Al Basrah university
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Mathematics and Statistics\ first stage

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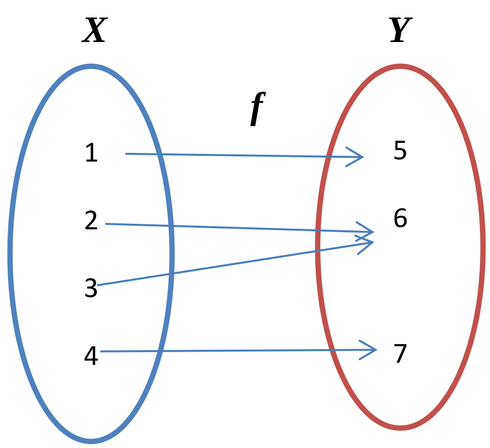
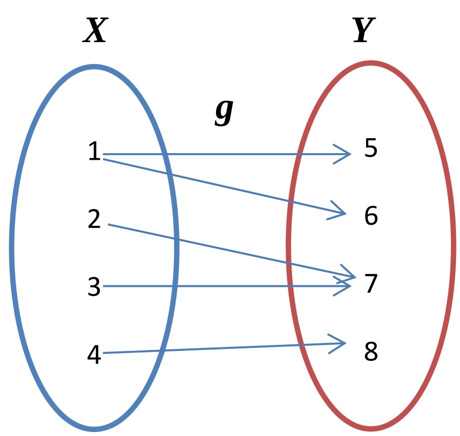
Functions:

Def : *Let each of X and Y be a non-empty set. The relationship f from X to Y is said to be a function from X to Y if for every $x \in X$, there is one element $y \in Y$ satisfy $y = f(x)$.The set x is called a Domain and is denoted by D_f and the set Y is called Range and is denoted by R_f.*





We note

- *f : X → Y is a function. Every element in X has associated with it exactly one element of Y .*
- *But g : X → Y is not a function. The element 1 in set X is assigned two elements, 5 and 6 in set Y .*
- *the domain of f is the set X.*
- *the range of f is the set of y-values such that y = f(x) for some x in X.*

Arithmetic Operations on functions:

- *Sum:* $(f + g)(x) = f(x) + g(x)$.
- *Difference :* $(f - g)(x) = f(x) - g(x)$.
- *Product:* $(f * g)(x) = f(x) * g(x)$.
- *Quotient::* $(f/g)(x) = \frac{f(x)}{g(x)}$ *,where* $g(x) \neq 0$.

The Limit of function at point:

If the value of a function $f(x)$ approach the value L as x approach b, we say f has limit L as x approach b and we write $lim_{x \to b} f(x) = L$.

properties of limits

1.let $f(x) = c$, c is constant then $\lim_{x \to b} f(x) = \lim_{x \to b} c = c$.
2. $\lim_{x \to b} c.f(x) = c.\lim_{x \to b} f(x)$.
3. $\lim_{x \to b} (f(x) \pm g(x)) = \lim_{x \to b} f(x) \pm \lim_{x \to b} g(x)$.
4. $\lim_{x \to b} (f(x).g(x)) = \lim_{x \to b} f(x).\lim_{x \to b} g(x)$.
5. $\lim_{x \to b} (\frac{f(x)}{g(x)}) = \frac{\lim_{x \to b} f(x)}{\lim_{x \to b} g(x)}$, where $\lim_{x \to b} g(x) \neq 0$.
6. $\lim_{x \to b} (\sqrt[n]{f(x)}) = \sqrt[n]{\lim_{x \to b} f(x)}$,where $\lim_{x \to b} f(x) \geq 0$.
7. $\lim_{x \to b} x = b$.
8. $\lim_{x \to b} x^n = b.^n$

Examples

1.Evaluating the Limit of a Constant Function at a Point:

Example 1 *Let $f(x) = 5$ Find $lim_{x \to 1} f(x)$?*
Solution: $lim_{x \to 1} 5=5$. *since there is no x in $f(x)$ anyway.*

2.Evaluating the Limit of a Polynomial Function at a Point:

Example 2 $f(x) = x^2 + x - 2$.*Calculate $lim_{x \to 2} f(x)$?*

Solution $lim_{x \to 2} f(x) = 2^2 + 2 - 2 = 4$.

3. Evaluating the Limit of a Rational Function at a Point:

Example 3 $f(x) = \frac{3x+2}{x-1}$.*Calculate $lim_{x \to 2} f(x)$?*

Note that f is a rational function with implied domain $Dom(f) = \{x \in R, x \neq 1\}$.
Solution $lim_{x \to 2} f(x) = \frac{3.2+2}{2-1}= 8$.

Example 4 $f(x) = \frac{x^2-4}{x-2}$, *Calculate $lim_{x \to 2} f(x)$?*

Notice that the function $f(x) = \frac{x^2-4}{x-2}$ is not defined when x =2,because the denominator is zero we use the simplification first and then we substitute the value of x.

Solution $lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2}(x + 2) = 4$

Example 5: Let $f(x) = \frac{1}{x^2}$ Find $\lim_{x\to 0} f(x)$ if it exists?

Solution: The value of $f(x)$ do not approach 0 number, so $\lim_{x\to 0}\frac{1}{x^2}$ does not exist.

4. Two sided limits:

Let $f(x)$ be a function then the right-limit define as $\lim_{x\to b^+} f(x)$(the limit of $f(x)$ as approaches a form the right).And the left-limit define as $\lim_{x\to b^-} f(x)$(the limit of $f(x)$ as approaches a form the left).

Remark

$\lim_{x\to b} f(x) = L$ if and only if $\lim_{x\to b^+} f(x) = L = \lim_{x\to b^-} f(x)$.

Example 6 Let $f(x) = \begin{cases} 2x - 1 & , x \geq -1 \\ 3x & , x < -1 \end{cases}$,

Find the left and the right $\lim_{x\to -1} f(x)$?and explain if $\lim_{x\to -1} f(x)$if it exists?

Solution :

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (2x - 1) = -3 = L_1$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 3x = -3 = L_2$$

$\therefore L_1 = L_2 \to \lim_{x\to -1} f(x)$is exists and equal -3.

Example 7 Let $f(x) = \begin{cases} -3x + 1 & , x < 1 \\ x^2 - 2 & , x \geq 1 \end{cases}$, Find the left and the right $\lim_{x\to 1} f(x)$? and explain if $\lim_{x\to 1} f(x)$ if it exists?

Solution :

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 - 2) = -1 = L_1$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-3x + 1) = -2 = L_2$$

We notice $L_1 \neq L_2 \to \lim_{x\to 1} f(x)$does not exist.

Example 8 Calculate the following :

a) $\lim_{x\to b}(x^3 + 4x^2 - 3)$

b) $\lim_{x\to b}\frac{x^2+3x+2}{x+1}$

c) $\lim_{x\to 2}\sqrt{2x^2 + 1}$

d) $\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$

f) $\lim_{x\to 9}\frac{\frac{1}{\sqrt{x}}-\frac{1}{3}}{x-9}$

Solution

$$a)\lim_{x\to b}(x^3+4x^2-3) = \lim_{x\to b}x^3+\lim_{x\to b}4x^2-\lim_{x\to b}3$$

$$=b^3+4b^2-3$$

$$b)\lim_{x\to b}\frac{x^2+3x+2}{x+1} = \frac{\lim_{x\to b}(x^2+3x+2)}{\lim_{x\to b}(x+1)} = \frac{b^2+3b+2}{b+1}$$

$$c)\lim_{x\to 2}\sqrt{2x^2+1} = \sqrt{\lim_{x\to 2}(2x^2+1)} = \sqrt{2.2^2+1} = 3$$

d) $\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$

We can Multiply both numerator and denominator by the conjugate radical expression $\sqrt{x^2+100}+10$ (obtained by changing the sign after the square root).

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$=\frac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)} = \frac{x^2}{x^2(\sqrt{x^2+100}+10)} = \frac{1}{(\sqrt{x^2+100}+10)}$$

Therefore,

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0}\frac{1}{(\sqrt{x^2+100}+10)} = \frac{1}{20}$$

$$\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6} = \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-3)(x-2)} = \frac{4}{-1} = -4$$

$$f)\ \lim_{x\to 9}\frac{\frac{1}{\sqrt{x}}-\frac{1}{3}}{x-9}=\lim_{x\to 9}\frac{\frac{3-\sqrt{x}}{3\sqrt{x}}}{(\sqrt{x}-3)(\sqrt{x}+3)}=\lim_{x\to 9}\frac{3-\sqrt{x}}{3\sqrt{x}}*\frac{1}{(\sqrt{x}-3)(\sqrt{x}+3)}$$

$$\lim_{x\to 9}\frac{-(\sqrt{x}\ -3)}{3\sqrt{x}}*\frac{1}{(\sqrt{x}-3)(\sqrt{x}+3)}=\frac{-1}{54}$$

Differentiation:

Derivative definition :

The derivative of a function f is the function f' whose value at x is defined by the equation: $f'(x\ )=lim_{h\to 0}\frac{f(x+h\ )-f(x\ )}{h}$

Notes

- *A function that has derivatives at a point x is said to be differentiable at x .*
- *A function that is differentiable at every point of its' domain is called differentiable.*
- *The differentiation operation is often denoted by $\frac{df}{dx}$, which read (the derivative of $f(x)$with respect x.*

***Example1:** Find the derivative of $f(x)=\sqrt{x}$by definition ?*

***Sol.** :* $f'(x)=lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

$$=lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}*\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=\frac{1}{2\sqrt{x}}$$

***Example2:** Find the derivative of $f(x)=x^2+3x+2$ by definition ?*

Sol: $f'(x)=lim_{h\to 0}\frac{(x+h)^2+3(x+h)+2-(x^2+3x+2}{h}$

$$=\lim_{h\to 0}\frac{x^2+2hx+h^2+3x+3h+2-x^2-3x-2}{h}$$

$$=\lim_{h\to 0}\frac{2hx+h^2+3h}{h}=\lim_{h\to 0}\frac{h(2x+h+3)}{h}$$

$= 2x + 0 + 3 = 2x + 3.$

Example3: Find the derivative of $f(x) = \frac{1}{x}$ by definition ?

Sol: $f'(x) = lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = lim_{h\to 0} \frac{1}{h} \left(\frac{x-(x+h)}{x(x+h)}\right)$

$= lim_{h\to 0} \frac{1}{h} \left(\frac{-h}{x(x+h)}\right) = lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}.$

Example 4: Find the derivative of $f(x) = x^2 + 3$ by definition ?

Sol: $f'(x) = lim_{h\to 0} \frac{(x+h)^2+3-(x^2+3)}{h}$

$$= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 + 3 - x^2 - 3}{h}$$

$= lim_{h\to 0} \frac{h(2x+h)}{h} = 2x + 0 = 2x$.

Slopes and Tangent Lines:

When the value $f'(x)$ is exists is called slope of the curve $y = f(x)$ at x . The line through the point $(x, f(x))$ with slope $f'(x) = m$ is the tangent to the curve at x.

Now, steps to find the equation of the tangent :-

1. Find a contact point (x_1, y_1).
2. Find the slope of the curve $m = f'(x)$.

3.Apply following relation $y - y_1 = m(x - x_1)$.

Example1: Find the equation for the tangent to the curve $f(x) = (2 - x^2)^2$ at $x = 2$?

Sol. :- from steps a bove:

1.Find a contact point $y_1 = f(2) = (2 - 2^2)^2 = (-2)^2 = 4$.

$(x_1, y_1) = (2,4)$

2.Find the slope $m = f'(x)$

$\rightarrow m = f'(x) = 2(2 - x^2) * (-2x) = -4x(2 - x^2)$

$\rightarrow m = f'(2) = -4(2)(2 - 2^2) = -8 * (-2) = 16$

3.Apply the relation $y - y_1 = m(x - x_1)$

$y - 4 = 16(x - 2) = 16x - 32$

$16x - y - 32 + 4 = 0$

$16x - y - 28 = 0$

$y = 16x - 28$

$\rightarrow$ The equation for tangent is $y = 16x - 28$

Example 2 :- Find equation for the tangent to the curve $y = \sqrt{x}$ at $x = 4$

Sol. :

1.firstly find (x_1, y_1) by

$y_1 = \sqrt{x_1} = \sqrt{4} = 2 \rightarrow (x_1, y_1) = (4,2)$

2.Then, find slope m= f′(x)

$\rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

$\rightarrow m = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

3.Now, Apply the equation :-

$y - y_1 = m\,(x - x_1)$

$y - 2 = \frac{1}{4}(x - 4)$

$y = \frac{1}{4}x - 1 + 2$

$\rightarrow$ *The equation for tangent is* $y = \frac{1}{4}x + 1$

Example 3:- find equation for the tangent to the curve $y = \sqrt[3]{3x+5}$ *at* $x = 1$*?*

Sol. :

1. find (x_1, y_1) *by* $y_1 = f(x_1) = f(1) = \sqrt[3]{3(1)+5} = \sqrt[3]{8} = 2$

$\rightarrow (x_1, y_1) = (1,2)$

2. find slope $m = f'(x)$

$\rightarrow f'(x) = \frac{1}{3}\,(3x+5)^{-2/3} * 3 = \frac{1}{\sqrt[3]{(3x+5)^2}}$

$m = f'(1) = \frac{1}{\sqrt[3]{(3(1)+5)^2}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$

3. Apply equation

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{4}(x - 1) \rightarrow 4y - 8 = x - 1$

$x - 4y + 8 - 1 = 0$

$x - 4y + 7 = 0$

So, equation of tangent is $y = \frac{1}{4}(x + 7)$.

Example 4: find equation for the tangent to the curve $y = x^2$ *at the point* $\left(\frac{-1}{2}, \frac{1}{4}\right)$

Sol. : firstly, find slope of curve:

$f'(x) = 2x \rightarrow m = f'(x) = 2\left(\frac{-1}{2}\right) = -1$

Apply equation $y - y_1 = m(x - x_1)$

$y - \frac{1}{4} = -1\left(x - \left(\frac{-1}{2}\right)\right)$

$y - \frac{1}{4} = -x - \frac{1}{2}$

$y = -x - \frac{1}{4}$

$\rightarrow$ *equation of tangent is* $y = -x - \frac{1}{4}$