Definite Integrals

We define the Definite Integration of the function $f(x)$ with respect to x from a to b to be

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

which is read as 'the integral from a to b of $f(x)dx$,where F(x) is the anti-derivative of f(x). We call a and b the lower and upper limits of integration respectively.

we introduce definite integrals, so called because the result will be a definite answer, usually a number, with no constant of integration .

$$\int_a^b f(x)dx = F(x)|_a^b = (F(b) + C) - (F(a) + C)$$

$$= F(b) + C - F(a) - C$$

$$= F(b) - F(a).$$

Properties of the Definite Integral

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, $a < c < b$.

3. $\int_a^b k.f(x)dx = k\int_a^b f(x)dx$, for any constant k.

4. $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

5. $\int_a^b k.dx = k(b - a)$.

6. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Examples:

1. $\int_{1}^{2} 3x^2 dx = \left[3\frac{x^3}{3}\right]_{1}^{2} = (2)^3 - (1)^3 = 7$

2. $\int_{-2}^{3} x^3 dx = \left[\frac{x^4}{4}\right]_{-2}^{3} = \frac{3^4}{4} - \frac{(-2)^4}{4}$

$$= \frac{81}{4} - \frac{16}{4} = \frac{65}{4}.$$

3. $\int_{4}^{9} 3\sqrt{t}\, dt = 3\int_{4}^{9} t^{1/2}\, dt = \left[3\frac{t^{3/2}}{\frac{3}{2}}\right]_{4}^{9} = \left[2\, t^{3/2}\right]_{4}^{9}$

$2\,(9)^{3/2} - 2\,(4)^{3/2} = 2(9^{1/2})^3 - 2(4^{1/2})^3$

$= 2 * 3^3 - 2 * 2^3 = 2 * 27 - 2 * 8 = 45 - 16 = 38.$

Example 4: *Let* $\int_{2}^{6} f(x)dx = -2$, *find* $\int_{6}^{2} f(x)dx$??

$$\therefore \int_{6}^{2} f(x)dx = -\int_{2}^{6} f(x)dx = -(-2) = 2.$$

Example 5: *If* $\int_{0}^{10} f(x)dx = 12$, $\int_{10}^{15} f(x)dx = 8$, *find* $\int_{0}^{15} f(x)dx$?

Sol: $\int_{0}^{15} f(x)dx = \int_{0}^{10} f(x)dx + \int_{10}^{15} f(x)dx = 12+8 = 20.$

Example 6 : *Calculate* $\int_{-1}^{1} 4\, dx = [4\, x]_{-1}^{1} = 4(1 - (-1)) = 8.$

Example 7 : *find* $\int_{-1}^{2}(2 - x^2 + x)dx = 2x - \frac{x^3}{3} + \frac{x^2}{2}\, |_{-1}^{2}$

$$= \left(2.2 - \frac{2^3}{3} + \frac{2^2}{2}\right) - (2.(-1) - \frac{(-1)^3}{3} + \frac{(-1)^2}{2})$$

$$= 4 - \frac{8}{3} + \frac{4}{2} - 2 + \frac{1}{3} + \frac{1}{2}$$

$$= 6 - \frac{4}{6} - \frac{5}{6} = \frac{36-9}{6} = \frac{27}{6} = \frac{9}{2}.$$

Example 8 : find $\int_2^{10} \frac{3dx}{\sqrt{5x-1}}$??

$\int_2^{10} \frac{3dx}{\sqrt{5x-1}} = \frac{5}{5}\int_2^{10} 3(5x-1)^{-\frac{1}{2}} dx = \frac{3}{5}\int_2^{10} 5(5x-1)^{-\frac{1}{2}} dx$

$$= \frac{3}{5}\left[\frac{(5x-1)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^{10} = \frac{6}{5}\left(\sqrt{49} - \sqrt{9}\right) = \frac{6}{5}\ (7-3) = \frac{24}{5}.$$

Example 9: find $\int_0^1 (2x+1)^3 dx = \frac{2}{2}\int_0^1 (2x+1)^3 dx = \frac{1}{2}\frac{(2x+1)^4}{4}\ |_0^1$

$$= \frac{1}{8}\ [(2.1+1)^4 - (2.0+1)^4]$$

$$= \frac{1}{8}\ [81-1] = \frac{80}{8} = 10.$$

Example 10 : Calculate $\int_{-3}^{-1} \frac{dx}{(2x+1)^3} = \frac{2}{2}\int_{-3}^{-1}(2x+1)^{-3}$

$$= \frac{1}{2}\frac{(2x+1)^{-2}}{-2}\ |_{-3}^{-1} = \frac{-1}{4}\frac{1}{(2x+1)^2}\ |_{-3}^{-1}$$

$$= \frac{-1}{4}\left[\frac{1}{(-2+1)^2} - \frac{1}{(-6+1)^2}\right]$$

$$= \frac{-1}{4}\left[\frac{1}{1} - \frac{1}{25}\right] = \frac{-1}{4}\left[\frac{24}{25}\right] = \frac{-6}{25}.$$

Example 11 : find $\int_{-1}^3 (3x^2 - 2x + 1)dx = \left[3\frac{x^3}{3} - 2\frac{x^2}{2} + x\right]_{-1}^3$

$$= [27 - 9 + 3] - [-1 - 1 - 1] = 24.$$

Example 12 :write the following sum or difference as a single integral in the form $\int_a^b f(x)\ dx$??

a) $\int_2^{10} f(x)\,dx \; - \int_2^7 f(x)\,dx$?

b) $\int_{-3}^{5} f(x)\,dx \; - \int_{-3}^{0} f(x)\,dx + \int_5^6 f(x)\,dx$?

Sol

a) $\int_2^{10} f(x)\,dx \; - (- \int_7^2 f(x))\,dx = \int_2^{10} f(x)\,dx \; + \int_7^2 f(x)\,dx = \int_7^{10} f(x)\,dx$

b) $\int_{-3}^{5} f(x)\,dx - (- \int_0^{-3} f(x))\,dx + \int_5^6 f(x)\,dx = \int_{-3}^{5} f(x)\,dx + \int_0^{-3} f(x)\,dx + \int_5^6 f(x)\,dx$

$$= \int_0^5 f(x)\,dx + \int_5^6 f(x)\,dx = \int_0^6 f(x)\,dx$$

Example 13 :Let $\int_0^2 f(x)\,dx = 3$ *, and* $\int_2^6 f(x)\,dx = -2,$

Find $2\int_0^2 f(x)\,dx - 5\int_2^6 f(x)\,dx$??

<u>*SOL*</u> $2\int_0^2 f(x)\,dx - 5\int_2^6 f(x)\,dx = 2(3) - 5(-2) = 6 + 10 = 16.$

Example 14 :Let $M = \int_a^b k.x\,dx$ *and* $N = \int_a^b k\,dx$ *,find Z ,where* $Z = \dfrac{M}{N}$?

Sol: $M = \int_a^b k.x\,dx = k \int_a^b x\,dx = k\left[\dfrac{x^2}{2} \right]_a^b = \dfrac{k}{2}(b^2 - a^2)$

$N = \int_a^b k\,dx = k \int_a^b dx = k[\ x]_a^b = k(b - a)$

$\to Z = \dfrac{M}{N} = \dfrac{\frac{k}{2}(b^2 - a^2)}{k(b-a)} = \dfrac{k(b-a)(b+a)}{2} * \dfrac{1}{k(b-a)}$

$$= \dfrac{b + a}{2}$$

Example 15:Find $\int_1^2 (\sqrt{x} + \frac{1}{\sqrt{x}})dx$?? *H.W*