \begin{cases} \sin(\pi/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

does not satisfy $\lim_{x\to 0} f(x) = f(0)$, since $\lim_{x\to 0} f(x)$ fails to exist.

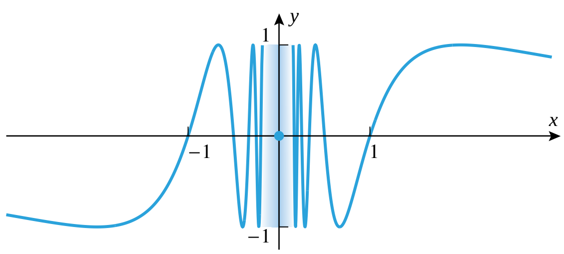


Figure 2.5.2

The term *continuous* is used to describe the useful circumstance where the calculation of a limit can be accomplished by mere evaluation of the function.

2.5.1 DEFINITION. A function f is said to be ***continuous at x = c*** provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$.

If one or more of the conditions of this definition fails to hold, then we will say that f has a ***discontinuity at x = c***. Each function drawn in Figure 2.5.3 illustrates a discontinuity at $x = c$. In Figure 2.5.3*a*, the function is not defined at c, violating the first condition of Definition 2.5.1. In Figures 2.5.3*b* and 2.5.3*c*, $\lim_{x\to c} f(x)$ does not exist, violating the second condition of Definition 2.5.1. In Figure 2.5.3*d*, the function is defined at c and $\lim_{x\to c} f(x)$ exists, but these two values are not equal, violating the third condition of Definition 2.5.1.

From such graphs we can develop an intuitive, geometric feel for where a function is continuous and where it is discontinuous. Observe that continuity at c may fail due to a "break" in the graph of the function, either due to a hole or to a jump as in Figure 2.5.3, or perhaps due to a wild oscillation as in Figure 2.5.2. Although the intuitive interpretation of "f is continuous at c" as "the graph of f is unbroken at c" lacks precision, it is a useful guide in most circumstances.

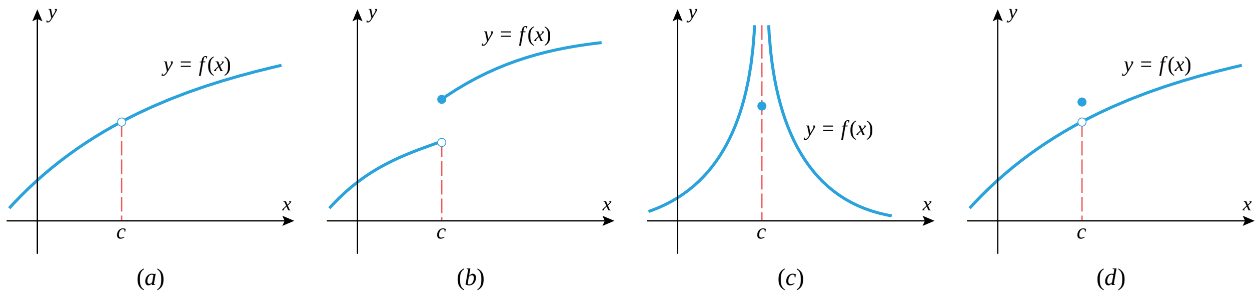


Figure 2.5.3

REMARK. Note that the third condition of Definition 2.5.1 really implies the first two conditions, since it is understood in the statement $\lim_{x\to c} f(x) = f(c)$ that the limit on the left exists, the expression $f(c)$ on the right is defined and has a finite value, and that quantitites on the two sides are equal. Thus, when we want to establish continuity of a function at a point our usual procedure will be to establish the validity of the third condition only.

Example 1 Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2-4}{x-2}, \qquad g(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

Solution. In each case we must determine whether the limit of the function as $x \to 2$ is the same as the value of the function at $x = 2$. In all three cases the functions are identical, except at $x = 2$, and hence all three have the same limit at $x = 2$, namely

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4$$

The function f is undefined at $x = 2$, and hence is not continuous at $x = 2$ (Figure 2.5.4*a*). The function g is defined at $x = 2$, but its value there is $g(2) = 3$, which is not the same as the limit as x approaches z; hence, g is also not continuous at $x = 2$ (Figure 2.5.4*b*). The value of the function h at $x = 2$ is $h(2) = 4$, which is the same as the limit as x approaches z; hence, h is continuous at $x = 2$ (Figure 2.5.4*c*). (Note that the function h could have been written more simply as $h(x) = x + 2$, but we wrote it in piecewise form to emphasize its relationship to f and g.) ◀

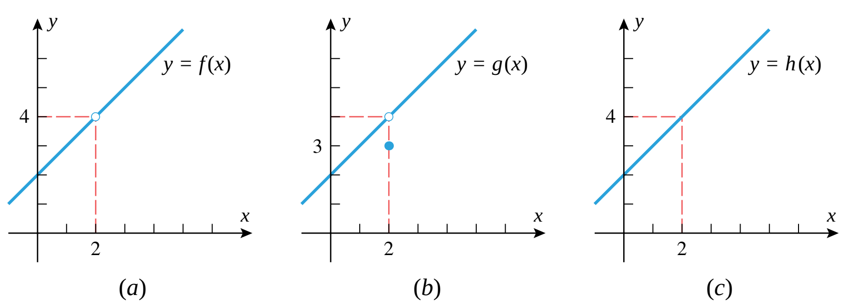


Figure 2.5.4

CONTINUITY IN APPLICATIONS

In applications, discontinuities often signal the occurrence of important physical phenomena. For example, Figure 2.5.5*a* is a graph of voltage versus time for an underground cable that is accidentally cut by a work crew at time $t = t_0$ (the voltage drops to zero when the line

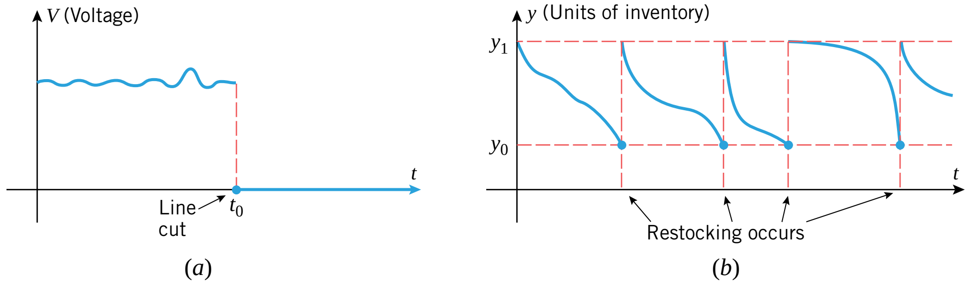


Figure 2.5.5

is cut). Figure 2.5.5*b* shows the graph of inventory versus time for a company that restocks its warehouse to y_1 units when the inventory falls to y_0 units. The discontinuities occur at those times when restocking occurs.

Given the possible physical significance of discontinuities, it is important to be able to identify discontinuities for specific functions, and to be able to make general statements about the continuity properties of entire families of functions. This is our next goal.

CONTINUITY ON AN INTERVAL AND CONTINUITY OF POLYNOMIALS

If a function f is continuous at each number in an open interval (a, b), then we say that f is ***continuous on* (a, b)**. This definition applies to infinite open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the case where f is continuous on $(-\infty, +\infty)$, we will say that f is ***continuous everywhere***.

The general procedure for showing that a function is continuous everywhere is to show that it is continuous at an *arbitrary* real number. For example, we showed in Theorem 2.2.3 that if $p(x)$ is a polynomial and a is any real number, then

$$\lim_{x \to a} p(x) = p(a)$$

Thus, we have the following result.

2.5.2 THEOREM. *Polynomials are continuous everywhere.*

Example 2 Show that $|x|$ is continuous everywhere (Figure 1.2.5).

Solution. We can write $|x|$ as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

so $|x|$ is the same as the polynomial x on the interval $(0, +\infty)$ and is the same as the polynomial $-x$ on the interval $(-\infty, 0)$. But polynomials are continuous everywhere, so $x = 0$ is the only possible discontinuity for $|x|$. Since $|0| = 0$, to prove the continuity at $x = 0$ we must show that

$$\lim_{x \to 0} |x| = 0 \tag{1}$$

Because the formula for $|x|$ changes at 0, it will be helpful to consider the one-sided limits at 0 rather than the two-sided limit. We obtain

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \quad \text{and} \quad \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Thus, (1) holds and $|x|$ is continuous at $x = 0$. ◀

SOME PROPERTIES OF CONTINUOUS FUNCTIONS

The following theorem, which is a consequence of Theorem 2.2.2, will enable us to reach conclusions about the continuity of functions that are obtained by adding, subtracting, multiplying, and dividing continuous functions.

2.5.3 THEOREM. *If the functions f and g are continuous at c, then*
(*a*) *$f + g$ is continuous at c.*
(*b*) *$f - g$ is continuous at c.*
(*c*) *fg is continuous at c.*
(*d*) *f/g is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c) = 0$.*

We will prove part (*d*). The remaining proofs are similar and will be omitted.

Proof. First, consider the case where $g(c) = 0$. In this case $f(c)/g(c)$ is undefined, so the function f/g has a discontinuity at c.

Next, consider the case where $g(c) \neq 0$. To prove that f/g is continuous at c, we must show that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)} \tag{2}$$

Since f and g are continuous at c,

$$\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c)$$

Thus, by Theorem 2.2.2(*d*)

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{f(c)}{g(c)}$$

which proves (2). ∎

CONTINUITY OF RATIONAL FUNCTIONS

Since polynomials are continuous everywhere, and since rational functions are ratios of polynomials, part (*d*) of Theorem 2.5.3 yields the following result.

2.5.4 THEOREM. *A rational function is continuous at every number where the denominator is nonzero.*

Example 3 For what values of x is there a hole or a gap in the graph of

$$y = \frac{x^2 - 9}{x^2 - 5x + 6}?$$

Solution. The function being graphed is a rational function, and hence is continuous at every number where the denominator is nonzero. Solving the equation

$$x^2 - 5x + 6 = 0$$

yields discontinuities at $x = 2$ and at $x = 3$. ◀

FOR THE READER. If you use a graphing utility to generate the graph of the equation in this example, then there is a good chance that you will see the discontinuity at $x = 2$ but not at $x = 3$. Try it, and explain what you think is happening.

CONTINUITY OF COMPOSITIONS

The following theorem, whose proof is given in Appendix G, will be useful for calculating limits of compositions of functions.

2.5.5 THEOREM. *If* $\lim_{x\to c} g(x) = L$ *and if the function* f *is continuous at* L, *then* $\lim_{x\to c} f(g(x)) = f(L)$. *That is,*

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$$

This equality remains valid if $\lim_{x\to c}$ *is replaced everywhere by one of* $\lim_{x\to c^+}$, $\lim_{x\to c^-}$, $\lim_{x\to +\infty}$, *or* $\lim_{x\to -\infty}$.

In words, this theorem states:

A limit symbol can be moved through a function sign provided the limit of the expression inside the function sign exists and the function is continuous at this limit.

Example 4 We know from Example 2 that the function $|x|$ is continuous everywhere; thus, it follows that if $\lim_{x\to a} g(x)$ exists, then

$$\lim_{x\to a} |g(x)| = \left|\lim_{x\to a} g(x)\right| \tag{3}$$

That is, a limit symbol can be moved through an absolute value sign, provided the limit of the expression inside the absolute value signs exists. For example,

$$\lim_{x\to 3} |5 - x^2| = \left|\lim_{x\to 3} (5 - x^2)\right| = |-4| = 4$$ ◀

The following theorem is concerned with the continuity of compositions of functions; the first part deals with continuity at a specific number, and the second part with continuity everywhere.

2.5.6 THEOREM.

(*a*) *If the function* g *is continuous at* c, *and the function* f *is continuous at* $g(c)$, *then the composition* $f\circ g$ *is continuous at* c.

(*b*) *If the function* g *is continuous everywhere and the function* f *is continuous everywhere, then the composition* $f\circ g$ *is continuous everywhere.*

Proof. We will prove part (*a*) only; the proof of part (*b*) can be obtained by applying part (*a*) at an arbitrary number c. To prove that $f\circ g$ is continuous at c, we must show that the value of $f\circ g$ and the value of its limit are the same at $x = c$. But this is so, since we can write

$$\lim_{x\to c} (f\circ g)(x) = \lim_{x\to c} f(g(x)) \underset{\text{Theorem 2.5.5}}{=} f(\lim_{x\to c} g(x)) \underset{g \text{ is continuous at } c.}{=} f(g(c)) = (f\circ g)(c)$$ ■

We know from Example 2 that the function $|x|$ is continuous everywhere. Thus, if $g(x)$ is continuous at c, then by part (*a*) of Theorem 2.5.6, the function $|g(x)|$ must also be continuous at c; and, more generally, if $g(x)$ is continuous everywhere, then so is $|g(x)|$. Stated informally:

The absolute value of a continuous function is continuous.

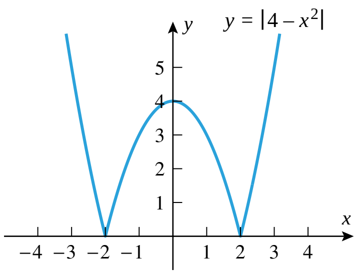


Figure 2.5.6

For example, the polynomial $g(x) = 4 - x^2$ is continuous everywhere, so we can conclude that the function $|4 - x^2|$ is also continuous everywhere (Figure 2.5.6).

FOR THE READER. Can the absolute value of a function that is not continuous be continuous? Justify your answer.

CONTINUITY FROM THE LEFT AND RIGHT

Because Definition 2.5.1 involves a two-sided limit, that definition does not generally apply at the endpoints of a closed interval $[a, b]$ or at the endpoint of an interval of the form $[a, b)$, $(a, b]$, $(-\infty, b]$, or $[a, +\infty)$. To remedy this problem, we will agree that a function is continuous at an endpoint of an interval if its value at the endpoint is equal to the appropriate one-sided limit at that endpoint. For example, the function graphed in Figure 2.5.7 is continuous at the right endpoint of the interval $[a, b]$ because

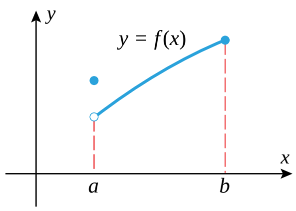


Figure 2.5.7

$$\lim_{x\to b^-} f(x) = f(b)$$

but it is not continuous at the left endpoint because

$$\lim_{x\to a^+} f(x) \neq f(a)$$

In general, we will say a function f is ***continuous from the left*** at c if

$$\lim_{x\to c^-} f(x) = f(c)$$

and is ***continuous from the right*** at c if

$$\lim_{x\to c^+} f(x) = f(c)$$

Using this terminology we define continuity on a closed interval as follows.

2.5.7 DEFINITION. A function f is said to be ***continuous on a closed interval* [*a*, *b*]** if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

FOR THE READER. We leave it for you to modify this definition appropriately so that it applies to intervals of the form $[a, +\infty)$, $(-\infty, b]$, $(a, b]$, and $[a, b)$.

Example 5 What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$?

Solution. Because the natural domain of this function is the closed interval $[-3, 3]$, we will need to investigate the continuity of f on the open interval $(-3, 3)$ and at the two endpoints. If c is any number in the interval $(-3, 3)$, then it follows from Theorem 2.2.2(*e*) that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \sqrt{9 - x^2} = \sqrt{\lim_{x\to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

which proves f is continuous at each number in the interval $(-3, 3)$. The function f is also continuous at the endpoints since

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x\to 3^-} (9 - x^2)} = 0 = f(3)$$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x\to -3^+} (9 - x^2)} = 0 = f(-3)$$

Thus, f is continuous on the closed interval $[-3, 3]$. ◀

THE INTERMEDIATE-VALUE THEOREM

Figure 2.5.8 shows the graph of a function that is continuous on the closed interval $[a, b]$. The figure suggests that if we draw any horizontal line $y = k$, where k is between $f(a)$ and $f(b)$, then that line will cross the curve $y = f(x)$ at least once over the interval $[a, b]$. Stated in numerical terms, if f is continuous on $[a, b]$, then the function f must take on every value k between $f(a)$ and $f(b)$ at least once as x varies from a to b. For example, the polynomial $p(x) = x^5 - x + 3$ has a value of 3 at $x = 1$ and a value of 33 at $x = 2$. Thus, it follows from the continuity of p that the equation $x^5 - x + 3 = k$ has at least one

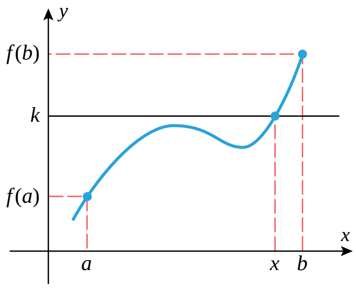


Figure 2.5.8

solution in the interval [1, 2] for every value of k between 3 and 33. This idea is stated more precisely in the following theorem.

> **2.5.8 THEOREM (*Intermediate-Value Theorem*).** *If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$.*

Although this theorem is intuitively obvious, its proof depends on a mathematically precise development of the real number system, which is beyond the scope of this text.

APPROXIMATING ROOTS USING THE INTERMEDIATE-VALUE THEOREM

A variety of problems can be reduced to solving an equation $f(x) = 0$ for its roots. Sometimes it is possible to solve for the roots exactly using algebra, but often this is not possible and one must settle for decimal approximations of the roots. One procedure for approximating roots is based on the following consequence of the Intermediate-Value Theorem.

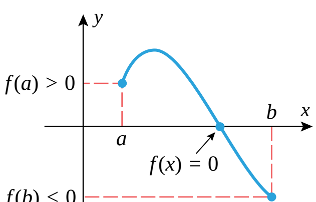


Figure 2.5.9

> **2.5.9 THEOREM.** *If f is continuous on $[a, b]$, and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).*

This result, which is illustrated in Figure 2.5.9, can be proved as follows.

Proof. Since $f(a)$ and $f(b)$ have opposite signs, 0 is between $f(a)$ and $f(b)$. Thus, by the Intermediate-Value Theorem there is at least one number x in the interval $[a, b]$ such that $f(x) = 0$. However, $f(a)$ and $f(b)$ are nonzero, so x must lie in the interval (a, b), which completes the proof. ∎

Before we illustrate how this theorem can be used to approximate roots, it will be helpful to discuss some standard terminology for describing errors in approximations. If x is an approximation to a quantity x_0, then we call

$$\epsilon = |x - x_0|$$

the ***absolute error*** or (less precisely) the ***error*** in the approximation. The terminology in Table 2.5.1 is used to describe the size of such errors:

Table 2.5.1

ERROR	DESCRIPTION
$\lvert x - x_0\rvert \leq 0.1$	x approximates x_0 with an error of at most 0.1.
$\lvert x - x_0\rvert \leq 0.01$	x approximates x_0 with an error of at most 0.01.
$\lvert x - x_0\rvert \leq 0.001$	x approximates x_0 with an error of at most 0.001.
$\lvert x - x_0\rvert \leq 0.0001$	x approximates x_0 with an error of at most 0.0001.
$\lvert x - x_0\rvert \leq 0.5$	x approximates x_0 to the nearest integer.
$\lvert x - x_0\rvert \leq 0.05$	x approximates x_0 to 1 decimal place (i.e., to the nearest tenth).
$\lvert x - x_0\rvert \leq 0.005$	x approximates x_0 to 2 decimal places (i.e., to the nearest hundredth).
$\lvert x - x_0\rvert \leq 0.0005$	x approximates x_0 to 3 decimal places (i.e., to the nearest thousandth).

Example 6 The equation

$$x^3 - x - 1 = 0$$

cannot be solved algebraically very easily because the left side has no simple factors. However, if we graph $p(x) = x^3 - x - 1$ with a graphing utility (Figure 2.5.10), then we are led to conjecture that there is one real root and that this root lies inside the interval [1, 2].

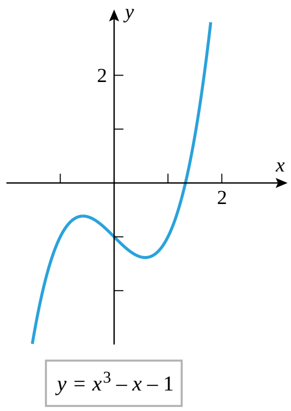


Figure 2.5.10

The existence of a root in this interval is also confirmed by Theorem 2.5.9, since $p(1) = -1$ and $p(2) = 5$ have opposite signs. Approximate this root to two decimal-place accuracy.

Solution. Our objective is to approximate the unknown root x_0 with an error of at most 0.005. It follows that if we can find an interval of length 0.01 that contains the root, then the midpoint of that interval will approximate the root with an error of at most $0.01/2 = 0.005$, which will achieve the desired accuracy.

We know that the root x_0 lies in the interval [1, 2]. However, this interval has length 1, which is too large. We can pinpoint the location of the root more precisely by dividing the interval [1, 2] into 10 equal parts and evaluating p at the points of subdivision using a calculating utility (Table 2.5.2). In this table $p(1.3)$ and $p(1.4)$ have opposite signs, so we know that the root lies in the interval [1.3, 1.4]. This interval has length 0.1, which is still too large, so we repeat the process by dividing the interval [1.3, 1.4] into 10 parts and evaluating p at the points of subdivision; this yields Table 2.5.3, which tells us that the root is inside the interval [1.32, 1.33] (Figure 2.5.11). Since this interval has length 0.01, its midpoint 1.325 will approximate the root with an error of at most 0.005. Thus, $x_0 \approx 1.325$ to two decimal-place accuracy. ◀

Table 2.5.2

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$f(x)$	–1	–0.77	–0.47	–0.10	0.34	0.88	1.50	2.21	3.03	3.96	5

Table 2.5.3

x	1.3	1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.4
$f(x)$	–0.103	–0.062	–0.020	0.023	0.066	0.110	0.155	0.201	0.248	0.296	0.344

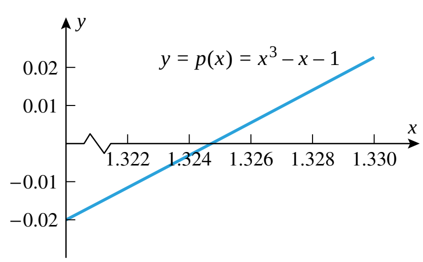


Figure 2.5.11

APPROXIMATING ROOTS BY ZOOMING WITH A GRAPHING UTILITY

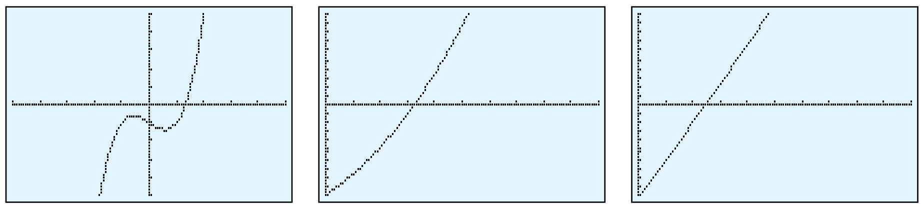
The method illustrated in Example 6 can also be implemented with a graphing utility as follows.

Step 1. Figure 2.5.12*a* shows the graph of f in the window $[-5, 5] \times [-5, 5]$ with xScl $= 1$ and yScl $= 1$. That graph places the root between $x = 1$ and $x = 2$.

Step 2. Since we know that the root lies between $x = 1$ and $x = 2$, we will zoom in by regraphing f over an x-interval that extends between these values and in which xScl $= 0.1$. The y-interval and yScl are not critical, as long as the y-interval extends above and below the x-axis. Figure 2.5.12*b* shows the graph of f in the window $[1, 2] \times [-1, 1]$ with xScl $= 0.1$ and yScl $= 0.1$. That graph places the root between $x = 1.3$ and $x = 1.4$.

Step 3. Since we know that the root lies between $x = 1.3$ and $x = 1.4$, we will zoom in again by regraphing f over an x-interval that extends between these values and in which $x\text{Scl} = 0.01$. Figure 2.5.12c shows the graph of f in the window $[1.3, 1.4] \times [-0.1, 0.1]$ with $x\text{Scl} = 0.01$ and $y\text{Scl} = 0.01$. That graph places the root between $x = 1.32$ and $x = 1.33$.

Step 4. Since the interval in Step 3 has length 0.01, its midpoint 1.325 approximates the root with an error of at most 0.005, so $x_0 \approx 1.325$ to two decimal-place accuracy.



[–5, 5] × [–5, 5]
*x*Scl = 1, *y*Scl = 1
(*a*)

[1, 2] × [–1, 1]
*x*Scl = 0.1, *y*Scl = 0.1
(*b*)

[1.3, 1.4] × [–0.1, 0.1]
*x*Scl = 0.01, *y*Scl = 0.01
(*c*)

Figure 2.5.12

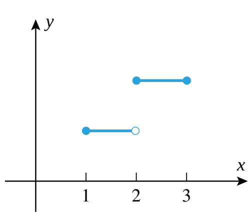
REMARK. To say that x approximates x_0 to n decimal places does *not* mean that the first n decimal places of x and x_0 will be the same when the numbers are rounded to n decimal places. For example, $x = 1.084$ approximates $x_0 = 1.087$ to two decimal places because $|x - x_0| = 0.003 (< 0.005)$. However, if we round these values to two decimal places, then we obtain $x \approx 1.08$ and $x_0 \approx 1.09$. Thus, if you approximate a number to n decimal places, then you should display that approximation to at least $n + 1$ decimal places to preserve the accuracy.

FOR THE READER. Use a graphing or calculating utility to show that the root x_0 in Example 6 can be approximated as $x_0 \approx 1.3245$ to three decimal-place accuracy.

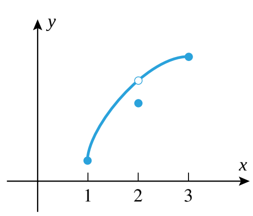
EXERCISE SET 2.5 Graphing Calculator

In Exercises 1–4, let f be the function whose graph is shown. On which of the following intervals, if any, is f continuous?
(a) $[1, 3]$ (b) $(1, 3)$ (c) $[1, 2]$
(d) $(1, 2)$ (e) $[2, 3]$ (f) $(2, 3)$
For each interval on which f is not continuous, indicate which conditions for the continuity of f do not hold.

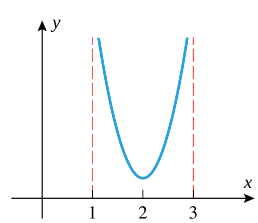
1.



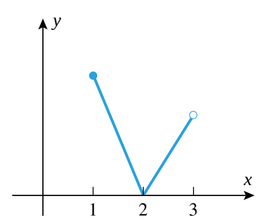
2.



3.



4.



In Exercises 5 and 6, find all values of c such that the specified function has a discontinuity at $x = c$. For each such value of c, determine which conditions of Definition 2.5.1 fail to be satisfied.

5. (a) The function f in Exercise 1 of Section 2.1.
(b) The function F in Exercise 5 of Section 2.1.
(c) The function f in Exercise 9 of Section 2.1.

6. (a) The function f in Exercise 2 of Section 2.1.
(b) The function F in Exercise 6 of Section 2.1.
(c) The function f in Exercise 10 of Section 2.1.

7. Suppose that f and g are continuous functions such that $f(2) = 1$ and $\lim_{x \to 2} [f(x) + 4g(x)] = 13$. Find
(a) $g(2)$ (b) $\lim_{x \to 2} g(x)$.

8. Suppose that f and g are continuous functions such that $\lim_{x \to 3} g(x) = 5$ and $f(3) = -2$. Find $\lim_{x \to 3} [f(x)/g(x)]$.

9. In each part sketch the graph of a function f that satisfies the stated conditions.
(a) f is continuous everywhere except at $x = 3$, at which point it is continuous from the right.
(b) f has a two-sided limit at $x = 3$, but it is not continuous at $x = 3$.
(c) f is not continuous at $x = 3$, but if its value at $x = 3$ is changed from $f(3) = 1$ to $f(3) = 0$, it becomes continuous at $x = 3$.
(d) f is continuous on the interval $[0, 3)$ and is defined on the closed interval $[0, 3]$; but f is not continuous on the interval $[0, 3]$.

10. Find formulas for some functions that are continuous on the intervals $(-\infty, 0)$ and $(0, +\infty)$, but are not continuous on the interval $(-\infty, +\infty)$.

11. A student parking lot at a university charges \$2.00 for the first half hour (or any part) and \$1.00 for each subsequent half hour (or any part) up to a daily maximum of \$10.00.
(a) Sketch a graph of cost as a function of the time parked.
(b) Discuss the significance of the discontinuities in the graph to a student who parks there.

12. In each part determine whether the function is continuous or not, and explain your reasoning.
(a) The Earth's population as a function of time
(b) Your exact height as a function of time
(c) The cost of a taxi ride in your city as a function of the distance traveled
(d) The volume of a melting ice cube as a function of time

In Exercises 13–24, find the values of x (if any) at which f is not continuous.

13. $f(x) = x^3 - 2x + 3$

14. $f(x) = (x - 5)^{17}$

15. $f(x) = \dfrac{x}{x^2 + 1}$

16. $f(x) = \dfrac{x}{x^2 - 1}$

17. $f(x) = \dfrac{x - 4}{x^2 - 16}$

18. $f(x) = \dfrac{3x + 1}{x^2 + 7x - 2}$

19. $f(x) = \dfrac{x}{|x| - 3}$

20. $f(x) = \dfrac{5}{x} + \dfrac{2x}{x + 4}$

21. $f(x) = |x^3 - 2x^2|$

22. $f(x) = \dfrac{x + 3}{|x^2 + 3x|}$

23. $f(x) = \begin{cases} 2x + 3, & x \le 4 \\ 7 + \dfrac{16}{x}, & x > 4 \end{cases}$

24. $f(x) = \begin{cases} \dfrac{3}{x - 1}, & x \ne 1 \\ 3, & x = 1 \end{cases}$

25. Find a value for the constant k, if possible, that will make the function continuous everywhere.
(a) $f(x) = \begin{cases} 7x - 2, & x \le 1 \\ kx^2, & x > 1 \end{cases}$
(b) $f(x) = \begin{cases} kx^2, & x \le 2 \\ 2x + k, & x > 2 \end{cases}$

26. On which of the following intervals is
$$f(x) = \frac{1}{\sqrt{x - 2}}$$
continuous?
(a) $[2, +\infty)$ (b) $(-\infty, +\infty)$ (c) $(2, +\infty)$ (d) $[1, 2)$

A function f is said to have a ***removable discontinuity*** at $x = c$ if $\lim_{x \to c} f(x)$ exists but f is not continuous at $x = c$, either because f is not defined at c or because the definition for $f(c)$ differs from the value of the limit. This terminology will be needed in Exercises 27–30.

27. (a) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is undefined.
(b) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is defined.

28. (a) The terminology *removable discontinuity* is appropriate because a removable discontinuity of a function f at $x = c$ can be "removed" by redefining the value of f appropriately at $x = c$. What value for $f(c)$ removes the discontinuity?
(b) Show that the following functions have removable discontinuities at $x = 1$, and sketch their graphs.
$$f(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad g(x) = \begin{cases} 1, & x > 1 \\ 0, & x = 1 \\ 1, & x < 1 \end{cases}$$
(c) What values should be assigned to $f(1)$ and $g(1)$ to remove the discontinuities?

In Exercises 29 and 30, find the values of x (if any) at which f is not continuous, and determine whether each such value is a removable discontinuity.

29. (a) $f(x) = \dfrac{|x|}{x}$ (b) $f(x) = \dfrac{x^2 + 3x}{x + 3}$
(c) $f(x) = \dfrac{x - 2}{|x| - 2}$

30. (a) $f(x) = \dfrac{x^2 - 4}{x^3 - 8}$

(b) $f(x) = \begin{cases} 2x - 3, & x \le 2 \\ x^2, & x > 2 \end{cases}$

(c) $f(x) = \begin{cases} 3x^2 + 5, & x \ne 1 \\ 6, & x = 1 \end{cases}$

31. (a) Use a graphing utility to generate the graph of the function $f(x) = (x + 3)/(2x^2 + 5x - 3)$, and then use the graph to make a conjecture about the number and locations of all discontinuities.
(b) Check your conjecture by factoring the denominator.

32. (a) Use a graphing utility to generate the graph of the function $f(x) = x/(x^3 - x + 2)$, and then use the graph to make a conjecture about the number and locations of all discontinuities.
(b) Use the Intermediate-Value Theorem to approximate the location of all discontinuities to two decimal places.

33. Prove that $f(x) = x^{3/5}$ is continuous everywhere, carefully justifying each step.

34. Prove that $f(x) = 1/\sqrt{x^4 + 7x^2 + 1}$ is continuous everywhere, carefully justifying each step.

35. Let f and g be discontinuous at c. Give examples to show that
(a) $f + g$ can be continuous or discontinuous at c
(b) fg can be continuous or discontinuous at c.

36. Prove Theorem 2.5.4.

37. Prove:
(a) part (*a*) of Theorem 2.5.3
(b) part (*b*) of Theorem 2.5.3
(c) part (*c*) of Theorem 2.5.3.

38. Prove: If f and g are continuous on $[a, b]$, and $f(a) > g(a)$, $f(b) < g(b)$, then there is at least one solution of the equation $f(x) = g(x)$ in (a, b). [*Hint:* Consider $f(x) - g(x)$.]

39. Give an example of a function f that is defined on a closed interval, and whose values at the endpoints have opposite signs, but for which the equation $f(x) = 0$ has no solution in the interval.

40. Use the Intermediate-Value Theorem to show that there is a square with a diagonal length that is between r and $2r$ and an area that is half the area of a circle of radius r.

41. Use the Intermediate-Value Theorem to show that there is a right circular cylinder of height h and radius less than r whose volume is equal to that of a right circular cone of height h and radius r.

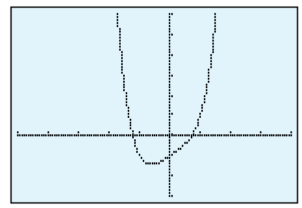
In Exercises 42 and 43, show that the equation has at least one solution in the given interval.

42. $x^3 - 4x + 1 = 0$; $[1, 2]$

43. $x^3 + x^2 - 2x = 1$; $[-1, 1]$

44. Prove: If $p(x)$ is a polynomial of odd degree, then the equation $p(x) = 0$ has at least one real solution.

45. The accompanying figure shows the graph of $y = x^4 + x - 1$. Use the method of Example 6 to approximate the x-intercepts with an error of at most 0.05.

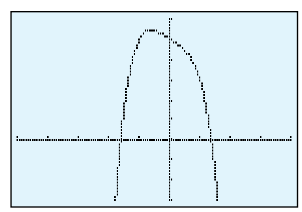


$[-5, 4] \times [-3, 6]$
xScl = 1, yScl = 1

Figure Ex-45

46. Use a graphing utility to solve the problem in Exercise 45 by zooming.

47. The accompanying figure shows the graph of $y = 5 - x - x^4$. Use the method of Example 6 to approximate the roots of the equation $5 - x - x^4 = 0$ to two decimal-place accuracy.



$[-5, 4] \times [-3, 6]$
xScl = 1, yScl = 1

Figure Ex-47

48. Use a graphing utility to solve the problem in Exercise 47 by zooming.

49. Use the fact that $\sqrt{5}$ is a solution of $x^2 - 5 = 0$ to approximate $\sqrt{5}$ with an error of at most 0.005.

50. Prove that if a and b are positive, then the equation

$$\frac{a}{x-1} + \frac{b}{x-3} = 0$$

has at least one solution in the interval $(1, 3)$.

51. A sphere of unknown radius x consists of a spherical core and a coating that is 1 cm thick (see the accompanying figure). Given that the volume of the coating and the volume of the core are the same, approximate the radius of the sphere to three decimal-place accuracy.

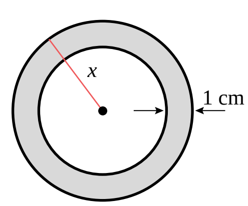


Figure Ex-51

52. A monk begins walking up a mountain road at 12:00 noon and reaches the top at 12:00 midnight. He meditates and rests until 12:00 noon the next day, at which time he begins walking down the same road, reaching the bottom at 12:00 midnight. Show that there is at least one point on the road that he reaches at the same time of day on the way up as on the way down.

53. Let f be defined at c. Prove that f is continuous at c if, given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ if $|x - c| < \delta$.

2.6 LIMITS AND CONTINUITY OF TRIGONOMETRIC FUNCTIONS

In this section we will investigate the continuity properties of the trigonometric functions, and we will discuss some important limits involving these functions.

CONTINUITY OF TRIGONOMETRIC FUNCTIONS

Before we begin, recall that in the expressions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ it is understood that x is in radian measure.

In trigonometry, the graphs of $\sin x$ and $\cos x$ are drawn as continuous curves (Figure 2.6.1). To actually prove that these functions are continuous everywhere, we must show that the following equalities hold for every real number c:

$$\lim_{x\to c} \sin x = \sin c \quad \text{and} \quad \lim_{x\to c} \cos x = \cos c \tag{1–2}$$

Although we will not formally prove these results, we can make them plausible by considering the behavior of the point $P(\cos x, \sin x)$ as it moves around the unit circle. For this purpose, view c as a fixed angle in radian measure, and let $Q(\cos c, \sin c)$ be the corresponding point on the unit circle. As $x \to c$ (i.e., as the angle x approaches the angle c), the point P moves along the circle toward Q, and this implies that the coordinates of P approach the corresponding coordinates of Q; that is, $\cos x \to \cos c$, and $\sin x \to \sin c$ (Figure 2.6.2).

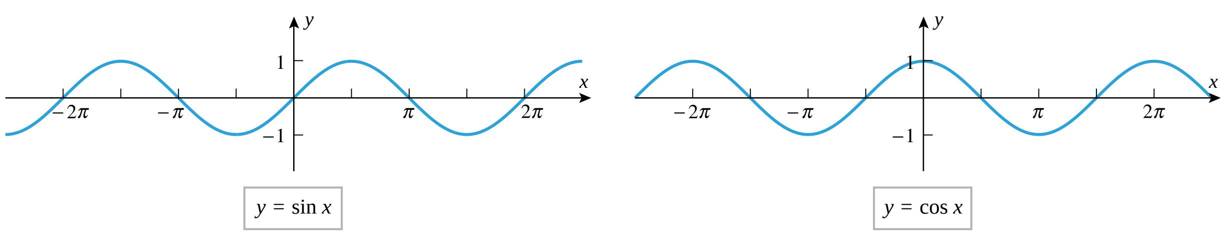


Figure 2.6.1

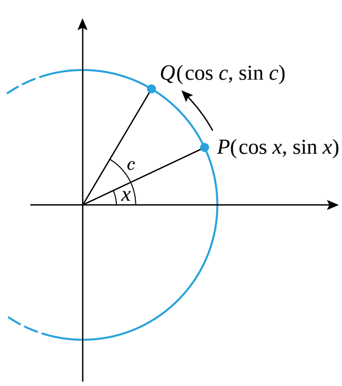


Figure 2.6.2

Formulas (1) and (2) can be used to find limits of the remaining trigonometric functions by expressing them in terms of $\sin x$ and $\cos x$; for example, if $\cos c \neq 0$, then

$$\lim_{x\to c} \tan x = \lim_{x\to c} \frac{\sin x}{\cos x} = \frac{\sin c}{\cos c} = \tan c$$

Thus, we are led to the following theorem.

2.6.1 THEOREM. *If c is any number in the natural domain of the stated trigonometric function, then*

$$\lim_{x\to c} \sin x = \sin c \qquad \lim_{x\to c} \cos x = \cos c \qquad \lim_{x\to c} \tan x = \tan c$$

$$\lim_{x\to c} \csc x = \csc c \qquad \lim_{x\to c} \sec x = \sec c \qquad \lim_{x\to c} \cot x = \cot c$$

It follows from this theorem, for example, that $\sin x$ and $\cos x$ are continuous everywhere and that $\tan x$ is continuous, except at the points where it is undefined.

Example 1 Find the limit

$$\lim_{x \to 1} \cos\left(\frac{x^2 - 1}{x - 1}\right)$$

Solution. Recall from the last section that since the cosine function is continuous everywhere,

$$\lim_{x \to 1} \cos(g(x)) = \cos(\lim_{x \to 1} g(x))$$

provided $\lim_{x \to 1} g(x)$ exists. Thus,

$$\lim_{x \to 1} \cos\left(\frac{x^2 - 1}{x - 1}\right) = \lim_{x \to 1} \cos(x + 1) = \cos\left(\lim_{x \to 1} (x + 1)\right) = \cos 2 \qquad \blacktriangleleft$$

OBTAINING LIMITS BY SQUEEZING

In Section 2.1 we used the numerical evidence in Table **??** to *conjecture* that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{3}$$

However, it is not a simple matter to establish this limit with certainty. The difficulty is that the numerator and denominator both approach zero as $x \to 0$. As discussed in Section 2.2, such limits are called indeterminate forms of type $0/0$. Sometimes indeterminate forms of this type can be established by manipulating the ratio algebraically, but in this case no simple algebraic manipulation will work, so we must look for other methods.

The problem with indeterminate forms of type $0/0$ is that there are two conflicting influences at work: as the numerator approaches 0 it drives the magnitude of the ratio toward 0, and as the denominator approaches 0 it drives the magnitude of the ratio toward $\pm\infty$ (depending on the sign of the expression). The limiting behavior of the ratio is determined by the precise way in which these influences offset each other. Later in this text we will discuss general methods for attacking indeterminate forms, but for the limit in (3) we can use a method called *squeezing*.

In the method of squeezing one proves that a function f has a limit L at a number c by trapping the function between two other functions, g and h, whose limits at c are known to be L (Figure 2.6.3). This is the idea behind the following theorem, which we state without proof.

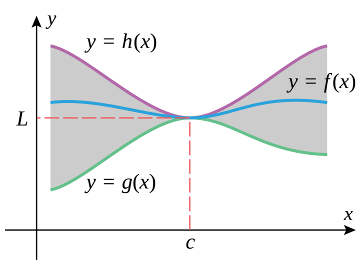


Figure 2.6.3

2.6.2 THEOREM (*The Squeezing Theorem*). *Let f, g, and h be functions satisfying*

$$g(x) \le f(x) \le h(x)$$

for all x in some open interval containing the number c, with the possible exception that the inequalities need not hold at c. If g and h have the same limit as x approaches c, say

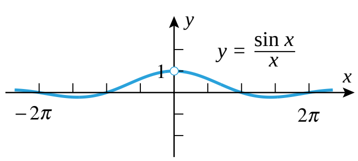
$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

then f also has this limit as x approaches c, that is,

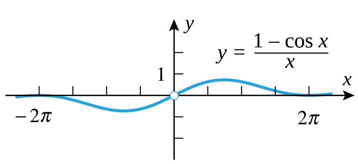
$$\lim_{x \to c} f(x) = L$$

FOR THE READER. The Squeezing Theorem also holds for one-sided limits and limits at $+\infty$ and $-\infty$. How do you think the hypotheses of the theorem would change in those cases?

The usefulness of the Squeezing Theorem will be evident in our proof of the following theorem (Figure 2.6.4).



$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$



$$\boxed{\lim_{x \to 0} \frac{1 - \cos x}{x} = 0}$$

Figure 2.6.4

2.6.3 THEOREM.

(a) $\displaystyle \lim_{x \to 0} \frac{\sin x}{x} = 1$ (b) $\displaystyle \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

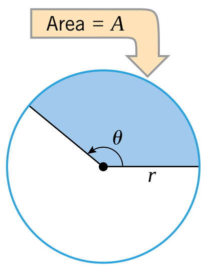


Figure 2.6.5

However, before giving the proof, it will be helpful to review the formula for the area A of a sector with radius r and a central angle of θ radians (Figure 2.6.5). The area of the sector can be derived by setting up the following proportion to the area of the entire circle:

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi} \qquad \left[\frac{\text{area of the sector}}{\text{area of the circle}} = \frac{\text{central angle of the sector}}{\text{central angle of the circle}}\right]$$

From this we obtain the formula

$$A = \tfrac{1}{2}r^2\theta \tag{4}$$

Now we are ready for the proof of Theorem 2.6.3.

Proof (a). In this proof we will interpret x as an angle in radian measure, and we will assume to start that $0 < x < \pi/2$. It follows from Formula (4) that the area of a sector of radius 1 and central angle x is $x/2$. Moreover, it is suggested by Figure 2.6.6 that the area of this sector lies between the areas of two triangles, one with area $(\tan x)/2$ and one with area $(\sin x)/2$. Thus,

$$\frac{\tan x}{2} \geq \frac{x}{2} \geq \frac{\sin x}{2}$$

Multiplying through by $2/(\sin x)$ yields

$$\frac{1}{\cos x} \geq \frac{x}{\sin x} \geq 1$$

and then taking reciprocals and reversing the inequalities yields

$$\cos x \leq \frac{\sin x}{x} \leq 1 \tag{5}$$

Moreover, these inequalities also hold for $-\pi/2 < x < 0$, since replacing x by $-x$ in (5) and using the identities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ leaves the inequalities unchanged (verify). Finally, since the functions $\cos x$ and 1 both have limits of 1 as $x \to 0$, it follows from the Squeezing Theorem that $(\sin x)/x$ also has a limit of 1 as $x \to 0$.

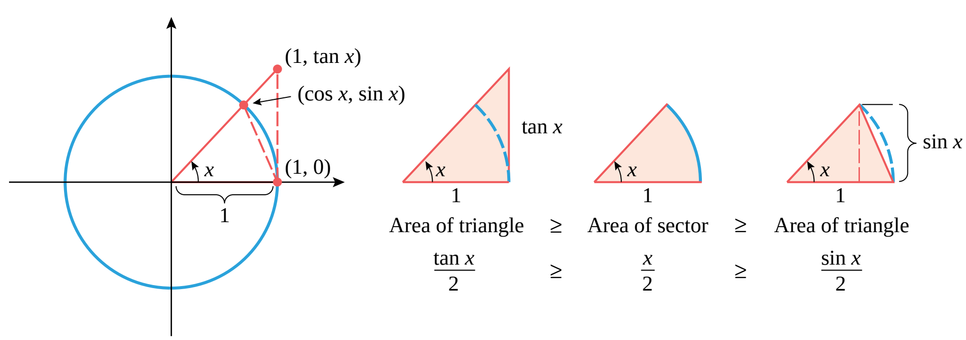


Figure 2.6.6

Proof (b). For this proof we will use the limit in part (*a*), the continuity of the sine function, and the trigonometric identity $\sin^2 x = 1 - \cos^2 x$. We obtain

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\left[\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\right] = \lim_{x\to 0}\frac{\sin^2 x}{(1+\cos x)x}$$
$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right) = (1)\left(\frac{0}{1+1}\right) = 0$$

Example 2 Find

(a) $\displaystyle\lim_{x\to 0}\frac{\tan x}{x}$ (b) $\displaystyle\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}$ (c) $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}$

Solution (a).

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)=(1)(1)=1$$

Solution (b). The trick is to multiply and divide by 2, which will make the denominator the same as the argument of the sine function [just as in Theorem 2.6.3(*a*)]:

$$\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}=\lim_{\theta\to 0}2\cdot\frac{\sin 2\theta}{2\theta}=2\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}$$

Now make the substitution $x=2\theta$, and use the fact that $x\to 0$ as $\theta\to 0$. This yields

$$\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}=2\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}=2\lim_{x\to 0}\frac{\sin x}{x}=2(1)=2$$

Solution (c).

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}=\lim_{x\to 0}\frac{\dfrac{\sin 3x}{x}}{\dfrac{\sin 5x}{x}}=\lim_{x\to 0}\frac{3\cdot\dfrac{\sin 3x}{3x}}{5\cdot\dfrac{\sin 5x}{5x}}=\frac{3\cdot 1}{5\cdot 1}=\frac{3}{5}$$ ◀

FOR THE READER. Use a graphing utility to confirm the limits in the last example graphically, and if you have a CAS, then use it to obtain the limits.

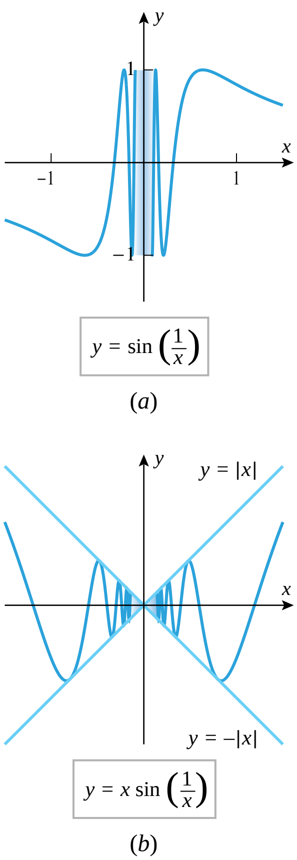


Figure 2.6.7

Example 3 Make conjectures about the limits

(a) $\displaystyle\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ (b) $\displaystyle\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)$

and confirm your conclusions by generating the graphs of the functions near $x=0$ using a graphing utility.

Solution (a). Since $1/x\to+\infty$ as $x\to 0^+$, we can view $\sin(1/x)$ as the sine of an angle that increases indefinitely as $x\to 0^+$. As this angle increases, the function $\sin(1/x)$ keeps oscillating between -1 and 1 without approaching a limit. Similarly, there is no limit from the left since $1/x\to-\infty$ as $x\to 0^-$. These conclusions are consistent with the graph of $y=\sin(1/x)$ shown in Figure 2.6.7*a*. Observe that the oscillations become more and more rapid as x approaches 0 because $1/x$ increases (or decreases) more and more rapidly as x approaches 0.

Solution (b). If $x>0$, $-x\le x\sin(1/x)\le x$, and if $x<0$, $x\le x\sin(1/x)\le -x$. Thus, for $x\ne 0$, $-|x|\le x\sin(1/x)\le |x|$. Since both $|x|\to 0$ and $-|x|\to 0$ as $x\to 0$, the Squeezing Theorem applies and we can conclude that $x\sin(1/x)\to 0$ as $x\to 0$. This is illustrated in Figure 2.6.7*b*. ◀

REMARK. It follows from part (b) of this example that the function

$$f(x)=\begin{cases}x\sin(1/x), & x\ne 0\\ 0, & x=0\end{cases}$$

is continuous at $x=0$, since the value of the function and the value of the limit are the same at 0. This shows that the behavior of a function can be very complex in the vicinity of an x-value c, even though the function is continuous at c.

EXERCISE SET 2.6 Graphing Calculator C CAS

In Exercises 1–10, find the discontinuities, if any.

1. $f(x) = \sin(x^2 - 2)$

2. $f(x) = \cos\left(\dfrac{x}{x-\pi}\right)$

3. $f(x) = \cot x$

4. $f(x) = \sec x$

5. $f(x) = \csc x$

6. $f(x) = \dfrac{1}{1+\sin^2 x}$

7. $f(x) = |\cos x|$

8. $f(x) = \sqrt{2+\tan^2 x}$

9. $f(x) = \dfrac{1}{1-2\sin x}$

10. $f(x) = \dfrac{3}{5+2\cos x}$

11. Use Theorem 2.5.6 to show that the following functions are continuous everywhere by expressing them as compositions of simpler functions that are known to be continuous.
(a) $\sin(x^3+7x+1)$ (b) $|\sin x|$
(c) $\cos^3(x+1)$ (d) $\sqrt{3+\sin 2x}$
(e) $\sin(\sin x)$ (f) $\cos^5 x - 2\cos^3 x + 1$

12. (a) Prove that if $g(x)$ is continuous everywhere, then so are $\sin(g(x))$, $\cos(g(x))$, $g(\sin(x))$, and $g(\cos(x))$.
(b) Illustrate the result in part (a) with some of your own choices for g.

Find the limits in Exercises 13–35.

13. $\lim\limits_{x\to+\infty} \cos\left(\dfrac{1}{x}\right)$

14. $\lim\limits_{x\to+\infty} \sin\left(\dfrac{2}{x}\right)$

15. $\lim\limits_{x\to+\infty} \sin\left(\dfrac{\pi x}{2-3x}\right)$

16. $\lim\limits_{h\to 0} \dfrac{\sin h}{2h}$

17. $\lim\limits_{\theta\to 0} \dfrac{\sin 3\theta}{\theta}$

18. $\lim\limits_{\theta\to 0^+} \dfrac{\sin\theta}{\theta^2}$

19. $\lim\limits_{x\to 0^-} \dfrac{\sin x}{|x|}$

20. $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{3x^2}$

21. $\lim\limits_{x\to 0^+} \dfrac{\sin x}{5\sqrt{x}}$

22. $\lim\limits_{x\to 0} \dfrac{\sin 6x}{\sin 8x}$

23. $\lim\limits_{x\to 0} \dfrac{\tan 7x}{\sin 3x}$

24. $\lim\limits_{\theta\to 0} \dfrac{\sin^2\theta}{\theta}$

25. $\lim\limits_{h\to 0} \dfrac{h}{\tan h}$

26. $\lim\limits_{h\to 0} \dfrac{\sin h}{1-\cos h}$

27. $\lim\limits_{\theta\to 0} \dfrac{\theta^2}{1-\cos\theta}$

28. $\lim\limits_{x\to 0} \dfrac{x}{\cos\left(\frac{1}{2}\pi - x\right)}$

29. $\lim\limits_{\theta\to 0} \dfrac{\theta}{\cos\theta}$

30. $\lim\limits_{t\to 0} \dfrac{t^2}{1-\cos^2 t}$

31. $\lim\limits_{h\to 0} \dfrac{1-\cos 5h}{\cos 7h - 1}$

32. $\lim\limits_{x\to 0^+} \sin\left(\dfrac{1}{x}\right)$

33. $\lim\limits_{x\to 0^+} \cos\left(\dfrac{1}{x}\right)$

34. $\lim\limits_{x\to 0} \dfrac{x^2 - 3\sin x}{x}$

35. $\lim\limits_{x\to 0} \dfrac{2x+\sin x}{x}$

In Exercises 36–39: (i) Construct a table to estimate the limit by evaluating the function near the limiting value. (ii) Find the exact value of the limit.

36. $\lim\limits_{x\to 5} \dfrac{\sin(x-5)}{x^2-25}$

37. $\lim\limits_{x\to 2} \dfrac{\sin(2x-4)}{x^2-4}$

38. $\lim\limits_{x\to -2} \dfrac{\sin(x^2+3x+2)}{x+2}$

39. $\lim\limits_{x\to -1} \dfrac{\sin(x^2+3x+2)}{x^3+1}$

40. Find a value for the constant k that makes

$$f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

continuous at $x = 0$.

41. Find a nonzero value for the constant k that makes

$$f(x) = \begin{cases} \dfrac{\tan kx}{x}, & x < 0 \\ 3x + 2k^2, & x \geq 0 \end{cases}$$

continuous at $x = 0$.

42. Is

$$f(x) = \begin{cases} \dfrac{\sin x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

continuous at $x = 0$?

43. In each part, find the limit by making the indicated substitution.
(a) $\lim\limits_{x\to+\infty} x\sin\dfrac{1}{x}$; $t = \dfrac{1}{x}$
(b) $\lim\limits_{x\to-\infty} x\left(1-\cos\dfrac{1}{x}\right)$; $t = \dfrac{1}{x}$
(c) $\lim\limits_{x\to\pi} \dfrac{\pi - x}{\sin x}$. [*Hint:* Let $t = \pi - x$.]

44. Find $\lim\limits_{x\to 2} \dfrac{\cos(\pi/x)}{x-2}$; $t = \dfrac{\pi}{2} - \dfrac{\pi}{x}$.

45. Find $\lim\limits_{x\to 1} \dfrac{\sin(\pi x)}{x-1}$.

46. Find $\lim\limits_{x\to\pi/4} \dfrac{\tan x - 1}{x - \pi/4}$.

47. Use the Squeezing Theorem to show that

$$\lim_{x\to 0} x\cos\frac{50\pi}{x} = 0$$

and illustrate the principle involved by using a graphing utility to graph $y = |x|$, $y = -|x|$, and $y = x\cos(50\pi/x)$ on the same screen in the window $[-1, 1] \times [-1, 1]$.

48. Use the Squeezing Theorem to show that

$$\lim_{x\to 0} x^2 \sin\left(\frac{50\pi}{\sqrt[3]{x}}\right) = 0$$

and illustrate the principle involved by using a graphing utility to graph $y = x^2$, $y = -x^2$, and $y = x^2 \sin(50\pi/\sqrt[3]{x})$ on the same screen in the window $[-0.5, 0.5] \times [-0.25, 0.25]$.

49. Sketch the graphs of $y = 1 - x^2$, $y = \cos x$, and $y = f(x)$, where f is a function that satisfies the inequalities

$$1 - x^2 \le f(x) \le \cos x$$

for all x in the interval $(-\pi/2, \pi/2)$. What can you say about the limit of $f(x)$ as $x \to 0$? Explain your reasoning.

50. Sketch the graphs of $y = 1/x$, $y = -1/x$, and $y = f(x)$, where f is a function that satisfies the inequalities

$$-\frac{1}{x} \le f(x) \le \frac{1}{x}$$

for all x in the interval $[1, +\infty)$. What can you say about the limit of $f(x)$ as $x \to +\infty$? Explain your reasoning.

51. Find formulas for functions g and h such that $g(x) \to 0$ and $h(x) \to 0$ as $x \to +\infty$ and such that

$$g(x) \le \frac{\sin x}{x} \le h(x)$$

for positive values of x. What can you say about the limit

$$\lim_{x \to +\infty} \frac{\sin x}{x}?$$

Explain your reasoning.

52. Draw pictures analogous to Figure 2.6.3 that illustrate the Squeezing Theorem for limits of the forms $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

> Recall that unless stated otherwise the variable x in trigonometric functions such as $\sin x$ and $\cos x$ is assumed to be in radian measure. The limits in Theorem 2.6.3 are based on that assumption. Exercises 53 and 54 explore what happens to those limits if degree measure is used for x.

53. (a) Show that if x is in degrees, then

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{\pi}{180}$$

(b) Confirm that the limit in part (a) is consistent with the results produced by your calculating utility by setting the utility to degree measure and calculating $(\sin x)/x$ for some values of x that get closer and closer to 0.

54. What is the limit of $(1 - \cos x)/x$ as $x \to 0$ if x is in degrees?

55. It follows from part (*a*) of Theorem 2.6.3 that if θ is small (near zero) and measured in radians, then one should expect the approximation

$$\sin\theta \approx \theta$$

to be good.
(a) Find $\sin 10^\circ$ using a calculating utility.
(b) Estimate $\sin 10^\circ$ using the approximation above.

56. (a) Use the approximation of $\sin\theta$ that is given in Exercise 55 together with the identity $\cos 2\alpha = 1 - 2\sin^2\alpha$ with $\alpha = \theta/2$ to show that if θ is small (near zero) and measured in radians, then one should expect the approximation

$$\cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

to be good.
(b) Find $\cos 10^\circ$ using a calculating utility.
(c) Estimate $\cos 10^\circ$ using the approximation above.

57. It follows from part (a) of Example 2 that if θ is small (near zero) and measured in radians, then one should expect the approximation

$$\tan\theta \approx \theta$$

to be good.
(a) Find $\tan 5^\circ$ using a calculating utility.
(b) Find $\tan 5^\circ$ using the approximation above.

58. Referring to the accompanying figure, suppose that the angle of elevation of the top of a building, as measured from a point L feet from its base, is found to be α degrees.
(a) Use the relationship $h = L\tan\alpha$ to calculate the height of a building for which $L = 500$ ft and $\alpha = 6^\circ$.
(b) Show that if L is large compared to the building height h, then one should expect good results in approximating h by $h \approx \pi L\alpha/180$.
(c) Use the result in part (b) to approximate the building height h in part (a).

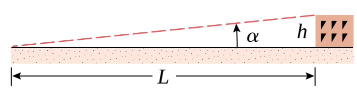


Figure Ex-58

59. (a) Use the Intermediate-Value Theorem to show that the equation $x = \cos x$ has at least one solution in the interval $[0, \pi/2]$.
(b) Show graphically that there is exactly one solution in the interval.
(c) Approximate the solution to three decimal places.

60. (a) Use the Intermediate-Value Theorem to show that the equation $x + \sin x = 1$ has at least one solution in the interval $[0, \pi/6]$.
(b) Show graphically that there is exactly one solution in the interval.
(c) Approximate the solution to three decimal places.

61. In the study of falling objects near the surface of the Earth, the ***acceleration g due to gravity*** is commonly taken to be $9.8\ \text{m/s}^2$ or $32\ \text{ft/s}^2$. However, the elliptical shape of the Earth and other factors cause variations in this constant that are latitude dependent. The following formula, known as the Geodetic Reference Formula of 1967, is commonly used to predict the value of g at a latitude of ϕ degrees (either north or south of the equator):

$$g = 9.7803185(1.0 + 0.005278895\sin^2\phi - 0.000023462\sin^4\phi)\ \text{m/s}^2$$

(a) Observe that g is an even function of ϕ. What does this suggest about the shape of the Earth, as modeled by the Geodetic Reference Formula?

(b) Show that $g = 9.8\ \mathrm{m/s^2}$ somewhere between latitudes of 38° and 39°.

62. Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

(a) Make a conjecture about the limit of $f(x)$ as $x \to 0$.
(b) Make a conjecture about the limit of $xf(x)$ as $x \to 0$.
(c) Prove your conjectures.

SUPPLEMENTARY EXERCISES

1. For the function f graphed in the accompanying figure, find the limit if it exists.
(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 2} f(x)$ (c) $\lim_{x\to 3} f(x)$
(d) $\lim_{x\to 4} f(x)$ (e) $\lim_{x\to +\infty} f(x)$ (f) $\lim_{x\to -\infty} f(x)$
(g) $\lim_{x\to 3^+} f(x)$ (h) $\lim_{x\to 3^-} f(x)$ (i) $\lim_{x\to 0} f(x)$

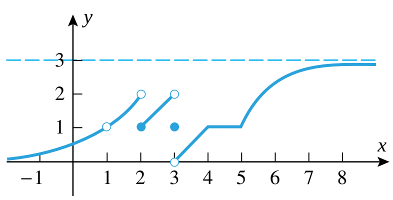


Figure Ex-1

2. (a) Find a formula for a rational function that has a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 2$.
(b) Check your work by using a graphing utility to graph the function.

3. (a) Write a paragraph or two that describes how the limit of a function can fail to exist at $x = a$. Accompany your description with some specific examples.
(b) Write a paragraph or two that describes how the limit of a function can fail to exist as $x \to +\infty$ or $x \to -\infty$. Also, accompany your description with some specific examples.
(c) Write a paragraph or two that describes how a function can fail to be continuous at $x = a$. Accompany your description with some specific examples.

4. Show that the conclusion of the Intermediate-Value Theorem may be false if f is not continuous on the interval $[a, b]$.

5. In each part, evaluate the function for the stated values of x, and make a conjecture about the value of the limit. Confirm your conjecture by finding the limit algebraically.
(a) $f(x) = \dfrac{x-2}{x^2-4}$; $\lim_{x\to 2^+} f(x)$; $x = 2.5, 2.1, 2.01, 2.001, 2.0001, 2.00001$
(b) $f(x) = \dfrac{\tan 4x}{x}$; $\lim_{x\to 0} f(x)$; $x = \pm 1.0, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001, \pm 0.00001$

6. In each part, find the horizontal asymptotes, if any.
(a) $y = \dfrac{2x-7}{x^2-4x}$ (b) $y = \dfrac{x^3-x^2+10}{3x^2-4x}$
(c) $y = \dfrac{2x^2-6}{x^2+5x}$

7. (a) Approximate the value for the limit
$$\lim_{x\to 0} \frac{3^x - 2^x}{x}$$
to three decimal places by constructing an appropriate table of values.
(b) Confirm your approximation using graphical evidence.

8. According to Ohm's law, when a voltage of V volts is applied across a resistor with a resistance of R ohms, a current of $I = V/R$ amperes flows through the resistor.
(a) How much current flows if a voltage of 3.0 volts is applied across a resistance of 7.5 ohms?
(b) If the resistance varies by ± 0.1 ohm, and the voltage remains constant at 3.0 volts, what is the resulting range of values for the current?
(c) If temperature variations cause the resistance to vary by $\pm\delta$ from its value of 7.5 ohms, and the voltage remains constant at 3.0 volts, what is the resulting range of values for the current?
(d) If the current is not allowed to vary by more than $\epsilon = \pm 0.001$ ampere at a voltage of 3.0 volts, what variation of $\pm\delta$ from the value of 7.5 ohms is allowable?
(e) Certain alloys become ***superconductors*** as their temperature approaches absolute zero (-273°C), meaning that their resistance approaches zero. If the voltage remains constant, what happens to the current in a superconductor as $R \to 0^+$?

9. Suppose that f is continuous on the interval $[0, 1]$ and that $0 \le f(x) \le 1$ for all x in this interval.
(a) Sketch the graph of $y = x$ together with a possible graph for f over the interval $[0, 1]$.
(b) Use the Intermediate-Value Theorem to help prove that there is at least one number c in the interval $[0, 1]$ such that $f(c) = c$.

10. Use algebraic methods to find
(a) $\lim_{\theta\to 0} \tan\left(\dfrac{1-\cos\theta}{\theta}\right)$ (b) $\lim_{t\to 1} \dfrac{t-1}{\sqrt{t}-1}$
(c) $\lim_{x\to +\infty} \dfrac{(2x-1)^5}{(3x^2+2x-7)(x^3-9x)}$
(d) $\lim_{\theta\to 0} \cos\left(\dfrac{\sin(\theta+\pi)}{2\theta}\right)$.

11. Suppose that f is continuous on the interval $[0, 1]$, that $f(0) = 2$, and that f has no zeros in the interval. Prove that $f(x) > 0$ for all x in $[0, 1]$.

12. Suppose that

$$f(x) = \begin{cases} -x^4 + 3, & x \le 2 \\ x^2 + 9, & x > 2 \end{cases}$$

Is f continuous everywhere? Justify your conclusion.

13. Show that the equation $x^4 + 5x^3 + 5x - 1 = 0$ has at least two real solutions in the interval $[-6, 2]$.

14. Use the Intermediate-Value Theorem to approximate $\sqrt{11}$ to three decimal places, and check your answer by finding the root directly with a calculating utility.

15. Suppose that f is continuous at x_0 and that $f(x_0) > 0$. Give either an ϵ-δ proof or a convincing verbal argument to show that there must be an open interval containing x_0 on which $f(x) > 0$.

16. Sketch the graph of $f(x) = |x^2 - 4|/(x^2 - 4)$.

17. In each part, approximate the discontinuities of f to three decimal places.

(a) $f(x) = \dfrac{x+1}{x^2 + 2x - 5}$

(b) $f(x) = \dfrac{x+3}{|2\sin x - x|}$

18. In Example 3 of Section 2.6 we used the Squeezing Theorem to prove that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$$

Why couldn't we have obtained the same result by writing

$$\begin{aligned} \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) &= \lim_{x\to 0} x \cdot \lim_{x\to 0} \sin\left(\frac{1}{x}\right) \\ &= 0 \cdot \lim_{x\to 0} \sin\left(\frac{1}{x}\right) = 0? \end{aligned}$$

In Exercises 19 and 20, find $\lim_{x\to a} f(x)$, if it exists, for $a = 0,\ 5^+,\ -5^-,\ -5,\ 5,\ -\infty,\ +\infty$

19. (a) $f(x) = \sqrt{5 - x}$ (b) $f(x) = (x^2 - 25)/(x - 5)$

20. (a) $f(x) = (x + 5)/(x^2 - 25)$

(b) $f(x) = \begin{cases} (x-5)/|x-5|, & x \ne 5 \\ 0, & x = 5 \end{cases}$

In Exercises 21–28, find the indicated limit, if it exists.

21. $\displaystyle\lim_{x\to 0} \frac{\tan ax}{\sin bx}$ $(a \ne 0, b \ne 0)$

22. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{\tan 3x}$

23. $\displaystyle\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta^2}$

24. $\displaystyle\lim_{x\to 0} \frac{x \sin x}{1 - \cos x}$

25. $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}}$

26. $\displaystyle\lim_{x\to 0} \frac{\sin^2(kx)}{x^2}$, $k \ne 0$

27. $\displaystyle\lim_{x\to 0} \frac{3x - \sin(kx)}{x}$, $k \ne 0$

28. $\displaystyle\lim_{x\to +\infty} \frac{2x + x\sin 3x}{5x^2 - 2x + 1}$

29. One dictionary describes a continuous function as "one whose value at each point is closely approached by its values at neighboring points."

(a) How would you explain the meaning of the terms "neighboring points" and "closely approached" to a nonmathematician?

(b) Write a paragraph that explains why the dictionary definition is consistent with Definition 2.5.1.

30. (a) Show by rationalizing the numerator that

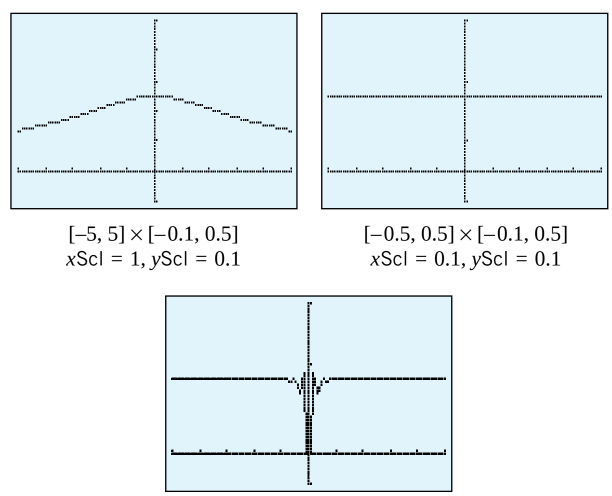
$$\lim_{x\to 0} \frac{\sqrt{x^2 + 4} - 2}{x^2} = \frac{1}{4}$$

(b) Evaluate $f(x)$ for

$$x = \pm 1.0, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001, \pm 0.00001$$

and explain why the values are not getting closer and closer to the limit.

(c) The accompanying figure shows the graph of f generated with a graphing utility and zooming in on the origin. Explain what is happening.



$[-5, 5] \times [-0.1, 0.5]$
xScl = 1, yScl = 0.1

$[-0.5, 0.5] \times [-0.1, 0.5]$
xScl = 0.1, yScl = 0.1

$[-5 \times 10^{-6}, 5 \times 10^{-6}] \times [-0.1, 0.5]$
xScl = 10^{-6}, yScl = 0.1

Figure Ex-30

In Exercises 31–36, approximate the limit of the function by looking at its graph and calculating values for some appropriate choices of x. Compare your answer with the value produced by a CAS.

C **31.** $\lim_{x \to 0} (1+x)^{1/x}$

C **32.** $\lim_{x \to 3} \dfrac{2^x - 8}{x - 3}$

C **33.** $\lim_{x \to 1} \dfrac{\sin x - \sin 1}{x - 1}$

C **34.** $\lim_{x \to 0^+} x^{-2}(1.001)^{-1/x}$

C **35.** $\lim_{x \to +\infty} \left(\sqrt{x + \sqrt{x}} - \sqrt{x}\right)$

C **36.** $\lim_{x \to +\infty} (3^x + 5^x)^{1/x}$

37. The limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

ensures that there is a number δ such that

$$\left| \frac{\sin x}{x} - 1 \right| < 0.001$$

if $0 < |x| < \delta$. Estimate the largest such δ.

38. If \$1000 is invested in an account that pays 7% interest compounded n times each year, then in 10 years there will be $1000(1 + 0.07/n)^{10n}$ dollars in the account. How much money will be in the account in 10 years if the interest is compounded quarterly ($n = 4$)? Monthly ($n = 12$)? Daily ($n = 365$)? Estimate the amount of money that will be in the account in 10 years if the interest is compounded *continuously*, that is, as $n \to +\infty$?

39. There are various numerical methods other than the method discussed in Section 2.5 to obtain approximate solutions of equations of the form $f(x) = 0$. One such method requires that the equation be expressed in the form $x = g(x)$, so that a solution $x = c$ can be interpreted as the value of x where the line $y = x$ intersects the curve $y = g(x)$, as shown in the accompanying figure. If x_1 is an initial estimate of c and the graph of $y = g(x)$ is not too steep in the vicinity of c, then a better approximation can be obtained from $x_2 = g(x_1)$ (see the figure). An even better approximation is obtained from $x_3 = g(x_2)$, and so forth. The formula $x_{n+1} = g(x_n)$ for $n = 1, 2, 3, \ldots$ generates successive approximations $x_2, x_3, x_4, \ldots$ that get closer and closer to c.

(a) The equation $x^3 - x - 1 = 0$ has only one real solution. Show that this equation can be written as

$$x = g(x) = \sqrt[3]{x + 1}$$

(b) Graph $y = x$ and $y = g(x)$ in the same coordinate system for $-1 \le x \le 3$.

(c) Starting with an arbitrary estimate x_1, make a sketch that shows the location of the successive iterates

$$x_2 = g(x_1), \quad x_3 = g(x_2), \ldots$$

(d) Use $x_1 = 1$ and calculate $x_2, x_3, \ldots,$ continuing until you obtain two consecutive values that differ by less than 10^{-4}. Experiment with other starting values such as $x_1 = 2$ or $x_1 = 1.5$.

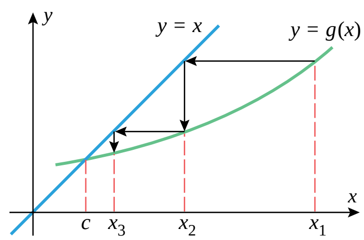


Figure Ex-39

40. The method described in Exercise 39 will not always work.

(a) The equation $x^3 - x - 1 = 0$ can be expressed as $x = g(x) = x^3 - 1$. Graph $y = x$ and $y = g(x)$ in the same coordinate system. Starting with an arbitrary estimate x_1, make a sketch illustrating the locations of the successive iterates $x_2 = g(x_1), x_3 = g(x_2), \ldots$.

(b) Use $x_1 = 1$ and calculate the successive iterates x_n for $n = 2, 3, 4, 5, 6$.

In Exercises 41 and 42, use the method of Exercise 39 to approximate the roots of the equation.

41. $x^5 - x - 2 = 0$

42. $x - \cos x = 0$