Unit Two: LAPLACE TRANSFORMATION

2.1 Definition, Basic Principles and Properties

The Laplace transform method is a powerful method for solving linear ODEs and corresponding initial value problems, as well as systems of ODEs arising in engineering. If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. It is a function of s, say, $F(s)$, and is denoted by $\mathscr{L}(f)$; thus

$$F(s) = \mathscr{L}(f) = \int_0^\infty e^{-st} f(t)dt$$

Note: The Laplace transform operation can be denoted by the symbol $\mathscr{L}$ or $\mathcal{L}$ or L.

EXAMPLE 1 Using Definition

Evaluate $\mathscr{L}\{1\}$.

SOLUTION

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{b\to\infty}\int_0^b e^{-st}dt$$

$$= \lim_{b\to\infty}\frac{-e^{-st}}{s}\bigg|_0^b = \lim_{b\to\infty}\frac{-e^{-sb}+1}{s} = \frac{1}{s}$$

EXAMPLE 2

Evaluate $\mathscr{L}\{t\}$.

SOLUTION From Definition ,we have $\mathscr{L}\{t\} = \int_0^\infty e^{-st}t\,dt$. Integrating by parts we obtain

$$\mathscr{L}\{t\} = \frac{-te^{-st}}{s}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s}\mathscr{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}.$$

EXAMPLE 3

Evaluate **(a)** $\mathscr{L}\{e^{-3t}\}$ **(b)** $\mathscr{L}\{e^{6t}\}$.

SOLUTION

(a) $$\mathscr{L}\{e^{-3t}\} = \int_0^\infty e^{-3t}e^{-st}dt = \int_0^\infty e^{-(s+3)t}dt = \frac{-e^{-(s+3)t}}{s+3}\bigg|_0^\infty = \frac{1}{s+3}$$

(b) $$\mathscr{L}\{e^{6t}\} = \int_0^\infty e^{6t}e^{-st}dt = \int_0^\infty e^{-(s-6)t}dt = \frac{-e^{-(s-6)t}}{s-6}\bigg|_0^\infty = \frac{1}{s-6}$$

EXAMPLE 4

Evaluate $\mathscr{L}\{\sin 2t\}$.

SOLUTION

$$\mathscr{L}\{\sin 2t\} = \int_0^\infty e^{-st}\sin 2t\,dt = \frac{-e^{-st}\sin 2t}{s}\bigg|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt$$

$$= \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt, \quad s > 0$$

$\lim_{t\to\infty} e^{-st}\cos 2t = 0,\ s > 0$ ↓ Laplace transform of $\sin 2t$ ↓

$$= \frac{2}{s}\left[\frac{-e^{-st}\cos 2t}{s}\bigg|_0^\infty - \frac{2}{s}\int_0^\infty e^{-st}\sin 2t\,dt\right]$$

$$= \frac{2}{s^2} - \frac{4}{s^2}\mathscr{L}\{\sin 2t\}.$$

At this point we have an equation with $\mathscr{L}\{\sin 2t\}$ on both sides of the equality. Solving for that quantity yields the result

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2 + 4}, \qquad s > 0.$$

2.2 Laplace Transforms of Some Basic Functions

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$1/s$
t	$1/s^2$
$t^n\ (n = 1,\ 2, \ldots)$	$n!/s^{n+1}$
e^{at}	$1/(s-a)$
$\sin at$	$a/(s^2+a^2)$
$\cos at$	$s/(s^2+a^2)$
$\sinh at$	$a/(s^2-a^2)$
$\cosh at$	$s/(s^2-a^2)$

2.2.1 Laplace of $e^{at} f(t)$ (The s-shift theorem)

The Laplace transform of $e^{at} f(t)$ is obtained from $F(s)$ by replacing s by $s - a$.

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$$

2.2.2 Laplace of $t^n f(t)$

Let $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

EXAMPLE find $\mathcal{L}\{e^{at} t^n\}$, $\mathcal{L}\{e^{at} \cos bt\}$, and $\mathcal{L}\{e^{at} t \sin bt\}$.

Solution

$$\mathcal{L}\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{e^{at} \cos bt\} = \frac{(s-a)}{[(s-a)^2 + b^2]}$$

and

$$\mathcal{L}\{e^{at} t \sin bt\} = \frac{2b(s-a)}{[(s-a)^2 + b^2]^2}$$

Example : Find the Laplace transform of $t^2 \cos at$.

Solution: $\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$

$$\mathcal{L}(t^2 \cos at) = \frac{d^2}{ds^2}\left[\frac{s}{s^2 + a^2}\right]$$

$$= \frac{d}{ds}\frac{(s^2 + a^2).1 - s(2s)}{(s^2 + a^2)^2} = \frac{d}{ds}\frac{a^2 - s^2}{(s^2 + a^2)^2}$$

$$= \frac{(s^2 + a^2)^2(-2s) - (a^2 - s^2).2(s^2 + a^2)(2s)}{(s^2 + a^2)^4}$$

$$= \frac{-2s^3 - 2a^2s - 4a^2s + 4s^3}{(s^2 + a^2)^3}$$

$$= \frac{2s(s^2 - 3a^3)}{(s^2 + a^2)^3}$$

Example: Find $\mathcal{L}\,[e^{3t}\sin(2t)]$.

Solution: *In this case,* $f(t) = \sin(2t)$

$$F(s) = \mathcal{L}[f(t)] = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$$

$$\mathcal{L}\left[e^{3t}\sin(2t)\right] = F(s-3) = \frac{2}{(s-3)^2+4}$$

2.3 Some Important Properties of Laplace Transforms

1. Linearity

Let the functions $f_1(t), f_2(t), \ldots, f_n(t)$ have Laplace transforms, and let $c_1, c_2, \ldots, c_n$ be any set of arbitrary constants. Then

$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\} + \cdots + c_n\mathcal{L}\{f_n(t)\}$

EXAMPLE Find the Laplace transform of $f(t) = c_1e^{at} + c_2e^{-at}$.

Solution $\mathcal{L}\{c_1e^{at} + c_2e^{-at}\} = c_1\mathcal{L}\{e^{at}\} + c_2\mathcal{L}\{e^{-at}\}$

$$= \frac{c_1}{(s-a)} + \frac{c_2}{(s+a)}$$

2. Laplace Transform of The Derivatives of $f(t)$

The Laplace transform of derivative of order n is

$$\mathcal{L}[f^n(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0)$$

So;

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0)$$

$$\mathcal{L}[f'''(t)] = s^3\mathcal{L}[f(t)] - s^2f(0) - sf'(0) - f''(0)$$

$$\mathcal{L}[f''''(t)] = s^4\mathcal{L}[f(t)] - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$$

4.1 Exercises (page: 217)

In Problems 19–36, find $\mathscr{L}\{f(t)\}$.

19. $f(t) = 2t^4$

20. $f(t) = t^5$

21. $f(t) = 4t - 10$

22. $f(t) = 7t + 3$

23. $f(t) = t^2 + 6t - 3$

24. $f(t) = -4t^2 + 16t + 9$

25. $f(t) = (t + 1)^3$

26. $f(t) = (2t - 1)^3$

27. $f(t) = 1 + e^{4t}$

28. $f(t) = t^2 - e^{-9t} + 5$

29. $f(t) = (1 + e^{2t})^2$

30. $f(t) = (e^t - e^{-t})^2$

31. $f(t) = 4t^2 - 5\sin 3t$

32. $f(t) = \cos 5t + \sin 2t$

33. $f(t) = \sinh kt$

34. $f(t) = \cosh kt$

35. $f(t) = e^t \sinh t$

36. $f(t) = e^{-t}\cosh t$

2.3 Inverse Laplace transforms and their Properties.

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathscr{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write $f(t) = \mathscr{L}^{-1}\{F(s)\}$.

Some Inverse Transforms

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

(b) $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, $n = 1, 2, 3, \ldots$

(c) $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(d) $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

(e) $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$

(f) $\mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt$

(g) $\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$

EXAMPLE 1

Evaluate **(a)** $\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\}$ **(b)** $\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$.

SOLUTION **(a)** we identify $n + 1 = 5$ or $n = 4$ and then multiply and divide by 4!:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4.$$

(b) we identify $k^2 = 7$ and so $k = \sqrt{7}$.

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}t.$$

Note: **$\mathscr{L}^{-1}$ Is a Linear Transform** The inverse Laplace transform is also a linear transform; that is, for constants α and β,

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathscr{L}^{-1}\{F(s)\} + \beta \mathscr{L}^{-1}\{G(s)\},$$

where F and G are the transforms of some functions f and g.

EXAMPLE 2

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{-2s + 6}{s^2 + 4}\right\}$.

SOLUTION

$$\mathscr{L}^{-1}\left\{\frac{-2s + 6}{s^2 + 4}\right\} = \mathscr{L}^{-1}\left\{\frac{-2s}{s^2 + 4} + \frac{6}{s^2 + 4}\right\}$$

$$= -2\mathscr{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} + \frac{6}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\}$$

$$= -2\cos 2t + 3\sin 2t.$$

Example 3 Compute the inverse Laplace transform of

$$Y(s) = \frac{2}{3s^4}.$$

Solution

$$Y(s) = \frac{2}{3s^4} = \frac{1}{9}.\frac{3!}{s^4}$$

Thus, by linearity,

$$y(t) = L^{-1}[\frac{1}{9}.\frac{3!}{s^4}]$$

$$= \frac{1}{9}L^{-1}[\frac{3!}{s^4}]$$

$$= \frac{1}{9}t^3$$

Example 4 Compute the inverse Laplace transform of Y (s) = $\frac{1}{3-4s}+\frac{3-2s}{s^2+49}$

Solution Adjust it as follows:

Y (s) = $\frac{1}{3-4s}+\frac{3-2s}{s^2+49}$

= $\frac{1}{-4}.\frac{1}{s-\frac{3}{4}}+\frac{3}{s^2+49}-\frac{2s}{s^2+49}$

= $\frac{1}{-4}.\frac{1}{s-\frac{3}{4}}+\frac{3}{7}.\frac{7}{s^2+49}-2.\frac{s}{s^2+49}$

y(t) = $L^{-1}[\frac{-1}{4}.\frac{1}{s-\frac{3}{4}}+\frac{3}{7}.\frac{7}{s^2+49}-2.\frac{s}{s^2+49}]$

= $-\frac{1}{4}L^{-1}[\frac{1}{s-\frac{3}{4}}]+\frac{3}{7}L^{-1}[\frac{7}{s^2+49}]-2L^{-1}[\frac{s}{s^2+49}]$

= $-\frac{1}{4}e^{(\frac{3}{4})t}+\frac{3}{7}sin7t-2cos7t$

Example 5 Compute the inverse Laplace transform of Y (s) = $\frac{5}{(s+2)^3}$

Solution The transform pair is: $t \Leftrightarrow \frac{2}{s^3}$

According to the proposition, $e^{-2t}t^2 \Leftrightarrow \frac{2}{(s+2)^3}$

Therefore, y(t) = $L^{-1}[\frac{5}{(s+2)^3}]$

= $L^{-1}[\frac{5}{2}.\frac{2}{(s+2)^3}]$

= $\frac{5}{2}L^{-1}[\frac{2}{(s+2)^3}]$

= $\frac{5}{2}e^{-2t}t^2$

Example 6 Compute the inverse Laplace transform of Y (s) = $\dfrac{4(s-1)}{(s-1)^2+4}$

Solution The transform pair is: $cos2t \Leftrightarrow \dfrac{s}{s^2+4}$

According to the proposition, $e^t cos2t \Leftrightarrow \dfrac{s-1}{(s-1)^2+4}$

Hence, y(t) = $L^{-1}[\dfrac{4(s-1)}{(s-1)^2+4}]$

= $4L^{-1}[\dfrac{s-1}{(s-1)^2+4}]$

= $4e^t cos2t$

2.3.1 Inverse Laplace Transforms Using Partial Fractions

Partial fractions play an important role in finding inverse Laplace transforms. The first step is to factor the denominator as much as possible. Then for each term in the denominator, we will use the following table to get terms for partial fraction decomposition, and then find the inverse Laplace transform with respect to the obtained values.

Term in denominator	Term in partial fraction decomposition
$as+b$	$\dfrac{A}{as+b}$
$(as+b)^n$	$\dfrac{A_1}{as+b}+\dfrac{A_2}{(as+b)^2}+\cdots+\dfrac{A_n}{(as+b)^n}$
as^2+bs+c	$\dfrac{As+B}{as^2+bs+c}$
$(as^2+bs+c)^n$	$\dfrac{A_1s+B_1}{as^2+bs+c}+\dfrac{A_2s+B_2}{(as^2+bs+c)^2}+\cdots+\dfrac{A_ns+B_n}{(as^2+bs+c)^n}$

Example 1: **Distinct Real Roots**; (**the cover-up method)**. Consider that:

$$F(s) = \frac{s+3}{s^3+7s^2+10s} = \frac{s+3}{s(s+2)(s+5)}$$

$$= \frac{A_1}{s} + \frac{A_2}{s+2} + \frac{A_3}{s+5}$$

Find A_1 by first "*covering-up*" the first term in the denominator (i.e., the term that is associated with A_1) with your finger (shown as a gray ellipse), and then letting s=0.

$$A_1 = \left.\frac{s+3}{s(s+2)(s+5)}\right|_{s=0} = \frac{3}{2 \cdot 5} = \frac{3}{10} = 0.3$$

Likewise, for A_2

$$A_2 = \left.\frac{s+3}{s(s+2)(s+5)}\right|_{s=-2} = \frac{1}{-2 \cdot 3} = -\frac{1}{6}$$

and A_3

$$A_3 = \left.\frac{s+3}{s(s+2)(s+5)}\right|_{s=-5} = \frac{-2}{-5 \cdot -3} = -\frac{2}{15}$$

EXAMPLE 2 Partial Fractions and Linearity

Evaluate $\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$.

SOLUTION There exist unique constants A, B, and C such that

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$= \frac{A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)}{(s-1)(s-2)(s+4)}.$$

Since the denominators are identical, the numerators are identical:

$$s^2 + 6s + 9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2).$$

$$16 = A(-1)(5), \quad 25 = B(1)(6), \quad 1 = C(-5)(-6),$$

and so $A = -\frac{16}{5}$, $B = \frac{25}{6}$, $C = \frac{1}{30}$. Hence the partial fraction decomposition is

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4},$$

and thus,

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}.$$

Example 3: Repeated Real Roots. Factorize $Y(s)$ if

$$Y(s) = \frac{2s^2}{(s-6)^3}$$

Solution: *The partial fraction expansion of* $Y(s)$ *is of the form*

$$\frac{2s^2}{(s-6)^3} = \frac{A}{(s-6)^3} + \frac{B}{(s-6)^2} + \frac{C}{s-6}$$

$$\frac{2s^2}{(s-6)^3} = \frac{A}{(s-6)^3} + \frac{B(s-6)}{(s-6)^2(s-6)} + \frac{C(s-6)^2}{(s-6)(s-6)^2}$$

$$= \frac{A + B(s-6) + C(s-6)^2}{(s-6)^3}$$

So we must have $\quad 2s^2 = A + B(s-6) + C(s-6)^2$

$$2s^2 + 0s + 0 = Cs^2 + (B-12C)s + (A-6B+36C)$$

This tells us that

$$C = 2$$

$$B - 12C = 0$$

$$A - 6B + 36C = 0$$

and then $A = 72$, $B = 24$ and $C = 2$

Thus,

$$Y(s) = \frac{2s^2}{(s-6)^3}$$

$$= \frac{A}{(s-6)^3} + \frac{B}{(s-6)^2} + \frac{C}{s-6}$$

$$= \frac{72}{(s-6)^3} + \frac{24}{(s-6)^2} + \frac{2}{s-6}$$

Example 4: Find inverse Laplace transform of

$$F(s) = \frac{s^2+8}{s(s+2)(s^2+4)}$$

Solution

$$F(s) = \frac{s^2+8}{s(s+2)(s^2+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow \frac{s^2+8}{s(s+2)(s^2+4)} = \frac{A(s+2)(s^2+4) + Bs(s^2+4) + (Cs+D)s(s+2)}{s(s+2)(s^2+4)}$$

$$\Rightarrow s^2+8 = A(s+2)(s^2+4) + Bs(s^2+4) + (Cs+D)s(s+2)$$

At $s = 0$, we get:

$$(0)^2 + 8 = A(0+2)((0)^2+4) + B(0)((0)^2+4) + (C(0)+D)(0)(0+2)$$

$$\Rightarrow 8 = A(2)(4) \Rightarrow A = \frac{8}{(2)(4)} = 1$$

At $s = -2$, we get:

$$(-2)^2 + 8 = A(-2+2)((-2)^2+4) + B(-2)((-2)^2+4) + (C(-2)+D)(-2)(-2+2)$$

$$\Rightarrow 12 = B(-2)(8) \Rightarrow B = \frac{12}{(-2)(8)} = -\frac{3}{4} = -0.75$$

Now, with $A = 1$ and $B = -0.75$:

$$s^2+8 = (s+2)(s^2+4) + (-0.75)s(s^2+4) + (Cs+D)s(s+2)$$

$$\Rightarrow s^2+8 = (s^3+4s+2s^2+8) + (-0.75s^3-3s) + (Cs^3+2Cs^2+Ds^2+2Ds)$$

$$\Rightarrow s^2+8 = (1-0.75+C)s^3 + (2+2C+D)s^2 + (4-3+2D)s + (8)$$

So, we get:

$$0 = 1-0.75+C \quad \Rightarrow C = -0.25$$

$$0 = 4-3+2D \Rightarrow D = -0.5$$

As a result,

$$F(s) = \frac{s^2+8}{s(s+2)(s^2+4)} = \frac{1}{s} - \left(\frac{0.75}{s+2}\right) - \left(\frac{0.25s+0.5}{s^2+4}\right)$$

$$= \frac{1}{s} - 0.75\left(\frac{1}{s+2}\right) - 0.25\left(\frac{s}{s^2+4}\right) - \frac{0.5}{2}\left(\frac{2}{s^2+4}\right)$$

$$\therefore \mathcal{L}^{-1}\{F(s)\} = 1 - 0.75e^{-2t} - 0.25\cos 2t - 0.25\sin 2t$$

2.3.2 The Solution of Differential Equations Using Laplace Transforms

The linear differential equation:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = g(t),$$

$$y(0) = y_0, y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1},$$

where the coefficients a_i, $i = 0, 1, \ldots, n$ and $y_0, y_1, \ldots, y_{n-1}$ are constants, can be solved by Laplace transform techniques. The procedure for solving this IVP is summarized in the following four steps:

1. Take the Laplace Transform of the differential equation.
2. Put initial conditions into the resulting equation.
3. Rearrange your equation to isolate $Y(s)$ equated to something.
4. Calculate the inverse Laplace transform, which will be your final solution to the original differential equation.

EXAMPLE 1 Solving a First–Order IVP

Use the Laplace transform to solve the initial-value problem

$$\frac{dy}{dt} + 3y = 13 \sin 2t, \qquad y(0) = 6.$$

SOLUTION We first take the transform of the differential equation:

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} + 3\mathscr{L}\{y\} = 13\mathscr{L}\{\sin 2t\}$$

$$\mathscr{L}\{dy/dt\} = sY(s) - y(0) = sY(s) - 6,$$

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2 + 4} \qquad \text{or} \qquad (s + 3)Y(s) = 6 + \frac{26}{s^2 + 4}$$

Solving the last equation for $Y(s)$, we get

$$Y(s) = \frac{6}{s + 3} + \frac{26}{(s + 3)(s^2 + 4)} = \frac{6s^2 + 50}{(s + 3)(s^2 + 4)}.$$

$$\frac{6s^2 + 50}{(s + 3)(s^2 + 4)} = \frac{A}{s + 3} + \frac{Bs + C}{s^2 + 4}.$$

$$6s^2 + 50 = A(s^2 + 4) + (Bs + C)(s + 3)$$

Setting $s = -3$ then yields $A = 8$

we equate the coefficients of s^2 and s: $\quad 6 = A + B$ and $0 = 3B + C$

$B = -2$, and $\quad C = 6$

$$Y(s) = \frac{6s^2 + 50}{(s + 3)(s^2 + 4)} = \frac{8}{s + 3} + \frac{-2s + 6}{s^2 + 4}.$$

$$y(t) = 8\mathscr{L}^{-1}\left\{\frac{1}{s + 3}\right\} - 2\mathscr{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} + 3\mathscr{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\}.$$

$$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

EXAMPLE 2 Solving a Second-Order IVP

Solve $y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5.$

SOLUTION

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathscr{L}\left\{\frac{dy}{dt}\right\} + 2\mathscr{L}\{y\} = \mathscr{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s + 4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s + 4}$$

$$Y(s) = \frac{s + 2}{s^2 - 3s + 2} + \frac{1}{(s^2 - 3s + 2)(s + 4)} = \frac{s^2 + 6s + 9}{(s - 1)(s - 2)(s + 4)}$$

The details of the decomposition of $Y(s)$ into partial fractions have already been carried out in Example 2 in the last section, and the solution is

$$y(t) = -\tfrac{16}{5}e^{t} + \tfrac{25}{6}e^{2t} + \tfrac{1}{30}e^{-4t}$$

Exercises 4.2 (page:225)

In Problems 1–30, find the given inverse transform.

In Problems 33–44, use the Laplace transform to solve the given initial-value problem.