Al Basrah university
college of Pharmacy

Mathematics and Statistics\ first stage

Dr . Rana Hasan

2022 /12 / 21

Derivatives of Inverse Trigonometric Function

Note:- $\sin^{-1}x \neq (\sin x)^{-1} = \frac{1}{\sin x}$

1. $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$
2. $\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$
3. $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2} \cdot \frac{du}{dx}$
4. $\frac{d}{dx}\cot^{-1}u = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$
5. $\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$
6. $\frac{d}{dx}\csc^{-1}u = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

Examples:- Find derivative of following:-

1. $y = \tan^{-1}(3\tan(x))$

$$y' = \frac{1}{\sqrt{1-(3\tan(x))^2}}(3\sec^2 x) = \frac{3\sec^2 x}{\sqrt{1-(3\tan(x))^2}}.\backslash$$

2. $y = \sin^{-1}(x^2+3)$

$$y' = \frac{1}{\sqrt{1-(x^2+3)^2}}(2x) = \frac{2x}{\sqrt{1-(x^2+3)^2}}$$

3. $y = \cos^{-1}(\sqrt{2x-1})$

$$y' = \frac{-1}{\sqrt{1-(2x-1)}}.(\frac{1}{2}(2x-1)^{\frac{-1}{2}}.\ 2)$$

$$= \frac{-1}{\sqrt{2-2x}}.\ \frac{1}{\sqrt{2x-1}}.$$

4. $y = \sec^{-1}(2x^2)$

$$y' = \frac{1}{|2x^2|\sqrt{(2x^2)^2-1}}.(4x) = \frac{4x}{2x^2\sqrt{4x^4-1}}$$

$5.\ y = \tan^{-1}(\cos\sqrt{x})$

$$y' = \frac{1}{1+(\cos^2\sqrt{x})} * \left(\frac{-\sin\sqrt{x}}{2\sqrt{x}}\right) = \frac{-\sin\sqrt{x}}{2\sqrt{x}\,(1+\cos^2\sqrt{x})}.$$

$6.\ y = (\sin^{-1} x)^4$

$$y' = 4\,(\sin^{-1} x)^3 * \frac{1}{\sqrt{1-x^2}} = \frac{4\,(\sin^{-1} x)^3}{\sqrt{1-x^2}}$$

$7.\ y = 3x^2 + \cot^{-1} x$

$$y' = 6x + \frac{-1}{1+x^2} = 6x - \frac{1}{1+x^2}$$

Hyperbolic Functions

Some important relation :-

1- $\cosh^2(x) - \sinh^2(x) = 1$

2- $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

3- $\coth^2(x) - 1 = \operatorname{csch}^2(x)$

Derivatives of Hyperbolic Functions

1- $\frac{d}{dx}\sinh(u) = \cosh(u) * \frac{du}{dx}$

2- $\frac{d}{dx}\cosh(u) = \sinh(u) * \frac{du}{dx}$

3- $\frac{d}{dx}\tanh(u) = \operatorname{sech}^2(u) * \frac{du}{dx}$

4- $\frac{d}{dx}\coth(u) = -\operatorname{csch}^2(u) * \frac{du}{dx}$

5- $\frac{d}{dx}\operatorname{sech}(u) = -\operatorname{sech}(u) * \tanh(u) * \frac{du}{dx}$

6- $\frac{d}{dx}\operatorname{csch}(u) = -\operatorname{csch}(u) * \coth(u) * \frac{du}{dx}$

Examples:- Find derivative of the following functions :-

$1.\ f(x) = \tanh(x^3)\coth(x^2)$

$\longrightarrow f'(x) = \tanh(x^3).\left(-\operatorname{csch}^2(x^2)\right).(2x) + \coth(x^2)\left(\operatorname{sech}^2(x^3)\right)(3x^2).$

$2.\ f(x) = \tanh(5x^2+3) \longrightarrow f'(x) = \operatorname{sech}^2(5x^2+3).(10x).$

$3.\ f(x) = \cosh(\sqrt{x}) + 5x^3 \longrightarrow f'(x) = \sinh(\sqrt{x}).\left(\frac{1}{2\sqrt{x}}\right) + 15x^2.$

$4.\ f(x) = \coth(\sin^{-1} x) \longrightarrow f'(x) = -\operatorname{csch}^2(\sin^{-1} x).\ \frac{1}{\sqrt{1-x^2}}$

5. $f(x) = sech(3x^2+1) + coth(5x)$

$\longrightarrow f'(x) = -sech\,(3x^2+1) * tanh\,(3x^2+1) * (6x) - csch^2(5x) * 5.$

6. $f(x) = cosh\left(\sqrt{x^2+1}\right)$

$\longrightarrow f'(x) = sinh\left(\sqrt{x^2+1}\right).\left(\frac{1}{2}\,(x^2+1)^{\frac{-1}{2}}\,2x\right) = \frac{x\,.\,sinh(\sqrt{x^2+1})}{\sqrt{x^2+1}}.$

The Natural Logarithm Function

Rules of arithmetic logarithms :- For any $a > 0$ *and* $x > 0$ **.**

1- $ln(a * x) = ln\,a + ln\,x$
2- $ln\left(\frac{a}{x}\right) = ln\,a - ln\,x$
3- $ln\left(\frac{1}{x}\right) = -\,ln\,x$
4- $ln(x^n) = n\,ln\,x$
5- $ln(\sqrt[m]{x}) = \frac{1}{m}ln\,x$

Derivative of Natural Logarithm Function

If u ***is a differentiable function of*** x ***, and*** $f = ln(u)$ ***then the derivative of*** f ***is :-***

$$\frac{df}{dx} = \frac{d}{dx}(ln(u)) = \frac{u'}{u}$$

Examples:- Find derivative of the following functions :-

1. $y = ln\left(sin^{-1}(2x)\right) \longrightarrow y' = \frac{1}{sin^{-1}(2x)}.\frac{2}{\sqrt{1-4x^2}} = \frac{2}{sin^{-1}(2x)\sqrt{1-4x^2}}.$

2. $y = ln\,(5x) \longrightarrow y' = \frac{5}{5x} = \frac{1}{x}.$

3. $y = ln\,(5x^2+8) \longrightarrow y' = \frac{10x}{5x^2+8}.$

4. $y = ln(tan^3x) \longrightarrow y' = \frac{3tan^2x\,.\,sec^2x}{tan^3x} = \frac{3\,sec^2x}{tan\,x}.$

5. $y = ln(sin(2x)) \longrightarrow y' = \frac{2\cos 2x}{sin\,2x} = 2\,cot\,2x\,.$

6. $y = ln(ln\,2x) \longrightarrow y' = \frac{\frac{2}{2x}}{ln\,2x} = \frac{1}{x * ln\,2x}.$

7. $y = \sin^{-1}(\ln 2x) \longrightarrow y' = \frac{1}{\sqrt{1-(\ln 2x)^2}} \cdot \frac{2}{2x}$.

8. $y = \ln(\cos^{-1}(x)) \longrightarrow y' = \frac{\frac{-1}{\sqrt{1-x^2}}}{\cos^{-1} x} = \frac{-1}{\sqrt{1-x^2}} \cdot \frac{1}{\cos^{-1}(x)}$.

The General Exponential Function

If $a > 0$ and $a \neq 1$, the function $f(x) = a^x$ is called the General Exponential Function with base a.

Note $a^x = e^{\ln a^x} = e^{x \ln a}$*, for example* $2^{\sqrt{3}} = e^{\sqrt{3}\ln 2}$.

Laws of exponents ($a > 0$, any x and y):-

1. $a^{x+y} = a^x * a^y$
2. $a^{x-y} = \frac{a^x}{a^y}$
3. $a^{-x} = \frac{1}{a^x}$
4. $(a^x)^y = a^{x*y}$.

Derivative of General Exponential Function:

If $a > 0$ and u is a differentiable function of x , then a^u is a differentiable function of x and:

$$\frac{d}{dx} a^u = a^u * \ln a * \frac{du}{dx}$$

Examples: Find y' of the following functions:

1. $y = 3^{2x-5} \longrightarrow y' = 3^{2x-5} . \ln 3 . 2$.

2. $y = 5^{\sin x} \longrightarrow y' = 5^{\sin x} . \ln 5 . \cos x$.

3. $y = 5^{\sqrt{x}} \longrightarrow y' = 5^{\sqrt{x}} . \ln 5 . \frac{1}{2\sqrt{x}}$.

4. $y = x^2 * \sin\left(2^{x^2}\right)$
$\longrightarrow y' = x^2 * \cos\left(2^{x^2}\right) . \left(2^{x^2} . \ln 2 * 2x\right) + \sin\left(2^{x^2}\right) . 2x$.

5. $y = 2^{\sinh x} \longrightarrow y' = 2^{\sinh x} * \ln 2 * \cosh x$.

6. $y = \tan^{-1}\left(2^{3x^2}\right) \longrightarrow y' = \frac{1}{1+(2^{3x^2})^2} * 2^{3x^2} . \ln 2 . (6x)$.

Natural Exponential Function (e^x)

Exponential function :defined as the inverse of the natural logarithm function .

Note:- $e = 2.718$

Law of exponential function (e^x) *For all real numbers* x *and* y :

1. $e^{x+y} = e^x * e^y$
2. $e^{x-y} = \frac{e^x}{e^y}$
3. $e^{-x} = \frac{1}{e^x}$
4. $(e^x)^y = e^{x*y}$.

Derivative of natural exponential function:-

If $u(x)$ *is a differentiable function of* x *, and let* $f(x) = e^{u(x)}$ *, then the derivative of* f *is :-*

$$\frac{df}{dx} = \frac{d}{dx}(e^u) = e^u * \frac{du}{dx}$$

Examples: Find y' *of the following functions:*

1. $y = e^{3\cos\sqrt{x}} \longrightarrow y' = e^{3\cos\sqrt{x}} . \left(-3.\frac{1}{2\sqrt{x}}\sin\sqrt{x}\right) = \frac{-3}{2\sqrt{x}}. e^{3\cos\sqrt{x}}\sin\sqrt{x}$.

2. $y = e^{5x^2+1} \longrightarrow y' = e^{5x^2+1} * 10x = 10x * e^{5x^2+1}$.

3. $y = \cos(e^{2x}) \longrightarrow y' = -\sin(e^{2x}) * e^{2x} * 2 = -2e^{2x} * \sin(e^{2x})$.

4. $y = e^{\tan(2x)} \longrightarrow y' = e^{\tan 2x} * \sec^2(2x) * 2 = 2 * \sec^2(2x) * e^{\tan 2x}$.

5. $y = e^{\sqrt{x}} \longrightarrow y' = e^{\sqrt{x}} . \left(\frac{1}{2\sqrt{x}}\right)$

5. $y = e^{\left(\sin x + \sqrt[3]{x}\right)} \longrightarrow y' = e^{(\sin x + \sqrt[3]{x})} * (\cos x + \frac{1}{3\sqrt[3]{x^2}})$.

7. $y = \tan^{-1}(e^{3x}) \longrightarrow y' = \frac{1}{\sqrt{1+(e^{3x})^2}} * e^{3x} * 3$.

8. $y = e^{\sin^{-1} x} \longrightarrow y' = e^{\sin^{-1} x} * \frac{1}{\sqrt{1-x^2}}$.