Unit Three: Sequences and Series

(Reference: THOMAS CALCULUS, 12th edition)

3.1 Sequences

A sequence $\{a_n\}$ is a function whose domain is the set of positive integers (n=1,2,3,...).

For example, the sequence $\{1 + i^n\}$ is

$$\begin{array}{ccccc} 1+i, & 0, & 1-i, & 2, & 1+i,\dots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ n=1, & n=2, & n=3, & n=4, & n=5,\dots . \end{array}$$

Notes:

1- The sequence may be finite or infinite, and defined by a rules.
2- The index n =1,2,3, ..., refers to the term's number.

Sequence	Defining Rule
$0,1,2,\dots,n-1,\dots$	$a_n = n-1$
$1,\frac{-1}{2},\frac{1}{3},\frac{-1}{4},\dots,(-1)^{n+1}\,\frac{1}{n},\dots$	$a_n = (-1)^{n+1}\,\frac{1}{n}$
$0,\frac{-1}{2},\frac{2}{3},\frac{-3}{4},\dots,(-1)^{n+1}\,(\frac{n-1}{n}),\dots$	$a_n = (-1)^{n+1}\,(\frac{n-1}{n})$
$1,1,2,3,5,8,13,21,34,55,89,\dots,a_{n-1}+a_{n-2},\dots$	$a_n = a_{n-1} + a_{n-2}$

3.1.1 Convergent and Divergent Sequences

A sequence is said to be *convergent* if it approaches some limit. If such a limit does not exist, the sequence is *divergent*.

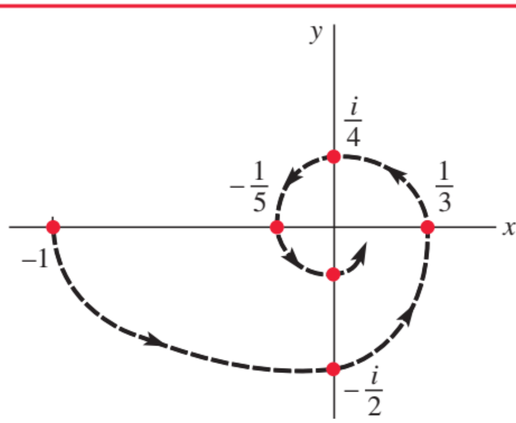
EXAMPLE 1 A Convergent Sequence

The sequence $\left\{\frac{i^{n+1}}{n}\right\}$ converges, since

$$\lim_{n\to\infty} \frac{i^{n+1}}{n} = 0.$$

As we see from $-1, -\frac{i}{2}, \frac{1}{3}, \frac{i}{4}, -\frac{1}{5},\dots,$

the terms of the sequence spiral toward 0.



Example 2: $a_n = \dfrac{3n^4 + 34n^3 + 14}{2n^2 + 15n + 8}$ is a divergent sequence.

Example 3: Examine the convergence of $a_n = \left(1 + \dfrac{1}{n}\right)^n$.

Solution:
$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{n}\right)}$$
$$= e^{\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)}$$
$$= e^{\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}}$$
$$= e^{\lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n}}\cdot\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}}}$$
$$= e$$

Exercises 10.1 (Page:541)

Which of the sequences $\{a_n\}$ in Exercises 27–90 converge, and which diverge? Find the limit of each convergent sequence.

27. $a_n = 2 + (0.1)^n$

28. $a_n = \dfrac{n + (-1)^n}{n}$

29. $a_n = \dfrac{1 - 2n}{1 + 2n}$

30. $a_n = \dfrac{2n + 1}{1 - 3\sqrt{n}}$

31. $a_n = \dfrac{1 - 5n^4}{n^4 + 8n^3}$

32. $a_n = \dfrac{n + 3}{n^2 + 5n + 6}$

33. $a_n = \dfrac{n^2 - 2n + 1}{n - 1}$

34. $a_n = \dfrac{1 - n^3}{70 - 4n^2}$

35. $a_n = 1 + (-1)^n$

36. $a_n = (-1)^n\left(1 - \dfrac{1}{n}\right)$

37. $a_n = \left(\dfrac{n + 1}{2n}\right)\left(1 - \dfrac{1}{n}\right)$

38. $a_n = \left(2 - \dfrac{1}{2^n}\right)\left(3 + \dfrac{1}{2^n}\right)$

39. $a_n = \dfrac{(-1)^{n+1}}{2n - 1}$

40. $a_n = \left(-\dfrac{1}{2}\right)^n$

41. $a_n = \sqrt{\dfrac{2n}{n + 1}}$

42. $a_n = \dfrac{1}{(0.9)^n}$

3.2 Infinite Series

For the sequence $\{a_n\}$ an expression of the form $a_1+a_2+a_3+ \dots +a_n+ \dots$ is called an *infinite series*.

For the sequence $\{S_n\}$:

$S_1=a_1$

$S_2=a_1+a_2$

⋮

$S_n=a_1+a_2+\dots+a_n=\sum_{k=1}^{n} a_k$

the sequence $\{S_n\}$ is called the sequence of partial sums of the series. If this sequence converges to a limit L, we say that the series is converges, and that its limit is L, or

$a_1+a_2+\dots+a_n+\dots=\sum_{n=1}^{\infty} a_n=L$

if this sequence does not converge, we say that the series *diverges*.

An easy example of a convergent series is

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

The partial sums look like $\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\cdots$, and we can see that they get closer and closer to 1.

3.3.1 Geometric Series

In this type of series, the ratio of each consecutive terms (a_{n+1}/a_n) is a constant.

For the geometric sequence $a, ar, ar^2, \dots, ar^{n-1}, \dots$, the corresponding geometric series is $a + ar + ar^2 + \dots, ar^{n-1} + \dots$, in which a and r are fixed real numbers and $a \neq 0$.

The n^{th} term of this series is $a_n= a\, r^{n-1}$.

3.3 Convergence Tests of Series

3.3.1 Geometric Series Test

The geometric series $S_n=\sum_{n=1}^{\infty} a\, r^{n-1}$ is:

1- Convergent to the sum $\frac{a}{1-r}$, if $|r|<1$ (or $-1<r<1$).
2- Divergent otherwise ($|r|\geq 1$).

a	r	Geometric series	
9	1/3	9 + 3 + 1 + 1/3 + 1/9 + ⋯	Convergent
1	−1/2	1 − 1/2 + 1/4 − 1/8 + 1/16 − 1/32 + ⋯	
3	1	3 + 3 + 3 + 3 + 3 + ⋯	Divergent
3	−1	3 − 3 + 3 − 3 + 3 − ⋯	
4	10	4 + 40 + 400 + 4000 + 40,000 + ⋯	

Exercises 10.2 (Page:551)

In Exercises 15–18, determine if the geometric series converges or diverges. If a series converges, find its sum.

15. $1 + \left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \cdots$

16. $1 + (-3) + (-3)^2 + (-3)^3 + (-3)^4 + \cdots$

17. $\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \left(\frac{1}{8}\right)^4 + \left(\frac{1}{8}\right)^5 + \cdots$

18. $\left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \left(\frac{-2}{3}\right)^5 + \left(\frac{-2}{3}\right)^6 + \cdots$

3.3.2 Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

EXAMPLE 1 Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \lim_{x\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^x$$

$$= \frac{1}{1-p}\lim_{x\to\infty}\left(\frac{1}{x^{p-1}} - 1\right)$$

$$= \frac{1}{1-p}(0-1) = \frac{1}{p-1},$$

the series converges by the Integral Test.

EXAMPLE 2 The series $\sum_{n=1}^{\infty}(1/(n^2+1))$ is not a p-series, but it converges by the Integral Test. The function $f(x) = 1/(x^2+1)$ is positive, continuous, and decreasing for $x \geq 1$, and

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{x\to\infty}\left[\arctan x\right]_1^x$$
$$= \lim_{x\to\infty}[\arctan x - \arctan 1]$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Exercises 10.3

Applying the Integral Test

Use the Integral Test to determine if the series in Exercises 1–10 converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

4. $\sum_{n=1}^{\infty} \frac{1}{n+4}$

5. $\sum_{n=1}^{\infty} e^{-2n}$

6. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

7. $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$

8. $\sum_{n=2}^{\infty} \frac{\ln (n^2)}{n}$

9. $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

10. $\sum_{n=2}^{\infty} \frac{n-4}{n^2-2n+1}$

3.3.3 The Ratio Test

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then **(a)** the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

EXAMPLE 1 Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$

Solution We apply the Ratio Test to each series.

(a) For the series $\sum_{n=0}^{\infty}(2^n + 5)/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}\right) \rightarrow \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges because $\rho = 2/3$ is less than 1. This does *not* mean that 2/3 is the sum of the series. In fact,

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1 - (2/3)} + \frac{5}{1 - (1/3)} = \frac{21}{2}.$$

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n + 2)!}{(n + 1)!(n + 1)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!}$$

$$= \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} = \frac{4n + 2}{n + 1} \rightarrow 4.$$

The series diverges because $\rho = 4$ is greater than 1.

(c) If $a_n = 4^n n!n!/(2n)!$, then

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n + 1)!(n + 1)!}{(2n + 2)(2n + 1)(2n)!} \cdot \frac{(2n)!}{4^n n!n!}$$

$$= \frac{4(n + 1)(n + 1)}{(2n + 2)(2n + 1)} = \frac{2(n + 1)}{2n + 1} \rightarrow 1.$$

Because the limit is $\rho = 1$, we cannot decide from the Ratio Test whether the series converges. When we notice that $a_{n+1}/a_n = (2n + 2)/(2n + 1)$, we conclude that a_{n+1} is always greater than a_n because $(2n + 2)/(2n + 1)$ is always greater than 1. and the nth term does not approach zero as $n \rightarrow \infty$. The series diverges.

3.3.4 The Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then **(a)** the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

Note: We will need the following fact to solve some problems:
For any positive real number a,

$$\lim_{n\to\infty} \sqrt[n]{n^a} = \lim_{n\to\infty} n^{a/n} = 1$$

EXAMPLE Which of the following series converge, and which diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Solution We apply the Root Test to each series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\displaystyle\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \to \frac{1^2}{2} < 1.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges because $\displaystyle\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{\left(\sqrt[n]{n}\right)^3} \to \frac{2}{1^3} > 1.$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges because $\displaystyle\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n} \to 0 < 1.$

Remarks:

1- If your terms contain factorials, or factorials and n^{th} powers, the Ratio Test might be helpful.

2- If your terms contain n^{th} powers, the Root Test may be helpful.

3- If $a_n = f(n)$ for some positive, decreasing function and $\int_a^{\infty} f(x)dx$ is easy to evaluate, then the Integral Test may work.

Exercises 10.5 (page 567)

Using the Ratio Test

In Exercises 1–8, use the Ratio Test to determine if each series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

2. $\sum_{n=1}^{\infty} \frac{n+2}{3^n}$

3. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$

4. $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$

5. $\sum_{n=1}^{\infty} \frac{n^4}{4^n}$

6. $\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$

7. $\sum_{n=1}^{\infty} \frac{n^2(n+2)!}{n!\,3^{2n}}$

8. $\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

Using the Root Test

In Exercises 9–16, use the Root Test to determine if each series converges or diverges.

9. $\sum_{n=1}^{\infty} \frac{7}{(2n+5)^n}$

10. $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

11. $\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$

12. $\sum_{n=1}^{\infty} \left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$

13. $\sum_{n=1}^{\infty} \frac{8}{(3+(1/n))^{2n}}$

14. $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$

15. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$

16. $\sum_{n=2}^{\infty} \frac{1}{n^{1+n}}$

(*Hint:* $\lim_{n\to\infty} (1 + x/n)^n = e^x$)

In Exercises 17– , use any method to determine if the series converges or diverges. Give reasons for your answer.

17. $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$

18. $\sum_{n=1}^{\infty} n^2 e^{-n}$

19. $\sum_{n=1}^{\infty} n!\,e^{-n}$

20. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

21. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

22. $\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$

23. $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$

24. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$

25. $\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^n$

26. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{3n}\right)^n$